Solved

A

Department of Mathematics
University of Toronto

Tuesday, November 1, 2011, 7:40 - 9:30 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it.

ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	SS1072	T0601A	R4A	RW142
T0101B	M9B	SS1073	T0601B	R4B	GB248
T0101C	M9C	SS1083	T0601C	R4C	AB107
T0201A	M3A	SS2106	T0601D	R4D	GB221
T0201B	M3B	MP134	T0701A	F2A	LM155
T0201C	M3C	RW143	T0701B	F2B	RW229
T0201D	M3D	UC328	T0701C	F2C	BA1210
T0301A	T3A	RW229	T0701D	F2D	MS4279
T0301B	T3B	SS1083	T0801A	F3A	SS1085
T0301C	T3C	SS1088	T0801B	F3B	SS2110
T0401A	W9A	SS1072	T0801C	F3C	WI 523
T0401B	W9B	SS1073	T5101A	M5A	SS1087
T0501A	W3A	SS1070	T5101B	M5B	MP134
T0501B	W3B	SS1088	T5101C	M5C	MP137
			T5201A	M6A	LM162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*

An account with a 4% nominal interest rate compounded monthly earns the same effective interest as an account with a continuously compounded interest rate of:

$\left(1+\frac{.04}{12}\right)^{12} = e^r$

$r = 12\ln\left(1+\frac{.04}{12}\right) \approx .0399$

3.99%

A. 0.33%

B. 3.92%

C. 3.96%

D. 3.99%

E. 4.07%

2. *[4 marks]*

How much money would a husband have to leave his wife in his will in order for her to receive \$50,000/year indefinitely, if the interest rate remains at 7% annually?
(round answers to nearest dollar)

$50{,}000 = .07A$

$A = \frac{50{,}000}{.07}$

$\approx \$714{,}286$

A. \$350,000

B. \$714,286

C. \$3,500

D. \$46,729

E. \$35,000

NAME: _____ STUDENT NO: _____

3. *[4 marks]*

Parents opened a college trust fund for their son on Sept 1, 1999 when he started grade 1. Every year on this date, beginning on Sept 1, 1999, they deposited $2000 into the fund. The last payment was made when their son started grade 12 on Sept 1, 2010. If the account earns 6% annually, how much money is there in the fund on Sept 1, 2011?

A. $31,943.29

B. $33,739.88

C. $33,859.88

D. $35,764.28

E. $37,764.28

There are 12 payments.
On Sept 1, 2010, there will be

$2000\, s_{\overline{12}|.06}$ in the fund

so on Sept 1, 2011

$$2000\, s_{\overline{12}|.06} \times (1.06)$$

$$= (1.06) \times (2000) \frac{[(1.06)^{12} - 1]}{.06}$$

$$= \boxed{\$35,764.28}$$

4. *[4 marks]*

A $2500 loan is amortized over 5 years with monthly payments of $48.33 at an interest rate of 6% compounded monthly. The difference between the interest paid in the first and last payments is:

A. $12.50

B. $0

C. $13.33

D. $12.26

E. $5.84

Interest is .005 per month.
P.O. at beginning of 1st month is $2500.
Interest in 1st payment $= .005 \times 2500$
$= \$12.50$

P.O. at beginning of last month
= P.V. of remaining payments, namely, P.V. of last payment

$= (1.005)^{-1} \times 48.33$

Interest in last payment $= .005 \times (1.005)^{-1} \times 48.33$
$= \$0.24$

$12.50 - 0.24 = \boxed{\$12.26}$

NAME: _____ STUDENT NO: _____

5. *[4 marks]*

On the day after a coupon payment, the price of a \$100 bond with 8 semi-annual coupon payments remaining, an annual coupon rate of 4%, and an annual yield to maturity of 3% is closest to

$rV = \$2 \quad V = \$100 \quad i = .015$

$n = 8$

$P = 100(1.015)^{-8} + 2a_{\overline{8}|.015}$

$\approx \boxed{\$103.74}$

A. \$97.16

B. \$99.38

C. \$100.00

D. \$102.95

E. \$103.74

6. *[4 marks]*

$$\left(\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}^T \begin{bmatrix} 4 & 3 \\ 7 & -5 \end{bmatrix} - 3\begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 7 & -5 \end{bmatrix} - \begin{bmatrix} 3 & -6 \\ 12 & 3 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \left(\begin{bmatrix} 4 & 3 \\ 1 & 11 \end{bmatrix} - \begin{bmatrix} 3 & -6 \\ 12 & 3 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 9 \\ -11 & 8 \end{pmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 10 \\ -3 \end{bmatrix}}$$

A. $\begin{bmatrix} -10 \\ 17 \end{bmatrix}$

B. $\begin{bmatrix} 10 \\ -3 \end{bmatrix}$

C. $[\ 7\ ]$

D. $\begin{bmatrix} 32 \\ -17 \end{bmatrix}$

E. $[\ 15\ ]$

NAME: _____ STUDENT NO: _____

7. *[4 marks]*

Which of the following matrices is in row-echelon form?

A. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

D. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

E. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

8. *[4 marks]*

For which real number b does the system

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + 2z &= 9 \\ x + 3y + 3z &= b \end{aligned}$$

have infinitely many solutions?

A. 10

B. 11

C. 13

D. 12

E. 14

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 1 & 2 & 2 & 9 \\ 1 & 3 & 3 & b \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & -1 & -1 & -4 \\ 0 & 2 & 2 & b-5 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_2]{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 0 & b-13 \end{array}\right)$$

$b = 13$

NAME: _____ STUDENT NO: _____

9. *[4 marks]*

The system of linear equations

$$\begin{array}{rcrcrcr} 3x & + & 6y & - & 3z & + & 9w = -3 \\ 2x & + & 4y & - & 2z & + & 5w = -2 \\ -x & - & y & + & 4z & & = 3 \end{array}$$

has

A. no solutions

B. a unique solution

C. a one-parameter family of solutions

D. a two-parameter family of solutions

E. a three-parameter family of solutions

$$\left(\begin{array}{cccc|c} 3 & 6 & -3 & 9 & -3 \\ 2 & 4 & -2 & 5 & -2 \\ -1 & -1 & 4 & 0 & 3 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_1]{R_1 \to \frac{1}{3}R_1,\ R_2 \to R_2 - 2R_1} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 3 & 3 & 2 \end{array}\right)$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 3 & -1 \\ 0 & 1 & 3 & 3 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

parameters = # variables − # non-zero rows

$= 4 - 3 = \boxed{1}$

10. *[4 marks]*

Let $\mathbf{0}$ denote the $n \times n$ zero matrix and I the $n \times n$ identity matrix. If A is an $n \times n$ matrix such that

$$(A - I)(A^2 + A + I) = \mathbf{0},$$

then $A^{-1} =$

[Hint: Expand the left hand side of the equation.]

A. $A + I$

B. A

C. $A - I$

D. $A^2 + I$

E. A^2

$$(A - I)(A^2 + A + I) = A^3 + A^2 + A - A^2 - A - I^2$$

$$= A^3 - I$$

So $A^3 - I = 0$

$A^3 = I$

$A(A^2) = I$

$A^{-1} = \boxed{A^2}$

NAME: _____ STUDENT NO: _____

PART B. Written-Answer Questions

1. *[16 marks]*

Consider a 15 year mortgage for \$150,000 with interest at 5% per year, compounded semi-annually.

[6] (a) If payments are monthly, and the first payment is at the end of the first month, what is the principal outstanding (to the nearest dollar) at the end of 10 years?

$(1.025)^2 = (1+i)^{12}$

$(1.025)^{30} = (1+i)^{180}$

$(1.025)^{10} = (1+i)^{60}$

$150{,}000 = R\, a_{\overline{180}|i}$. After 10 yrs, 60 payments remain.

$P.O. = R\, a_{\overline{60}|i} = \dfrac{150{,}000}{a_{\overline{180}|i}}\, a_{\overline{60}|i}$

$= 150{,}000\, \dfrac{[1-(1.025)^{-10}]}{[1-(1.025)^{-30}]} \approx \boxed{\$62{,}723}$

[6] (b) If, instead, payments are weekly (assume 52 weeks in each year), and the first payment is at the end of the first week, what are the weekly payments (to the nearest dollar)?

$(1.025)^2 = (1+i')^{52}$

$(1.025)^{30} = (1+i')^{780}$

$150{,}000 = R'\, a_{\overline{780}|i'}$

$R' = 150{,}000\, \dfrac{[1.025^{\frac{1}{26}} - 1]}{[1-(1.025)^{-30}]}$

$R' \approx \boxed{\$272}$

[4] (c) In case b), what is the principal outstanding (to the nearest dollar) at the end of 10 years?

$(1.025)^{10} = (1+i')^{260}$

$150{,}000 = R'\, a_{\overline{780}|i'}$

$P.O. = R\, a_{\overline{260}|i'} = 150{,}000\, \dfrac{a_{\overline{780}|i'}}{a_{\overline{260}|i'}}$

$= 150{,}000\, \dfrac{[1-(1.025)^{-30}]}{[1-(1.025)^{-10}]}$. But this is the same expression as in part (a), so $\boxed{\$62{,}723}$.

Note: $\boxed{R \approx \$1182.19}$ in (a), but we don't need it.

NAME: _____ STUDENT NO: _____

2. *[16 marks]*

A loan of \$50,000 is amortized at 6% per year compounded monthly, with monthly payments of \$1000 and one smaller last payment. The first payment is due one month after the loan is made.

[8] (a) How many payments does the debtor need to make in total?

$$50{,}000 = 1000\, a_{\overline{n}|.005} = 1000\left[\frac{1-(1.005)^{-n}}{.005}\right]$$

$$50 = \frac{1-(1.005)^{-n}}{.005}$$

$$.25 = 1-(1.005)^{-n}$$

$$(1.005)^{-n} = .75$$

$$-n \ln 1.005 = \ln .75$$

$$n = -\frac{\ln .75}{\ln 1.005} \approx 57.68$$

So **58 payments** in all.

[8] (b) How much is the last payment (to the nearest cent)?

$$50{,}000 = 1000\, a_{\overline{57}|.005} + X(1.005)^{-58}$$

$$\boxed{X = \$680.68}$$

If you use the approximation $.68 \times 1000 = \$680$, it comes out close to the right answer.

NAME: _____ STUDENT NO: _____

3. *[13 marks]*

A 16 litre can is filled with a mixture of 3 fuels: heavy hydrocarbon (which costs \$1 per litre and weighs 0.5 kilograms per litre), light hydrocarbon (which costs \$2 per litre and weighs 0.4 kilograms per litre), and alcohol (which costs \$5 per litre and weighs 0.7 kilograms per litre). If the mixture in the can costs \$34 total and weighs 8 kilograms total, find the number of litres of each fuel used to make the mixture.

Let H = # litres heavy hydrocarbon
L = # " light "
A = # " alcohol

$$H + L + A = 16$$
$$H + 2L + 5A = 34$$
$$.5H + .4L + .7A = 8$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 1 & 2 & 5 & 34 \\ .5 & .4 & .7 & 8 \end{array}\right) \xrightarrow[R_3 \to R_3 - .5R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 4 & 18 \\ 0 & -.1 & .2 & 0 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + .1R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & .6 & 1.8 \end{array}\right) \xrightarrow[R_2 \to R_2 - 4R_3]{R_3 \to \frac{1}{.6}R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$R_1 \to R_1 - R_3 - R_2$

Hence $H = 7$, $L = 6$, $A = 3$

or

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 16 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

$A = 3$
$L = 18 - 4A = 18 - 4(3) = 6$
$H = 16 - L - A = 16 - 6 - 3 = 7$

$A = 3$, $L = 6$, $H = 7$ as before.

Of course, you could have multiplied through the third line by 10 at any point to get rid of the decimal. That wouldn't change the answer.

NAME: _____ STUDENT NO: _____

4. *[15 marks]*

Let A be the following matrix

$$A = \begin{pmatrix} 1 & a-1 & -2a-2 \\ -1 & 0 & 2a+2 \\ 1 & a-1 & -a-1 \end{pmatrix},$$

where a is a real number.

[5] (a) For what values of a is the matrix A **not** invertible?

$$\left(\begin{array}{ccc|ccc} 1 & a-1 & -2a-2 & 1 & 0 & 0 \\ -1 & 0 & 2a+2 & 0 & 1 & 0 \\ 1 & a-1 & -a-1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & a-1 & -2a-2 & 1 & 0 & 0 \\ 0 & a-1 & 0 & 1 & 1 & 0 \\ 0 & 0 & a+1 & -1 & 0 & 1 \end{array}\right)$$

A^{-1} does not exist if $\boxed{a=1 \text{ or } a=-1}$

and A^{-1} exists if $a-1 \neq 0$ and $a+1 \neq 0$.

[6] (b) What is A when $a = 2$? Find A^{-1} when $a = 2$. (Use part (a).) : The first step is already done for any a. If $a=2$, we are at

$$\left(\begin{array}{ccc|ccc} 1 & 1 & -6 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to \frac{1}{3}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -1 & 2 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{3} & 0 & \frac{1}{3} \end{array}\right)$$

$R_1 \to R_1 - R_2 + 6R_3$

$$\boxed{A^{-1} = \begin{pmatrix} -2 & -1 & 2 \\ 1 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}}$$

[4] (c) Solve the following system of linear equations. (Use part (b).)

$$\begin{array}{rcrcrcr} x & + & y & - & 6z & = & -3 \\ -x & & & + & 6z & = & 2 \\ x & + & y & - & 3z & = & 3 \end{array}$$

When $a=2$ $A = \begin{pmatrix} 1 & 1 & -6 \\ -1 & 0 & 6 \\ 1 & 1 & -3 \end{pmatrix}$ which is the matrix of coefficients.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A^{-1}\begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 & -1 & 2 \\ 1 & 1 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}\begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-2+6 \\ -3+2+0 \\ 1+0+1 \end{pmatrix} = \boxed{\begin{pmatrix} 10 \\ -1 \\ 2 \end{pmatrix}}$$

$\boxed{\begin{array}{l} x = 10 \\ y = -1 \\ z = 2 \end{array}}$