A

Department of Mathematics
University of Toronto

Tuesday, October 29, 2013, 6:10-8:00 PM
MAT 133Y TERM TEST #1

Soln

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ________________

GIVEN NAME: ________________

STUDENT NO: ________________

SIGNATURE: ________________

TUTORIAL TIME and ROOM: ________________

REGCODE and TIMECODE: ________________

T.A.'S NAME: ________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1084	T0501B	W3B	UCA101
T0101B	M9B	SS1086	T0601A	R4A	BA1210
T0101C	M9C	SS1080	T0601B	R4B	BA1220
T0201A	M3A	WI524	T0601C	R4C	AB107
T0201B	M3B	UC52	T0601D	R4D	GB120
T0201C	M3C	UC152	T0701A	F2A	LM157
T0201D	M3D	GB220	T0701B	F2B	BF215
T0301A	T3A	RW229	T0701C	F2C	MP137
T0301B	T3B	BA1230	T0701D	F2D	SS2127
T0301C	T3C	AP120	T0801A	F3A	BF215
T0301D	T3D	WW126	T0801B	F3B	MP118
T0401A	W9A	SS1084	T5101A	M5A	SS2105
T0401B	W9B	SS1088	T5101B	M5B	UC87
T0401C	W9C	LM155	T5201A	M6A	LM162
T0501A	W3A	UC85			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________ STUDENT NO: ______________

PART A. Multiple Choice

1. *[4 marks]*

If money is invested at a nominal rate of 6% compounded continuously, then after 6 months it will have increased by:

$P_0 e^{.06 \times \frac{1}{2}} = P(1+x)$

$x = e^{.03} - 1$

$x = .03045$

B

A. 3%

B. 3.045%

C. 3.053%

D. 3.074%

E. 3.092%

2. *[4 marks]*

How much money is needed to generate a perpetuity of $50,000 per year if the effective annual rate is 5%?

$50{,}000 = .05X$

$1{,}000{,}000 = X$

E

A. $2500

B. $10,000

C. $100,000

D. $10,000,000

E. $1,000,000

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3. *[4 marks]*

Sarah deposits \$2000 into a savings account today and then deposits \$1000 at the end of each year for 19 years. How much will there be in the account at the end of 20 years if the account pays an effective annual rate of 3.5%?

A. \$31, 259.26

B. \$15, 709.84

C. \$30, 336.76

D. \$30, 291.16

E. \$30, 269.47

0 1 19 20

2000 1000 - - - - 1000

There are many ways to get the right answer. One way:

$$2000(1.035)^{20} + \underbrace{1000\, s_{\overline{19}|.035}}_{\text{ordinary annuity of 19 deposits accumulated at "19" then moved forward to 20}} (1.035)$$

First deposit

$$= \$31,259.26$$ A

4. *[4 marks]*

A \$35, 000 loan for 5 years with interest at 4% compounded quarterly has payments every 3 months of \$1939.54. The interest portion of the 16th payment is:

A. \$75.68

B. \$189.54

C. \$94.13

D. \$350.00

E. \$82.50

Check: $35,000 = R\, a_{\overline{20}|.01}$ $R = 1939.54$ ✓

Interest in 16th paymt is for money owed after 15th paymt until 16th

0 15 16 20

R . . . R

so 5 payments remain

$$P.O. = R\, a_{\overline{5}|.01}$$

Interest is $.01 R\, a_{\overline{5}|.01}$

$$= 1939.54\left[1-(1.01)^{-5}\right]$$

$$= \$94.13$$ C

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5. *[4 marks]*

A bond sells for \$567.89. If there are 16 semiannual interest payments of \$15 each remaining (the first in 6 months time) and the semiannual yield rate is 2% (annual yield rate of 4%), then the face value of the bond to the nearest dollar is:

A. \$407

B. \$550

C. \$600

D. \$500

E. \$450

$$567.89 = X(1.02)^{-16} + 15\, a_{\overline{16}|.02}$$

$$X = \left[567.89 - 15\, a_{\overline{16}|.02}\right](1.02)^{16}$$

$$= 500.00 \quad \text{D}$$

6. *[4 marks]*

If a \$100 bond has semiannual coupons with semiannual coupon rate 1.6% (annual coupon rate of 3.2%), 12 years until maturity, and sells for \$102, then its semiannual yield rate (not annual yield rate) is closest to:

A. 1.55%

B. 1.60%

C. 1.45%

D. 1.50%

E. 1.65%

Semi-annual coupon rate is 1.6%

$P > V$ so $i < r$: semi-annual yield rate is < 1.6%

So only A or D are possible.

Try A: 1.55%

$P = 100(1.0155)^{-24} + 1.60\, a_{\overline{24}|.0155}$

$\approx \$101$ too low

so yield is too high

Must be D

Check: $P = 100(1.015)^{-24} + 1.60\, a_{\overline{24}|.015}$

$= 102.00$ ✓

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7. *[4 marks]*

$$\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}^T = \begin{pmatrix} 5 & -9 \\ 3 & -4 \end{pmatrix} - 3\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 0 \\ -3 & -7 \end{pmatrix} \quad B$$

A. $\begin{bmatrix} 3 & -1 \\ -5 & 6 \end{bmatrix}$

B. $\begin{bmatrix} 2 & 0 \\ -3 & -7 \end{bmatrix}$

C. $\begin{bmatrix} -1 & 5 \\ 6 & -2 \end{bmatrix}$

D. $\begin{bmatrix} -4 & 2 \\ 11 & -9 \end{bmatrix}$

E. $\begin{bmatrix} 2 & -15 \\ 12 & -7 \end{bmatrix}$

8. *[4 marks]*

If

$$\begin{aligned} 2x + 3y + 3z &= 1 \\ x + y + z &= 1 \\ 3x + 4y + z &= 2 \end{aligned}$$

Interchange first and second row

then $y =$

A. -3

B. 2

C. 3

D. 1

E. -1

$$\begin{array}{ccc} x & y & z \end{array}$$
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & 1 \\ 3 & 4 & 1 & 2 \end{array}\right)$$

$$\xrightarrow[R_3 \to -3R_1 + R_3]{R_2 \to -2R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & -2 & -1 \end{array}\right) \xrightarrow{R_3 \to -R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & 0 \end{array}\right)$$

So $-3z = 0$ and $z = 0$

$y + z = -1$ and $y = -1$ E

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9. *[4 marks]*

Evaluate: $\lim\limits_{x\to 1^+} \dfrac{x^2+x-2}{\sqrt{x-1}} - \dfrac{1}{x}$. $= \lim\limits_{x\to 1^+} \dfrac{(x-1)(x+2)}{\sqrt{x-1}} - \dfrac{1}{x}$

$= \lim\limits_{x\to 1^+} \sqrt{x-1}\,(x+2) - \dfrac{1}{x}$

$= 0 - 1$

$= -1$ B

A. 0

B. -1

C. 1

D. 2

E. does not exist

10. *[4 marks]*

$\lim\limits_{x\to -\infty} \dfrac{5}{e^x+1}\sqrt{\dfrac{4x^6+7x-1}{2x^6-x^3+8}}$ is

$e^x \to 0$ as $x \to -\infty$
and the fraction inside the square root behaves like
$\dfrac{4x^6}{2x^6} = 2$
So the limit is
$5\sqrt{2}$ A

A. $5\sqrt{2}$

B. $10\sqrt{2}$

C. $\frac{5}{\sqrt{2}}$

D. 0

E. not defined

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PART B. Written-Answer Questions

1. *[15 marks]*

A 15 year mortgage for \$600,000 has monthly payments with interest at 4% per year compounded semiannually.

[5] (a) Find the amount of each of the payments.

$$(1.02)^2 = (1+i)^{12}$$
$$(1.02)^{-30} = (1+i)^{-180}$$

$$600{,}000 = R\,a_{\overline{180}|i}$$

$$R = \frac{600{,}000}{a_{\overline{180}|i}} = \frac{600{,}000\,i}{1-(1+i)^{-180}} = 600{,}000\,\frac{\left[(1.02)^{\frac{1}{6}}-1\right]}{1-(1.02)^{-30}}$$

$$\boxed{R = \$\,4428.23}$$

[5] (b) What is the principal outstanding just after the 120th payment is made?

After 120 payments, 60 payments remain

$$P.O. = R\,a_{\overline{60}|i} = 4428.23\left(\frac{1-(1+i)^{-60}}{i}\right)$$

$$= 4428.23\left(\frac{1-(1.02)^{-10}}{(1.02)^{\frac{1}{6}}-1}\right) = \boxed{\$240{,}642.90}$$

Alternatively: $P.O. = R\,a_{\overline{60}|i} = \frac{600{,}000}{a_{\overline{180}|i}}\,a_{\overline{60}|i} = 600{,}000\,\frac{\left[1-(1.02)^{-10}\right]}{\left[1-(1.02)^{-30}\right]} = \boxed{\$240{,}643.03}$

[5] (c) Just after the 120th payment of the mortgage is made, the interest rate changes to 3.6% compounded semiannually. Find the new monthly payment necessary to repay the rest of the loan in the remaining 5 years.

If i' is the new interest rate and R' the new payment

$$240{,}642.90 = R'\,a_{\overline{60}|i'} \qquad (1.018)^2 = (1+i')^{12}$$

$$R' = \frac{240{,}642.90}{a_{\overline{60}|i'}} = 240{,}642.90\,\frac{\left[(1.018)^{\frac{1}{6}}-1\right]}{1-(1.018)^{-10}}$$

$$= \boxed{\$\,4385.62}$$

(The other P.O. gives the same new monthly payment ± one cent.)

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2. *[12 marks]*

[6] (a) A \$100 bond has semiannual coupons worth \$1.30 each (annual coupon rate of 2.6%), matures in 8 years, and yields 3% **annually**. To the nearest cent, what is its market price?

$$P = 100(1.015)^{-16} + 1.30\, a_{\overline{16}|.015}$$

$$= \boxed{\$97.17}$$

[6] (b) A similar \$100 bond also has semiannual coupons, matures in 8 years, and yields 3% annually but, unlike the bond in part (a), sells for \$102.12. To the nearest cent, how much is each coupon worth?

$$102.12 = 100(1.015)^{-16} + X\, a_{\overline{16}|.015}$$

$$X = \frac{102.12 - 100(1.015)^{-16}}{a_{\overline{16}|.015}}$$

$$\boxed{X = \$1.65}$$

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3. *[16 marks]*

[10] (a) Let $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$. Find M^{-1}.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to -R_1+R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to -R_2+R_3]{R_2 \to -R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \xrightarrow[\substack{R_2 \to R_3+R_2 \\ R_1 \to -R_2+R_1}]{R_3 \to \frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$M^{-1} = \boxed{\begin{pmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}} = \boxed{\frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}}$$

[6] (b) A company ships three types of boxes: type A, type B, and type C. In their first shipment, they ship only types A and B for a total of 70 boxes. In their second shipment, they ship type C and the same number of type A boxes as before, for a total of 80 boxes. In their third shipment, they ship the same number of type B boxes as before and the same number of type C boxes as before for a total of 50 boxes. Use the matrix M^{-1} from part (a) to find the number of type A boxes, type B boxes, and type C boxes in each shipment.

If A = # boxes type A
B = # " " B
C = # " " C

$$\begin{aligned} A + B &= 70 \\ A + C &= 80 \\ B + C &= 50 \end{aligned}$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 70 \\ 80 \\ 50 \end{pmatrix} \qquad M\begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 70 \\ 80 \\ 50 \end{pmatrix}$$

so $$\begin{pmatrix} A \\ B \\ C \end{pmatrix} = M^{-1}\begin{pmatrix} 70 \\ 80 \\ 50 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 70 \\ 80 \\ 50 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 100 \\ 40 \\ 60 \end{pmatrix} = \begin{pmatrix} 50 \\ 20 \\ 30 \end{pmatrix}$$

$$\boxed{\begin{aligned} A &= 50 \\ B &= 20 \\ C &= 30 \end{aligned}}$$

A

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4. *[17 marks]*

[2] (a) Write the augmented coefficient matrix for the following system of equations:

$$\begin{aligned} x + y - z &= 2 \\ 2z + 2w &= 3 \\ 2x + 2z - w &= 0 \end{aligned}$$

$$\begin{array}{cccc} x & y & z & w \end{array}$$
$$\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & 0 & 2 & 2 & 3 \\ 2 & 0 & 2 & -1 & 0 \end{array}\right)$$

[10] (b) Find all solutions to the system of equations in part (a) using row reduction (complete or not).

$$\xrightarrow[R_2 \leftrightarrow R_3]{R_3 \to -2R_1 + R_3} \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & -2 & 4 & -1 & -4 \\ 0 & 0 & 2 & 2 & 3 \end{array}\right) \xrightarrow[R_3 \to \frac{1}{2}R_3]{R_2 \to -\frac{1}{2}R_2} \begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & \frac{1}{2} & 2 \\ 0 & 0 & 1 & 1 & \frac{3}{2} \end{array}\right) \end{array}$$

Back subst:

$z = \frac{3}{2} - w$

$y = 2 - \frac{1}{2}w + 2z = 2 - \frac{1}{2}w + 2(\frac{3}{2} - w) = 5 - \frac{5}{2}w$

$x = 2 + z - y = 2 + (\frac{3}{2} - w) - (5 - \frac{5}{2}w)$

$= -\frac{3}{2} + \frac{3}{2}w$

$$\boxed{\begin{aligned} x &= -\tfrac{3}{2} + \tfrac{3}{2}w \\ y &= 5 - \tfrac{5}{2}w \\ z &= \tfrac{3}{2} - w \end{aligned}}$$

or: $\xrightarrow{R_2 \to R_2 + 2R_3}$ $R_1 \to R_1 - R_2 + R_3$ $\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{5}{2} & 5 \\ 0 & 0 & 1 & 1 & \frac{3}{2} \end{array}\right)$

which gives the same soln

[5] (c) Does the following system of equations have a unique solution? Explain why or why not; you may refer to part (b) if you wish.

$$\begin{aligned} x + y - z &= 5 \\ 2z + 2w &= 1 \\ 2x + 2z - w &= 3 \end{aligned}$$

There cannot be a unique solution.

Argument 1 (no reference to (b)): There might be no solution, but there are at most 3 non-zero rows, hence at least 1 parameter; i.e., either no solutions or ∞-many solutions

Argument 2 (reference to (b)): The row reduction is the same in both parts, except for the augmented column. Hence there will still be a 1-parameter solution, no matter what the R.H.S.

Argument 3: The student can redo part (b) and get a 1-parameter solution.

NO UNIQUE SOLUTION