Soln

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Department of Mathematics
University of Toronto

Tuesday, October 30, 2012, 6:10 - 8:00 PM
MAT 133Y TERM TEST #1
Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1084	T0501C	W3C	UC328
T0101B	M9B	SS1086	T0501D	W3D	BA1180
T0201A	M3A	LM 155	T0601A	R4A	MP137
T0201B	M3B	LM 123	T0601B	R4B	MS2173
T0201C	M3C	RW142	T0601C	R4C	BA1220
T0201D	M3D	BA1180	T0601D	R4D	BA B024
T0301A	T3A	RW143	T0701A	F2A	SS1073
T0301B	T3B	WI524	T0701B	F2B	RW229
T0301C	T3C	MP134	T0701C	F2C	RW142
T0401A	W9A	SS1070	T0701D	F2D	MP134
T0401B	W9B	SS1072	T0801A	F3A	SS1087
T0401C	W9C	SS1074	T0801B	F3B	LM 157
T0501A	W3A	SS1087	T5101A	M5A	SS1069
T0501B	W3B	AB114	T5201A	M6A	LM 158

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

If \$150 is deposited at the end of each month into a savings account earning 2.4% per year compounded monthly, then at the end of 8 years, the account will contain approximately:

$i = .024/12 = .002$

A. \$14,400.00

$S = 150 s_{\overline{96}|.002}$

B. \$15,857.87

$= 150 \left[\frac{(1.002)^{96} - 1}{.002}\right]$

C. \$13,090.16

D. \$181.72

$= 15,857.87$ (B)

E. \$15,669.44

2. *[4 marks]*

If \$450 grows to \$1800 in 10 years in a savings account that pays r% per year compounded **continuously**, then r% is closest to:

A. 1.487%

$1800 = 450 e^{10i}$

B. 1.386%

$10i = \ln(1800/450)$

C. 3%

$i = .1386$

D. 14.87%

$r = 13.86\%$ (E)

E. 13.86%

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3. *[4 marks]*

A person leaves $500,000 to a college in order to generate a yearly scholarship indefinitely. If a return of 2.5% per year can be expected on this money, then the size of the yearly scholarship will be:

["indefinitely" means "forever"]

A. $1,250

B. $2,000,000

C. $125,000

D. $12,500

E. $20,000,000

This is a perpetuity.

$R = rA$

$= .025 \times 500{,}000$

$= 12{,}500$ (D)

4. *[4 marks]*

A $10,000 bond has semiannual interest payments (coupons), an **annual** coupon rate of 4.6%, and an **annual** yield to maturity — in 4 years — of 5.0%. If a coupon has just been redeemed, the market price (to the nearest $) of the bond is:

A. $9,857

B. $9,698

C. $9,934

D. $9,715

E. $9,629

$P = V(1+i)^{-n} + rV\, a_{\overline{n}|i}$

$= 10{,}000(1.025)^{-8} + 230\, a_{\overline{8}|.025}$

$= 9856.597$

$\simeq 9857$ (A)

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5. *[4 marks]*

A \$100 bond has 8 years until maturity, with semi-anunual interest payments. If a coupon has just been redeemed and the market price is \$105 at an annual yield to maturity of 6%, then the annual coupon rate is closest to:

[Hint: trial and error (educated guessing) is not necessary]

A. 6.8%

B. 3%

C. 2.5%

D. 3.4%

E. 5.0%

$105 = 100(1.03)^{-16} + 100r\, a_{\overline{16}|.03}$

$100r = \dfrac{105 - 100(1.03)^{-16}}{a_{\overline{16}|.03}}$

$= 3.398$

This is semi-annual so the annual coupon rate is 6.8% (A)

6. *[4 marks]*

If A is a 1×2 matrix and B is a 2×2 matrix, which of the following is not defined?

A. AB^T OK result is 1×2

B. A^2 not OK (B)

C. AB OK 1×2

D. $A^T A$ OK 2×2

E. B^2 OK 2×2

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7. *[4 marks]*

If $A = \begin{bmatrix} 2 & -5 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 3 \\ -1 & 7 \end{bmatrix}$, then $A^2 - BA + 3A^T =$

A. $\begin{bmatrix} 10 & -41 \\ -13 & 9 \end{bmatrix}$

B. $\begin{bmatrix} 2 & -33 \\ 1 & -13 \end{bmatrix}$

C. $\begin{bmatrix} 12 & 15 \\ 13 & 7 \end{bmatrix}$

D. $\begin{bmatrix} 2 & -15 \\ -17 & -13 \end{bmatrix}$

E. $\begin{bmatrix} 8 & -15 \\ -17 & 11 \end{bmatrix}$

$A^2 = \begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -15 \\ 3 & -4 \end{pmatrix}$

$-BA = \begin{pmatrix} 0 & 3 \\ -1 & 7 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 5 & 12 \end{pmatrix}$

$3A^T = 3\begin{pmatrix} 2 & 1 \\ -5 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -15 & 3 \end{pmatrix}$

(D) $\begin{pmatrix} 2 & -15 \\ -17 & -13 \end{pmatrix}$

8. *[4 marks]*

Which of the following systems of equations has infinitely many solutions?

A. $\left[\begin{array}{cc|c} 2 & -10 & 1 \\ 0 & 1 & 0 \end{array}\right]$ unique soln

B. $\left[\begin{array}{ccc|c} 1 & -5 & 9 & 0 \\ 0 & 1 & 11 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right]$ unique soln.

C. $\left[\begin{array}{cc|c} 0 & 1 & 2 \\ -1 & 0 & 3 \\ 0 & 0 & 0 \end{array}\right]$ unique soln

D. $\left[\begin{array}{cccc|c} 2 & 3 & 0 & -7 & 8 \\ 0 & 0 & -1 & 4 & 10 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$ no soln

E. $\left[\begin{array}{ccc|c} 5 & 9 & -2 & 8 \\ 0 & 0 & -1 & 4 \end{array}\right]$ 1-parameter (E)

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9. *[4 marks]*

For what value of b does the following system have no solution:

$$\begin{array}{rcrcrcr} 4x & + & 7y & - & 3z & = & -1 \\ -x & - & 2y & + & z & = & 0 \\ 2x & + & 5y & + & bz & = & 4 \end{array}$$

A. 0

B. -3

C. 1

D. 3

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 4 & 7 & -3 & -1 \\ 2 & 5 & b & 4 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & b+2 & 4 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - R_2]{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & b+3 & 3 \end{array}\right)$$

No soln only when $b+3=0$

$b = -3$ (B)

E. The system always has a solution.

10. *[4 marks]*

Consider the linear system

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 8 \\ 3x & - & 4y & - & z & = & 4 \\ x & - & y & - & 2z & = & -2 \end{array}$$

All solutions of this system are given by:

A. $x = 8 - 3z, y = 0, z \in \mathbb{R}$

B. $x = 7y - 10, 5z = y + 10, y \in \mathbb{R}$

C. $x = 8 + 2y, y \in \mathbb{R}, z = 0$

D. $x = 7z - 12, y = 5z - 10, z \in \mathbb{R}$

E. There are no solutions to this system.

$$\begin{array}{c} x \quad y \quad z \\ \left(\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 3 & -4 & -1 & 4 \\ 1 & -1 & -2 & -2 \end{array}\right) \end{array}$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 0 & 2 & -10 & -20 \\ 0 & 1 & -5 & -10 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|c} 1 & -2 & 3 & 8 \\ 0 & 1 & -5 & -10 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \begin{array}{c} x \quad y \quad z \\ \left(\begin{array}{ccc|c} 1 & 0 & -7 & -12 \\ 0 & 1 & -5 & -10 \end{array}\right) \end{array}$$

$y = -10 + 5z$

$x = -12 + 7z$ (D)

$z \in \mathbb{R}$.

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PART B. Written-Answer Questions

1. *[15 marks]*

A $600,000, 15 year mortgage has monthly payments with interest at 4% per year compounded semiannually.

[7] (a) Find the amount of each payment to the nearest cent.

$(1.02)^2 = (1+i)^{12}$

$600{,}000 = R\, a_{\overline{180}|i}$

$R = \dfrac{600{,}000}{a_{\overline{180}|i}} = \dfrac{600{,}000\, i}{1-(1+i)^{-180}}$

$= 600{,}000\dfrac{[1.02^{\frac{1}{6}}-1]}{1-(1.02)^{-30}} = \boxed{\$4428.23}$

[8] (b) Find the amount of interest included in the 84th payment, to the nearest cent.

After the 83 payment, there are 97 payments remaining.

$P.O. = R\, a_{\overline{97}|i}$

Interest is $i R\, a_{\overline{97}|i}$

$= 4428.23[1-(1+i)^{-97}]$

$= 4428.23[1-(1.02)^{-97/6}]$

$= \boxed{\$1213.13}$

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2. *[15 marks]*

[7] (a) A $20,000 loan is to paid off by quarterly payments over a period of 5 years. The first payment is made 3 months after the loan is taken. If the interest charged on the loan is 3% compounded quarterly, find the payment.

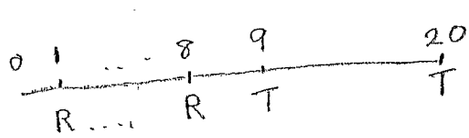
$$20{,}000 = R\,a_{\overline{20}|.0075}$$

$$R = \frac{20{,}000}{a_{\overline{20}|.0075}} = \frac{20000 \times .0075}{1-(1.0075)^{-20}}$$

$$\boxed{R = \$1080.61}$$

[8] (b) If after 2 years, the interest rate changes to 4% compounded quarterly, find the new quarterly payment in order to still pay off the loan by the end of 5 years.

$R = 1080.61$



There are many ways to set this up.

20,000

At 8: $20{,}000(1.0075)^8 - 1080.61\,s_{\overline{8}|.0075} = T\,a_{\overline{12}|}$ solve for T.

or $P.O. = R\,a_{\overline{12}|.0075} = T.a_{\overline{12}|.01}$ solve for T

$\boxed{T = \$1098}$ is close enough.

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3. *[14 marks]*

Two Porsches, two BMWs and three Mercedes can be rented for a total of \$100 per hour. At the same rates, one Porsche, four BMWs and two Mercedes can be rented for a total of \$90 per hour, whereas three Porsches, one BMW and three Mercedes cost a total of \$110 per hour. Find the hourly rental rate of each kind of car.

Let r_P be the hourly rate for Porsche
r_B " " " " " BMW
r_M " " " " " Mercedes

$$2r_P + 2r_B + 3r_M = 100$$
$$r_P + 4r_B + 2r_M = 90$$
$$3r_P + r_B + 3r_M = 110$$

Writing the middle eqn. first,

$$\begin{array}{ccc|c} r_P & r_B & r_M & \\ 1 & 4 & 2 & 90 \\ 2 & 2 & 3 & 100 \\ 3 & 1 & 3 & 110 \end{array} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{array}{ccc|c} & & & \\ 1 & 4 & 2 & 90 \\ 0 & -6 & -1 & -80 \\ 0 & -11 & -3 & -160 \end{array}$$

$$\xrightarrow[R_3 \to R_3 + 11R_2]{R_2 \to -\frac{1}{6}R_2} \begin{array}{ccc|c} & & & \\ 1 & 4 & 2 & 90 \\ 0 & 1 & \frac{1}{6} & \frac{40}{3} \\ 0 & 0 & -\frac{7}{6} & -\frac{40}{3} \end{array} \xrightarrow{R_3 \to -\frac{6}{7}R_3} \begin{array}{ccc|c} r_P & r_B & r_M & \\ 1 & 4 & 2 & 90 \\ 0 & 1 & \frac{1}{6} & \frac{40}{3} \\ 0 & 0 & 1 & \frac{80}{7} \end{array}$$

$$r_M = \frac{80}{7}$$

$$r_B = \frac{40}{3} - \frac{1}{6} r_M = \frac{40}{3} - \frac{1}{6} \cdot \frac{80}{7} = \frac{80}{7}$$

$$r_P = 90 - 2r_M - 4r_B$$
$$= 90 - \frac{160}{7} - \frac{320}{7} = \frac{150}{7}$$

$r_P = \frac{150}{7}$

$r_B = \frac{80}{7}$

$r_M = \frac{80}{7}$

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4. *[16 marks]*

[11] (a) Find the inverse of $A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 1 & -3 \\ 2 & 4 & -1 \end{bmatrix}$.

$$\left(\begin{array}{ccc|ccc} -1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 2 & 4 & -1 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 2R_1]{R_1 \to -R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 4 & 3 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 4R_2]{} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & -1 & 0 & 0 \\ 0 & 1 & -3 & 0 & 1 & 0 \\ 0 & 0 & 15 & 2 & -4 & 1 \end{array}\right) \xrightarrow[R_2 \to R_2 + 3R_3]{R_3 \to \frac{1}{15}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{11}{15} & -\frac{8}{15} & \frac{2}{15} \\ 0 & 1 & 0 & \frac{6}{15} & \frac{3}{15} & \frac{3}{15} \\ 0 & 0 & 1 & \frac{2}{15} & -\frac{4}{15} & \frac{1}{15} \end{array}\right)$$

$R_1 \to R_1 + 2R_3$

$$A^{-1} = \begin{pmatrix} -\frac{11}{15} & -\frac{8}{15} & \frac{2}{15} \\ \frac{6}{15} & \frac{3}{15} & \frac{3}{15} \\ \frac{2}{15} & -\frac{4}{15} & \frac{1}{15} \end{pmatrix} = \frac{1}{15}\begin{pmatrix} -11 & -8 & 2 \\ 6 & 3 & 3 \\ 2 & -4 & 1 \end{pmatrix}$$

[5] (b) Use A^{-1} to find X, if $AX = B$, where $B = \begin{bmatrix} 3 & 1 \\ -6 & 0 \\ -1 & -1 \end{bmatrix}$ (Note the shape of X).

If $AX = B$, $X = A^{-1}B$

$$X = \frac{1}{15}\begin{pmatrix} -11 & -8 & 2 \\ 6 & 3 & 3 \\ 2 & -4 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ -6 & 0 \\ -1 & -1 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 13 & -13 \\ -3 & 3 \\ 29 & 1 \end{pmatrix}$$

or $X = \begin{pmatrix} \frac{13}{15} & -\frac{13}{15} \\ -\frac{3}{15} & \frac{3}{15} \\ \frac{29}{15} & \frac{1}{15} \end{pmatrix}$