Solved

Department of Mathematics
University of Toronto

WEDNESDAY, October 31, 2007 6:10-8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: _____

GIVEN NAME: _____

STUDENT NO: _____

SIGNATURE: _____

TUTORIAL TIME and ROOM: _____

REGCODE and TIMECODE: _____

T.A.'S NAME: _____

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS1084	T0501D	W3D	SS1088
T0101B	M9B	SS1086	T0601A	R4A	LM 157
T0101C	M9C	SS1087	T0601B	R4B	SS1083
T0201A	M3A	SS2108	T0701A	F2A	SS1086
T0201B	M3B	RW 143	T0701B	F2B	SS2106
T0201C	M3C	SS1083	T0701C	F2C	SS2108
T0201D	M3D	RW 142	T0801A	F3A	MP 134
T0301A	T3A	SS1084	T0801B	F3B	MP 118
T0301B	T3B	SS2108	T5101A	M5A	MP 118
T0401A	W9A	SS1084	T5101B	M5B	WI 523
T0401B	W9B	SS1073	T5201A	M6A	LM 162
T0501A	W3A	SS1086			
T0501B	W3B	SS1083			
T0501C	W3C	SS2106			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: _____ STUDENT NO: _____

PART A. Multiple Choice

1. *[4 marks]*
What nominal annual rate compounded weekly is most nearly equivalent to 5% per year compounded quarterly? [For this question, 1 year=52 weeks.]

A. 4.89%
(B.) 4.97%
C. 5.09%
D. 5.00%
E. 4.93%

Let r = nominal annual rate.

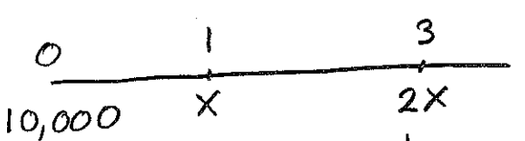
$$\left(1+\frac{r}{52}\right)^{52} = \left(1+\frac{.05}{4}\right)^{4}$$

$$r = 52\left[\left(1+\frac{.05}{4}\right)^{\frac{1}{13}} - 1\right]$$

$$r = \boxed{.04971}$$

2. *[4 marks]*
A loan of \$10,000 is to be repaid with payments of $\$X$ one year from now and $\$2X$ three years from now. If the effective annual rate of the loan is 10%, then $X =$

(A.) 4146.42
B. 3908.54
C. 3769.47
D. 4561.06
E. 5017.17



$$10{,}000 = X(1.10)^{-1} + 2X(1.10)^{-3}$$

$$X = \frac{10{,}000}{(1.10)^{-1} + 2(1.10)^{-3}}$$

$$= \boxed{4146.417}$$

NAME: _____ STUDENT NO: _____

3. *[4 marks]*
If interest is compounded continuously, what annual rate (to the nearest 0.01%) is required if the amount in an account is to double in 8 years?

A. 12.50%
B. 7.92%
C. 9.05%
D. 10.63%
(E.) 8.66%

$P = P_0 e^{rt}$

$2P_0 = P_0 e^{8r}$

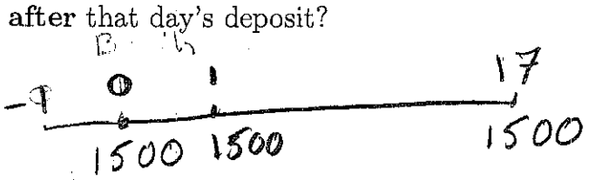
$\ln 2 = 8r$

$r = \frac{1}{8}\ln 2$

$= \boxed{.08664\ldots}$

4. *[4 marks]*
A father deposits $1,500 in an account on the day of his son's birth and continues to make similar deposits every year on his son's birthday up to and including his 17th birthday. If the account earns an effective annual rate of 5% then how much will there be in the account on his 17th birthday just **after** that day's deposit?

A. $40,698.58
B. $17,534.38
(C.) $42,198.58
D. $38,760.55
E. $37,260.55



Note: there are 18 payments.

$1500\, s_{\overline{18}|.05}$

$= 1500\dfrac{[(1.05)^{18}-1]}{.05}$

$= \boxed{42,198.577\ldots}$

Other ways, e.g.

$1500(1.05)^{17} + 1500\, s_{\overline{17}|.05}$

will get the same answer.

NAME: _____ STUDENT NO: _____

5. *[4 marks]*
How many semi-annual interest payments are remaining for a \$100 bond with annual coupon rate of 4% and annual yield rate very close to 4.5% if the bond is selling for \$96 per \$100 of face value and the next interest payment is in 6 months?

A. 18
(B.) 20
C. 10
D. 15
E. 24

$96 = 100(1.0225)^{-n} + 2a_{\overline{n}|.0225}$

$96 = 100(1.0225)^{-n} + 2\left[\frac{1-(1.0225)^{-n}}{.0225}\right]$

Solve for $(1.0225)^{-n}$:

$96 \times .0225 - 2 = (2.25-2)(1.0225)^{-n}$

$.16 = .25(1.0225)^{-n}$

$\ln\frac{16}{25} = -n \ln 1.0225$

$n = -\frac{\ln\frac{16}{25}}{\ln 1.0225} = \boxed{20.057}$

6. *[4 marks]*
To purchase a \$450,000 house a person pays \$50,000 down and takes on a 25 year mortgage with monthly payments and interest at 6% compounded semi-annually. The monthly mortgage payments will be closest to:

A. \$2,899
B. \$2,167
C. \$2,577
(D.) \$2,559
E. \$2,599

$450,000 - 50,000 = R\,a_{\overline{300}|i}$

$(1+i)^{12} = (1.03)^2$

$R = \frac{400,000\,i}{1-(1+i)^{-300}}$

$= \frac{400,000\left[(1.03)^{\frac{1}{6}}-1\right]}{1-(1.03)^{-50}}$

$= \boxed{2559.23}$

NAME: _____ STUDENT NO: _____

7. *[4 marks]*
A 20 year loan for \$100,000 is to be amortized by equal semi-annual payments. If interest is at the nominal rate of 10% per year compounded semi-annually, then the semi-annual payments are \$5,827.82 (You can check this if you have time to waste.) The interest in the 21st payment is closest to

A. \$2,914
B. \$3,631
C. \$3,847
D. \$3,914
E. \$4,047

P.O. after 20th payment is $5827.82\left[\frac{1-(1.05)^{-20}}{.05}\right]$

because there are 20 payments remaining:

Interest in the next payment is $.05 \times$ P.O.

$= 5827.82[1-(1.05)^{-20}]$

$= \boxed{3631.37\ldots}$

8. *[4 marks]*
Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$ $B = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$ $C = (1 \quad 3 \quad 4)$

Which (one) of the following products exists?

A. $C^{-1}B$
B. C^TAB
C. BB^TA
D. $A^{-1}B$
E. BCB^{-1}

A. C^{-1} cannot exist

B. $C^t = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$ cannot multiply A.

C. $_2B^2\ _2B^t\ _2A^3$ exists

D. A^{-1} cannot exist

E. $_2B^2\ _1C^3$ cannot be multiplied

NAME: _____ STUDENT NO: _____

9. *[4 marks]*

The system of equations

$$\begin{array}{rcrcrcl} x & + & y & +2z & = & 1 \\ x & & & -5z & = & -1 \\ 3x & & +2y & - \ z & = & 2 \end{array}$$

has

A. the complete solution $x = -1,\ y = 2,\ z = 0$

B. the complete solution $x = 5z - 1,\ y = 2 - 7z,\ z$ any real number

C. the complete solution $x = 4,\ y = -5,\ z = 1$

D. the complete solution $x = (3 - 5y)/7,\ y$ any real number, $z = (2 - y)/7$

E. no solution

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 0 & -5 & -1 \\ 3 & 2 & -1 & 2 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & -1 & -7 & -2 \\ 0 & -1 & -7 & -1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + R_2 \text{ (new)}]{R_2 \to -R_2} \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & 7 & 2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

no solution

10. *[4 marks]*

The a_{32} entry in the inverse of the matrix $\begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$ is

A. 1

B. −2

C. 0

D. −1

E. 2

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 - 2R_2 \text{ (new)}]{R_2 \to -R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & -1 & -2 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 \to -R^3} \left(\begin{array}{ccc|ccc} \text{not needed} & & & \text{not needed} & & \\ 0 & 0 & 1 & 2 & -2 & 1 \end{array}\right)$$

$a_{32} = -2$

NAME: _____ STUDENT NO: _____

PART B. Written-Answer Questions

1. *[15 marks]*

A pile of coins consists of nickels, dimes, and quarters. There are 18 coins in the pile. The total value is \$2.55. The number of quarters is one more than the number of nickels.
How many nickels, dimes, and quarters are in the pile?
[For full marks, make sure you use row-reduction to solve the system of equations.]

Let N = # nickels
D = # dimes
Q = # quarters

$$\left.\begin{array}{l} N+D+Q=18 \\ 5N+10D+25Q=255 \\ Q=N+1 \end{array}\right\} \quad \left.\begin{array}{l} N+D+Q=18 \\ N+2D+5Q=51 \\ -N\quad\quad +Q=1 \end{array}\right\}$$

$$\begin{array}{ccc|c} N & D & Q & \\ \hline 1 & 1 & 1 & 18 \\ 1 & 2 & 5 & 51 \\ -1 & 0 & 1 & 1 \end{array} \longrightarrow \begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ \hline 0 & 1 & 4 & 33 \\ 0 & 1 & 2 & 19 \end{array}$$

$$\longrightarrow \begin{array}{ccc|c} 1 & 1 & 1 & 18 \\ \hline 0 & 1 & 4 & 33 \\ 0 & 0 & -2 & -14 \end{array} \longrightarrow \begin{array}{ccc|c} N & D & Q & \\ \hline 1 & 1 & 1 & 18 \\ 0 & 1 & 4 & 33 \\ 0 & 0 & 1 & 7 \end{array}$$

$Q=7 \quad D=33-4Q=5 \quad N=18-Q-D=6$

6 nickels, 5 dimes, 7 quarters

Alternately, substitute $Q=N+1$ into 1st two equations:

$$N+D+N+1=18 \qquad 2N+D=17$$
$$5N+10D+25(N+1)=255 \qquad 30N+10D=230$$

$$2N+D=17$$
$$3N+D=23$$

$$\begin{array}{cc|c} 2 & 1 & 17 \\ \hline 3 & 1 & 23 \end{array}$$

Better still $D+2N=17$
$D+3N=23$

$$\begin{array}{cc|c} D & N & \\ \hline 1 & 2 & 17 \\ 1 & 3 & 23 \end{array} \longrightarrow \begin{array}{cc|c} 1 & 2 & 17 \\ \hline 0 & -1 & -6 \end{array}$$

$$\longrightarrow \begin{array}{cc|c} D & N & \\ \hline 1 & 2 & 17 \\ 0 & 1 & 6 \end{array}$$

so $N=6 \quad D=17-2N=5$
and $Q=N+1=7$ as before

or $\longrightarrow \begin{array}{cc|c} 1 & 0 & 5 \\ \hline 0 & 1 & 6 \end{array}$ same answer too.

NAME: _____ STUDENT NO: _____

2. *[16 marks]*

The input/output matrix for the industries A and B is given by:

	Industry A	Industry B		Final Demand
Industry A	100	300	⋮	100
Industry B	300	225	⋮	225
Other Production Factors	100	225	⋮	–

If the final demand changes to 160 for Industry A and to 200 for Industry B then find:

[12] (a) The new total outputs for the industries A and B.

Total output A = 500 Total output B = 750

Tech matrix = $\begin{pmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{3}{10} \\ \cdots & \cdots \\ \text{Other} \ \frac{1}{5} & \frac{3}{10} \end{pmatrix}$ Leontief = $\begin{pmatrix} \frac{4}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{7}{10} \end{pmatrix}$

$\frac{4}{5}A - \frac{2}{5}B = 160$
$-\frac{3}{5}A + \frac{7}{10}B = 200$

$\left(\begin{array}{cc|c} \frac{4}{5} & -\frac{2}{5} & 160 \\ -\frac{3}{5} & \frac{7}{10} & 200 \end{array}\right) \xrightarrow[R_2 \to \frac{5}{3}R_2]{R_1 \to \frac{5}{4}R_1} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 200 \\ -1 & \frac{7}{6} & \frac{1000}{3} \end{array}\right)$

$\xrightarrow[R_2 \to R_2 + R_1]{} \left(\begin{array}{cc|c} 1 & -\frac{1}{2} & 200 \\ 0 & \frac{2}{3} & \frac{1600}{3} \end{array}\right) \xrightarrow[R_1 \to R_1 + \frac{1}{2}R_2 \text{ new}]{R_2 \to \frac{2}{3}R_2} \left(\begin{array}{cc|c} 1 & 0 & 600 \\ 0 & 1 & 800 \end{array}\right)$

A = 600 and B = 800

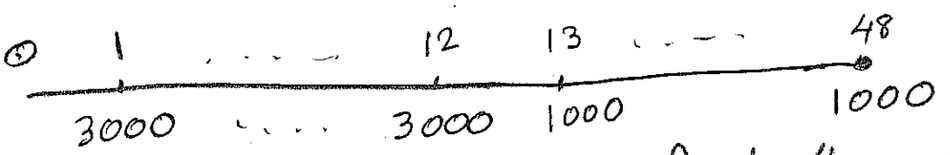
[4] (b) The new other production factors for industries A and B.

For industry A $\frac{1}{5} \times 600 = 120$
For industry B $\frac{3}{10} \times 800 = 240$

NAME: _____ STUDENT NO: _____

3. *[14 marks]*

Veronica has won a lottery which will pay her \$3,000 each month for 1 year, and then \$1,000 each month for the next 3 years. She will receive her first \$1,000 payment 1 month after her last \$3,000 payment. If she invests each lottery cheque just after receiving it, at 4.5% compounded monthly, what will be the amount of her investment immediately after she deposits her last lottery payment?



We need the value at 48 of all these payments

using $i = .045/12 = .00375$

Could do: ① $3000\, s_{\overline{48}|i} - 2000\, s_{\overline{36}|i}$

or ② $(3000\, s_{\overline{12}|i})(1+i)^{36} + 1000\, s_{\overline{36}|i}$

or ③ $(3000\, s_{\overline{12}|i} + 1000\, a_{\overline{36}|i})(1+i)^{36}$

All of these come to

\$80,519.32

Explanation:

① An ordinary annuity of \$3000 for 48 periods take away (to make \$1000) an ordinary annuity of \$2000 for 36 periods.

② Accumulate the 12 \$3000 payments and send them as a lump to the end, then add the annuity of 36 \$1000 payments

③ Accumulate all payments to the moment marked "12" on the time line, then send as a lump to the end.

NAME: _____ STUDENT NO: _____

4. *[15 marks]*

A bond with an 8% annual coupon rate and semi-annual coupons has 25 coupons remaining with the first one cashable 6 months from now.

[10] (a) If the bond is currently selling at \$120 per \$100 of face value, find the annual yield to maturity. [The price should be correct to within \$1 of the actual price per \$100 of face value.]

$120 = 100(1+i)^{-25} + 4\,a_{\overline{25}|i}$ $P>100$ $i<.04$

$i = .03 \Rightarrow P = 117.41$ too low; yield too high

$i = .025 \Rightarrow P = 127.64$ too high; yield too low

$i = .029 \Rightarrow P =$ 119.37 this is within #1 of 120

Yield = 5.8%

5.6% or 5.9% are not acceptable since they give 121.37 and 118.39.

Any yield to maturity greater than 5.6% and less than 5.9% will do. 5.7%, for example, gives #120.36 which is better than the given answer.

[5] (b) On the same day as in part (a), the issuers of the bond announce that those who so choose will be able, in 5 years time right after the coupon payment, to exchange each \$100 of face value with its remaining coupons for \$105 in cash. If the annual yield to maturity remains the same as in (a), what happens to the price of the bond? [Numerical calculation, with explanation please]

The issuers are offering #105 in cash 5 yrs. from now plus 10 coupon payments of #4 each.
The present value of this offer is

Using 5.8%, $105(1.029)^{-10} + 4\,a_{\overline{10}|.029} = 113.19$

Using 5.7%, $105(1.0285)^{-10} + 4\,a_{\overline{10}|.0285} = 113.66$

In either case, **nothing happens to the price of the bond**, because no one will plan to exchange a bond worth 120.36 for one worth 113 + pennies; and since they don't have to, they won't.