Solved

Department of Mathematics
University of Toronto

TUESDAY, October 27, 2009 6:10-8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1072	T0501D	W3D	BF 323
T0101B	M9B	SS1074	T0601A	R4A	SS2127
T0101C	M9C	SS2111	T0601B	R4B	LM 123
T0201A	M3A	SS1086	T0701A	F2A	MP 118
T0201B	M3B	RW 142	T0701B	F2B	SS2105
T0201C	M3C	LM 157	T0701C	F2C	LM 155
T0201D	M3D	SS2110	T0701D	F2D	RW 143
T0301A	T3A	SS2105	T0801A	F3A	SS2111
T0301B	T3B	SS1074	T0801B	F3B	SS1088
T0301C	T3C	UC 163	T0801C	F3C	RW 143
T0401A	W9A	SS1072	T5101A	M5A	SS2110
T0401B	W9B	SS1088	T5101B	M5B	SS2128
T0501A	W3A	BA B024	T5101C	M5C	RW 229
T0501B	W3B	BA3004	T5101D	M5D	LM 123
T0501C	W3C	GB 404	T5201A	M6A	LM 162

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*

If interest is 8% compounded daily, how many days are required for a given principal to earn 4% in interest?

$P\left(1+\frac{.08}{365}\right)^n = P(1.04)$

$n \ln\left(1+\frac{.08}{365}\right) = \ln(1.04)$

$n = \frac{\ln 1.04}{\ln\left(1+\frac{.08}{365}\right)} = 178.96\ldots$

$n \approx 179$ (B)

A. 175

B. 179

C. 183

D. 177

E. 181

2. *[4 marks]*

If interest is 10% compounded continuously, the effective annual rate is closest to

$1 + r_e = e^{.10}$

$r_e = e^{.10} - 1 = .10517\ldots$

$r_e \approx 10.5\%$ (E)

A. 10.3%

B. 9.8%

C. 10.0%

D. 10.8%

E. 10.5%

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3. *[4 marks]*

Monthly deposits are made into an account which earns 6% compounded monthly. The first 12 deposits are \$500 each, but beginning with the 13^{th}, each deposit is just \$300. Just after the 24^{th} deposit, the account (to the nearest dollar) has

A. \$10,372

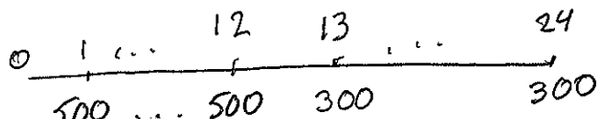
B. \$10,449

C. \$10,249

D. \$10,081

E. \$10,168

$i = \frac{.06}{12} = .005$



There are many ways to do this:

$500\, s_{\overline{24}|.005} - 200\, s_{\overline{12}|.005}$

or

$300\, s_{\overline{24}|.005} + 200\, s_{\overline{12}|.05}\,(1.005)^{12}$

or

$500\, s_{\overline{12}|.005}\,(1.005)^{12} + 300\, s_{\overline{12}|.005}$

All come to

\$10,248.8.....

so \$10,249 (C)

4. *[4 marks]*

If a \$400,000 mortgage has monthly payments for 15 years and interest at 4.5% compounded semi-annually, then each payment is closest to

A. \$3,030

B. \$3,050

C. \$3,070

D. \$3,060

E. \$3,040

Let i = monthly rate

$n = 12 \times 15 = 180$

$(1+i)^{12} = (1.0225)^2$

$400{,}000 = R\, a_{\overline{180}|i}$

$R = \frac{400{,}000\, i}{1-(1+i)^{-180}}$

$\frac{400{,}000\left[(1.0225)^{\frac{1}{6}}-1\right]}{1-(1.0225)^{-30}}$

$= \$3051.47$

closest to \$3050 (B)

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5. *[4 marks]*

A bond selling for \$187.91 has 16 semi-annual coupon payments remaining and an annual coupon rate of 6%. If the yield to maturity is 7%, then the face value of the bond is closest to:

A. \$200

B. \$195

C. \$205

D. \$175

E. \$194

$187.91 = V(1.035)^{-16} + .03V\,a_{\overline{16}|.035}$

$V = \dfrac{187.91}{(1.035)^{-16} + .03\,a_{\overline{16}|.035}}$

$V \approx 200.004$

$V = 200$ (A)

6. *[4 marks]*

A bond with a face value of \$100 matures in 5 years and has 10 semi-annual coupons of \$5 each remaining. If the bond is selling for \$145, then its annual yield to maturity is closest to

A. 1%

B. 2%

C. 3%

D. 4%

E. 6%

$P = 100(1+i)^{-10} + 5\,a_{\overline{10}|i}$ where i is semi-annual yield.

$P = 145$ is much larger than 100

so i is much lower than .05

Try $i = .01$ (which makes annual yield 2%)

$P = 137.88$

Price too low, so yield too high.

Try $i = .005$ (annual yield 1%)

then $P = 143.79$ this is the closest

so annual yield is closest to 1%

(A)

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7. *[4 marks]*

Let X, Y and Z be matrices of sizes 2×3, 4×3 and 1×4, respectively. Then the size of $XY^\top Z^\top$ (where $^\top$ denotes transpose of a matrix) is

A. the product is not defined

B. 2×1

C. 2×4

D. 4×3

E. 1×2

XY^TZ^T

$= 2(\ \ 3\ \ )\ 3(\ \ 4\ \ )\ 4(\ \ 1\ \ )$

so multiplication is possible

and the resulting size is 2×1

(B)

8. *[4 marks]*

Which of the matrices below is **not** in completely reduced form?

A. $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

B. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & -3 \end{pmatrix}$

C. $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

D. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

E. $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

(E)

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9. *[4 marks]*

When the matrix

$$\begin{pmatrix} 2 & 3 \\ 1 & 4 \\ 0 & 1 \\ 4 & 6 \\ 6 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_1} \begin{pmatrix} 1 & 4 \\ 2 & 3 \\ 0 & 1 \\ 4 & 6 \\ 6 & 2 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 4 \\ 0 & 1 \\ 2 & 3 \\ 4 & 6 \\ 6 & 2 \end{pmatrix}$$

is brought to reduced form, the number of zero rows is

$$\xrightarrow{R_1 \to R_1 - 4R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \\ 4 & 6 \\ 6 & 2 \end{pmatrix}$$

A. none

B. 1

C. 2

D. 3

E. 4

Can change last 3 rows to zero rows

by $R_3 \to R_3 - 3R_2 - 2R_1$

$R_4 \to R_4 - 6R_2 - 4R_1$

$R_5 \to R_5 - 2R_2 - 6R_1$

so 3 zero rows

(D)

10. *[4 marks]*

Consider the system of equations

$$\begin{array}{rcrcrcrcl} x & + & y & + & 2z & - & w & = & 0 \\ 3x & - & y & + & 4z & + & w & = & 0 \\ x & - & y & + & z & + & w & = & 0 \end{array}$$

The system has

A. only the trivial solution

B. a one-parameter family of solutions

C. a two-parameter family of solutions

D. a three-parameter family of solutions

E. no solutions

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 3 & -1 & 4 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix} \begin{matrix} R_2 \to R_2 - 3R_1 \\ \longrightarrow \\ R_3 \to R_3 - R_1 \end{matrix} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -4 & -2 & 4 \\ 0 & -2 & -1 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2 \to -\frac{1}{4}R_2} \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & \frac{1}{2} & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\xrightarrow{R_3 \to R_3 + 2R_2}$

Number of parameters = Number of variables − number of non-zero rows

$= 4 - 2 = 2$ (C)

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PART B. Written-Answer Questions

1. *[17 marks]*

A $15,000 loan at the interest rate of 9% per year compounded monthly is amortized over the next 10 years.

[5] (a) Find the monthly payment of the loan (to the nearest cent). (You may assume payments begin at the end of the first month.)

$i = \frac{.09}{12}$

$15{,}000 = R\,a_{\overline{120}|.0075}$

$R = \dfrac{15{,}000 \times .0075}{1-(1.0075)^{-120}} = \boxed{\$190.01}$

For items (b) and (c), assume that after 5 years, the interest rate changes to 12% compounded monthly.

[6] (b) If the entire loan must still be paid off in 10 years, find the new monthly payment for the last 5 years (to the nearest cent).

Principal outstanding $= R\,a_{\overline{60}|.0075}$ with R from (a)

$= \$9153.60 = T\,a_{\overline{60}|.01}$ (T is new payment)

$T = \dfrac{9153.60 \times .01}{1-(1.01)^{-60}}$

$\approx \boxed{\$203.62}$

[6] (c) If, instead of (b), the payment remains the same as in (a), how many **more** full payments would have to be made to repay the loan?

$9153.60 = 190.01\,a_{\overline{n}|.01}$

$\dfrac{9153.60}{190.01} = \dfrac{1-(1.01)^{-n}}{.01}$

$1 - \dfrac{.01 \times 9153.60}{190.01} = (1.01)^{-n}$

$-\dfrac{\ln\left(1 - \frac{.01 \times 9153.60}{190.01}\right) = n}{\ln(1.01)}$

$n \approx 66.06$ There would be 66 full payments

i.e. **6 more** than in the arrangement in (a)

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2. *[16 marks]*

A $500,000 25-year mortgage has **weekly** payments and an interest rate of 6% compounded semi-annually.

[Assume that every year has 52 weeks exactly.]

[9] (a) Find the weekly payment (to the nearest cent).

If i is weekly interest

$(1+i)^{52} = (1.03)^2$

$500{,}000 = R\, a_{\overline{25\times 52}|}\, i$

$R = \dfrac{500{,}000\, i}{1-(1+i)^{-1300}} = 500{,}000\, \dfrac{\left[(1.03)^{\frac{1}{26}}-1\right]}{1-(1.03)^{-50}}$

$\boxed{R = 736.84}$

[7] (b) How much *total interest* would be saved if the amortization period were reduced to 15 years?

If amortization were reduced to 15 yrs, payments T,

$500{,}000 = T\, a_{\overline{15\times 52}|}\, i$

$T = \dfrac{500{,}000\, i}{1-(1+i)^{-780}} = 500\, \dfrac{\left[(1.03)^{\frac{1}{26}}-1\right]}{1-(1.03)^{-30}}$

$T = 967.26$

In (a) Total Interest is $1300 \times 736.84 - 500{,}000$

In (b) " " " $780 \times 967.26 - 500{,}000$

Taking the difference (or noticing directly that since the original amount of the mortgage is the same, the difference in total payments must equal the difference in interest payments)

Total Interest saved $= 1300 \times 736.84 - 780 \times 967.26$

$= \boxed{\$\ 203{,}429.20}$

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3. *[13 marks]*

Use the method of row reduction to find all solutions of the system

$$\begin{array}{rcrcrcl} 2x & + & 3y & + & 12z & = & 4 \\ 3x & - & 2y & + & 5z & = & 5 \\ 4x & + & y & + & 14z & = & 6 \end{array}$$

[**No marks** will be given for any method other than row reduction.]

$$\left(\begin{array}{ccc|c} 2 & 3 & 12 & 4 \\ 3 & -2 & 5 & 5 \\ 4 & 1 & 14 & 6 \end{array}\right) \xrightarrow[R_3 \to R_3 - 4R_1]{R_1 \to \frac{1}{2}R_1,\ R_2 \to R_2 - 3R_1} \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & 6 & 2 \\ 0 & -\frac{13}{2} & -13 & -1 \\ 0 & -5 & -10 & -2 \end{array}\right)$$

$$\xrightarrow{R_2 \to -\frac{2}{13}R_2} \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & 6 & 2 \\ 0 & 1 & 2 & \frac{2}{13} \\ 0 & -5 & -10 & -2 \end{array}\right)$$

$$\xrightarrow{R_3 \to R_3 + 5R_2} \left(\begin{array}{ccc|c} 1 & \frac{3}{2} & 6 & 2 \\ 0 & 1 & 2 & \frac{2}{13} \\ 0 & 0 & 0 & \frac{10}{13} - 2 \end{array}\right)$$

Since $\frac{10}{13} - 2 \neq 0$, There are no solutions.

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4. *[14 marks]*

A furniture manufacturer produces two types of chairs, A and B. Chair A requires 7 nails and 4 brackets. Chair B requires 3 nails and 5 brackets. How many of each type of chair can be produced if 92 nails and 69 brackets are available and all **must** be used?

[Solve the linear equations using the method of row reduction. Many marks will be deducted for any other method of solution.]

Let A = # of chairs of type A
B = # " " " " B

$$\left.\begin{aligned} 7A + 3B &= 92 \\ 4A + 5B &= 69 \end{aligned}\right\}$$

$$\left(\begin{array}{cc|c} 7 & 3 & 92 \\ 4 & 5 & 69 \end{array}\right) \xrightarrow{R_1 \to \frac{1}{7}R_1} \left(\begin{array}{cc|c} 1 & \frac{3}{7} & \frac{92}{7} \\ 4 & 5 & 69 \end{array}\right)$$

$$\xrightarrow{R_2 \to R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & \frac{3}{7} & \frac{92}{7} \\ 0 & \frac{23}{7} & \frac{115}{7} \end{array}\right) \qquad -\frac{368}{7} + 69 = \frac{69\times 7 - 368}{7}$$

$$\xrightarrow{R_2 \to \frac{7}{23}R_2} \left(\begin{array}{cc|c} 1 & \frac{3}{7} & \frac{92}{7} \\ 0 & 1 & 5 \end{array}\right) \qquad \text{So } B = 5$$

$$\text{and } A = \frac{92}{7} - \frac{3}{7}B$$

$$= \frac{92-15}{7} = \frac{77}{7} = 11$$

OK

or completing the reduction

$$\xrightarrow{R_1 \to R_1 - \frac{3}{7}R_2} \left(\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & 5 \end{array}\right) \qquad \frac{92}{7} - \frac{15}{7} = \frac{77}{7} = 11$$

In either case and $\boxed{\begin{aligned} A &= 11 \\ B &= 5 \end{aligned}}$