

Soln

Department of Mathematics  
University of Toronto

WEDNESDAY, November 1, 2006 6:10-8:00 PM  
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce  
Duration: 1 hour 50 minutes

**Aids Allowed:** A non-graphing calculator, with empty memory, to be supplied by student.

**Instructions:** Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions. For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. **ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.**

**TOTAL MARKS: 100**

FAMILY NAME: \_\_\_\_\_

GIVEN NAME: \_\_\_\_\_

STUDENT NO: \_\_\_\_\_

SIGNATURE: \_\_\_\_\_

TUTORIAL TIME and ROOM: \_\_\_\_\_

REGCODE and TIMECODE: \_\_\_\_\_

T.A.'S NAME: \_\_\_\_\_

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	LM 155	T0501D	W3D	BF 323
T0101B	M9B	SS1074	T0601A	R4A	MP 137
T0101C	M9C	SS1084	T0601B	R4B	RW 143
T0201A	M3A	MS3163	T0701A	F2A	MP 137
T0201B	M3B	SS1087	T0701B	F2B	SS1087
T0201C	M3C	MP 134	T0701C	F2C	LM 162
T0201D	M3D	MP 137	T0801A	F3A	SS1083
T0301A	T3A	SS2110	T0801B	F3B	SS1073
T0301B	T3B	SS1083	T5101A	M5A	SS1069
T0401A	W9A	SS1084	T5101B	M5B	SS2108
T0401B	W9B	SS2127	T5201A	M6A	SS2110
T0501A	W3A	SS2102			
T0501B	W3B	RW 110			
T0501C	W3C	MP 203			

FOR MARKER ONLY	
Multiple Choice	
<b>B1</b>	
<b>B2</b>	
<b>B3</b>	
<b>B4</b>	
<b>TOTAL</b>	

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**PART A. Multiple Choice**

1. [4 marks]

An interest rate of 8% compounded semi-annually corresponds most nearly to an effective rate of

- A. 8%
- B. 8.16%
- C. 8.203%
- D. 8.310%
- E. 12%

$$1 + r_e = \left(1 + \frac{.08}{2}\right)^2 = 1.0816$$

$$r_e = .0816 \quad \textcircled{B}$$

2. [4 marks]

If 24 quarterly payments of \$200 are deposited into a savings account earning 4% interest per year compounded quarterly starting **immediately**, then at the end of 6 years, the amount in the account will be closest to:

- A. \$5,648.64
- B. \$6,094.73
- C. \$5,448.64
- D. \$4,848.00
- E. \$5,394.69

$$i = .01$$



$$F.V. = 200 S_{\overline{24}|.01} (1.01)$$

$$\approx 5448.64 \quad \textcircled{C}$$



$$F.V. = 200 S_{\overline{24}|.01} - 200$$

$$\approx 5448.64 \quad \textcircled{C}$$

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3. [4 marks]

A \$10,000 loan is amortized over 5 years with equal semi-annual payments. If the interest rate is 8% per year compounded semi-annually, then the principal repaid in the first payment is

- A. \$762.47
- B. \$795.38
- C. \$806.21
- D. \$832.91
- E. \$853.64

$$10,000 = R a_{\overline{10}|.04}$$

$$R = \frac{10,000}{a_{\overline{10}|.04}} \approx \frac{10,000 \times .04}{1 - (1.04)^{-10}}$$

$$R \approx 1232.91$$

Interest in the first period is

$$.04 \times 10,000 = 400$$

$$\text{Principal repaid} = 1232.91 - 400 = 832.91 \quad \textcircled{D}$$

4. [4 marks]

A \$100,000 loan is amortized over 10 years at 5% per year compounded quarterly with quarterly payments. The interest in the last payment is

- A. \$0
- B. \$39.41
- C. \$107.63
- D. \$235.81
- E. \$1,250.00

If  $R$  is the size of the quarterly payments, then at the beginning of the last period the principal outstanding is

$$R(1.0125)^{-1}$$

and the interest in the last payment is

$$.0125 R(1.0125)^{-1}$$

$$\text{Since } R = \frac{100,000}{a_{\overline{40}|.0125}} = 3192.14$$

the interest in the last payment is \$39.41  $\textcircled{B}$

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5. [4 marks]

What is the market price of a \$1,000 bond having 8 years until maturity, semiannual interest payments of \$36 each, and an annual yield rate of 8%?

- A. \$942.60
- B. \$953.39
- C. \$935.21
- D. \$967.85
- E. \$929.46

$$P = 1000(1.04)^{-16} + 36a_{\overline{16}|.04}$$

$$\approx 953.39 \quad \textcircled{B}$$

6. [4 marks]

Let  $A = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ ,  $B = \begin{pmatrix} 7 & 17 \\ 18 & -3 \\ 12 & 2 \end{pmatrix}$ ,  $C = (5 \ 5 \ 5)$ ,  $D = (1 \ 2)$ .

Which of the following products of matrices doesn't exist?

- A.  $A^T C B$   $A^t$  is  $1 \times 3$ ;  $C B$  is  $1 \times 2$   $(\begin{matrix} 3 \\ 2 \end{matrix})$  Impossible  $\textcircled{A}$
- B.  $A C B$   $A C$  is  $3 \times 3$ ;  $B$  is  $3 \times 2$ ; so  $A C B$  is OK
- C.  $B^T A C$   $B^t$  is  $2 \times 3$ ;  $A C$  is  $3 \times 3$ ; so  $B^t A C$  is OK
- D.  $(A D)^T B$   $A D$  is  $3 \times 2$ ;  $A D^t$  is  $2 \times 3$ ; so  $(A D)^t B$  OK
- E.  $C A D$   $C A$  is  $1 \times 1$ ;  $D$  is  $1 \times 2$ ; so  $C A D$  is OK

7. [4 marks]

Let  $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ -1 & 3 & 5 \end{pmatrix}$ ,  $B = \begin{pmatrix} 2 & -1 & 3 \\ 1 & -1 & 0 \\ 1 & -2 & -3 \end{pmatrix}$ ,  $C = \begin{pmatrix} 0 & -1 & 3 \\ 1 & 3 & -1 \\ -1 & 2 & 0 \end{pmatrix}$ .

Which two matrices have the same number of non-zero rows when they are put in row-echelon form (otherwise known as reduced form)?

- A. A and B
- B. A and C
- C. B and C
- D. They all have the same number of non-zero rows in row-echelon form.
- E. No two of them have the same number of non-zero rows in row-echelon form.

$A \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 2 & 1 & 0 \\ -1 & 3 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 3 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 4 \\ 0 & 0 & -9 \end{pmatrix}$  3 non-zero rows  
 $B \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 3 \\ 1 & -2 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & -1 & -3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$  2 non-zero rows  
 $C \rightarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ -1 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 5 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & -1 \\ 0 & -1 & 3 \\ 0 & 0 & 14 \end{pmatrix}$  3 non-zero rows

(B)

8. [4 marks]

Consider the system

$$\begin{cases} 2x - 2y + 3z + w = 0 & \text{Row 2} \\ x - 9y + 7z + w = 0 & \text{Row 1} \\ 3x + 5y - z + w = 0 & \text{Row 3} \\ x + 7y - 4z = 0 & \text{Row 4} \end{cases}$$

Which of the following statements is true?

- A. The system has a unique solution
- B. The system has no solutions
- C. The system has a 1-parameter family of solutions
- D. The system has a 2-parameter family of solutions
- E. The system has a 3-parameter family of solutions

R.H.S all zero will remain all zero no matter what.

$\begin{pmatrix} 1 & -9 & 7 & 1 \\ 2 & -2 & 3 & 1 \\ 3 & 5 & -1 & 1 \\ 1 & 7 & -4 & 0 \end{pmatrix} \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1 \\ R_4 \rightarrow R_4 - R_1}} \begin{pmatrix} 1 & -9 & 7 & 1 \\ 0 & 16 & -11 & -1 \\ 0 & 32 & -22 & -2 \\ 0 & 16 & -11 & -1 \end{pmatrix} \xrightarrow{\substack{R_3 \rightarrow R_3 - 2R_2 \\ R_4 \rightarrow R_4 - R_2}}$

$\begin{pmatrix} 1 & -9 & 7 & 1 \\ 0 & 16 & -11 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

no. of variables = 4  
 no. of non-zero rows in row-echelon form = 2  
 no. of parameters = 4 - 2 = 2 (D)

9. [4 marks]

For which value of the parameter  $a$  does the system  $\begin{cases} ax + y = 0 \\ x + 2y = a - \frac{1}{2} \end{cases}$  have **no** solution?

- A.  $a = 2$   
 B.  $a = \frac{1}{2}$   
 C.  $a = 0$   
 D.  $a = 1$   
 E. There is no such  $a$ .

$$\begin{pmatrix} a & 1 & | & 0 \\ 1 & 2 & | & a - \frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & | & a - \frac{1}{2} \\ a & 1 & | & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & | & a - \frac{1}{2} \\ 0 & 1 - 2a & | & \frac{1}{2}a - a^2 \end{pmatrix}$$

The only difficulty arises if  $a = \frac{1}{2}$   
 since  $1 - 2a = 0$ .

But if  $a = \frac{1}{2}$ ,  $\frac{1}{2}a - a^2 = 0$ , so there are  $\infty$ -many solns.

(E)

10. [4 marks]

What is the value of  $x$  in a solution of the following system?

$$7x - 3y = 27$$

$$11x + 5y = 23$$

Hint:  $\begin{bmatrix} 7 & -3 \\ 11 & 5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ -11 & 7 \end{bmatrix} = \begin{bmatrix} 68 & 0 \\ 0 & 68 \end{bmatrix}$

- A. 6  
 B. 4  
 C. 3  
 D. 5  
 E. 2

$$\begin{pmatrix} 7 & -3 \\ 11 & 5 \end{pmatrix}^{-1} = \frac{1}{68} \begin{pmatrix} 5 & 3 \\ -11 & 7 \end{pmatrix}$$

$$\text{so } \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{68} \begin{pmatrix} 5 & 3 \\ -11 & 7 \end{pmatrix} \begin{pmatrix} 27 \\ 23 \end{pmatrix}$$

$$\text{We only need } x = \frac{1}{68} (5 \cdot 27 + 3 \cdot 23)$$

$$= 3 \quad \text{(C)}$$

## PART B. Written-Answer Questions

1. [17 marks]

A 10 year, \$60,000 mortgage has monthly payments.

[5] (a) Find the amount, \$R, of each payment if interest is 8% compounded semiannually.

$$60,000 = R a_{\overline{120}|i} \quad \text{where} \quad (1.04)^2 = (1+i)^{12}$$

$$R = \frac{60,000 [(1.04)^{\frac{1}{2}} - 1]}{1 - (1.04)^{-20}} \approx \boxed{\$723.85}$$

[6] (b) Immediately after the 5th year of the mortgage, the interest

rate changes to 6% compounded semiannually. The debtor may change his monthly payment from \$R (see part (a)) to a different amount in order to repay the loan in a total of 10 years, as originally agreed. Find the new monthly payment required if the debtor chooses this plan.

$$\text{Principal outstanding} = R a_{\overline{60}|i} \quad i \text{ as before.}$$

The new interest rate is  $r$  where  $(1.03)^2 = (1+r)^{12}$

$$\text{New payments are } T, \text{ then } R a_{\overline{60}|i} = T a_{\overline{60}|r}$$

$$T = \frac{R a_{\overline{60}|i}}{a_{\overline{60}|r}} = 723.85 \frac{[1 - (1.04)^{-10}]}{[(1.04)^{\frac{1}{2}} - 1]} \cdot \frac{[(1.03)^{\frac{1}{2}} - 1]}{[1 - (1.03)^{-10}]} \approx \boxed{\$691.06}$$

[6] (c) Alternatively, when the rate changes as in part (b), the debtor may continue monthly payments of \$R each (see part (a)) until outstanding principal is less than \$R. One month after the last full payment of \$R, he would make a smaller final payment. Find the total number of payments required if the debtor chooses this plan. Remember to include the 60 payments made in the first 5 years of the mortgage.

After the change

$$\cancel{R} a_{\overline{60}|i} = \cancel{R} a_{\overline{n}|r}, \quad n \text{ unknown}$$

$$R \text{ s cancel} \quad \frac{1 - (1.04)^{-10}}{(1.04)^{\frac{1}{2}} - 1} = \frac{1 - (1.03)^{-\frac{n}{6}}}{(1.03)^{\frac{1}{2}} - 1}$$

$$1 - (1.03)^{-\frac{n}{6}} \approx .244315009$$

$$-\frac{n}{6} = \frac{\ln[1 - .244315009]}{\ln(1.03)}$$

$$n \approx 56.86$$

57 payments remain

50  $\boxed{117 \text{ in all}}$

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2. [16 marks]

XYZ Income Fund issues a bond with \$50,000,000 in face value, maturing in 20 years, with semi-annual coupons and an annual coupon rate of 4.8%. Just 5 years after the bond is issued it is trading in the market at \$104 per \$100 of face value.

[11] (a) To within an accuracy of \$0.10 in the price for \$100 of face value, what is the current annual yield to maturity of the bond?

\$50,000,000 is a red herring.

$$104 = 100(1+i)^{-30} + 2.4 a_{\overline{30}|i} \quad P > 100 \quad i < .024$$

Try  $i = .02$   $P = 108.96$  way too high

Half-way between is  $i = .022$ ; then  $P = 104.36$  ... almost,

Price a little bit too high; yield a little bit too low.

If try  $i = .0221$ , get  $P = 104.13$ ; not good enough

But  $i = .0222$ ,  $P = 103.91$ . Good enough!

Annual yield to maturity 4.44%

[5] (b) On the same day XYZ announces that at maturity it will pay \$110 per \$100 of face value (a premium of \$10 per \$100). Assuming that this announcement has no effect on the yield to maturity, exactly what happens to the price of the bond?

The extra \$10 is discounted using the yield rate.

The price will increase by

$$10(1.0222)^{-30}$$

$$= \text{\$ } 5.175 \text{ per } \$100.$$



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3. [14 marks]

[10] (a) Let

$$\left( \begin{array}{ccc|ccc} 3 & -1 & 0 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_1 \leftrightarrow R_3}$$

Find  $A^{-1}$  or show that  $A^{-1}$  does not exist.

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 2 & -5 & 1 & 0 & 1 & 0 \\ 3 & -1 & 0 & 1 & 0 & 0 \end{array} \right) \xrightarrow{\substack{R_2 \rightarrow R_2 - 2R_1 \\ R_3 \rightarrow R_3 - 3R_1}} \left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & -5 & 3 & 0 & 1 & -2 \\ 0 & -1 & 3 & 1 & 0 & -3 \end{array} \right) \xrightarrow{\substack{R_2 \leftrightarrow R_3 \\ \text{row } R_2 \rightarrow -\text{row } R_2}}$$

$$\left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -1 & 0 & 3 \\ 0 & -5 & 3 & 0 & 1 & -2 \end{array} \right) \xrightarrow{R_3 \rightarrow R_3 + 5R_2} \left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -1 & 0 & 3 \\ 0 & 0 & -12 & -5 & 1 & 13 \end{array} \right) \xrightarrow{R_3 \rightarrow -\frac{1}{12}R_3} \left( \begin{array}{ccc|ccc} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & -3 & -1 & 0 & 3 \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{12} & -\frac{13}{12} \end{array} \right)$$

$$\begin{array}{l} R_1 \rightarrow R_1 + R_3 \\ R_2 \rightarrow R_2 + 3R_3 \end{array} \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{12} & -\frac{1}{12} & -\frac{1}{12} \\ 0 & 1 & 0 & \frac{3}{12} & -\frac{3}{12} & -\frac{3}{12} \\ 0 & 0 & 1 & \frac{5}{12} & -\frac{1}{12} & -\frac{13}{12} \end{array} \right)$$

$$A^{-1} = \frac{1}{12} \begin{pmatrix} 5 & -1 & -1 \\ 3 & -3 & -3 \\ 5 & -1 & -13 \end{pmatrix}$$

[4] (b) Find all solutions (if any) to the system of equations

$$\begin{array}{rcl} 3x & - & y & = & 0 \\ 2x & - & 5y & + & z & = & 1 \\ x & & & - & z & = & -1 \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 5 & -1 & -1 \\ 3 & -3 & -3 \\ 5 & -1 & -13 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{12} \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\begin{array}{l} x = 0 \\ y = 0 \\ z = 1 \end{array}$$

Check of  $A^{-1}$   $\begin{pmatrix} 3 & -1 & 0 \\ 2 & -5 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 5 & -1 & -1 \\ 3 & -3 & -3 \\ 5 & -1 & -13 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 12 \end{pmatrix}$

as required.

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4. [13 marks]

A room contains a collection of ducks, insects, and spiders, but no other living creatures. Each duck has 1 mandible (moving jaw part), 1 pair of eyes, and 1 pair of legs; each insect has 2 mandibles, and 1 pair of eyes and 3 pairs of legs; and each spider has 2 mandibles, 2 pairs of eyes, and 4 pairs of legs. **Set up and solve** a system of linear equations to determine the number of ducks, the number of insects, and the number of spiders the room contains, if there are 22 mandibles, 16 pairs of eyes, and 34 pairs of legs in the room.

Let  $D$ ,  $I$ ,  $S$  be the number of ducks, insects and spiders respectively.

$$\left. \begin{array}{l} \text{mandibles:} \\ \text{pairs of eyes:} \\ \text{pairs of legs:} \end{array} \right\} \begin{array}{l} D + 2I + 2S = 22 \\ D + I + 2S = 16 \\ D + 3I + 4S = 34 \end{array}$$

By inspection, the first 2 eqns say  $I = 6$

$$\text{so } D + 2S = 10$$

$$D + 2S = 10$$

$$D + 4S = 16$$

$$\left. \begin{array}{l} D + 2S = 10 \\ D + 4S = 16 \end{array} \right\} 2S = 6 \Rightarrow S = 3 \Rightarrow D = 4$$

so ducks = 4, insects = 6, spiders = 3

or by row-reduction.