Soln.

Department of Mathematics
University of Toronto

WEDNESDAY, November 2, 2005 6:10-8:00 PM

MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce

Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	LM 123	T0501D	W3D	UCA101
T0101B	M9B	LM 157	T0601A	R4A	LM 123
T0101C	M9C	WI 523	T0601B	R4B	LM 157
T0201A	M3A	WO 35	T0701A	F2A	LM 157
T0201B	M3B	SS2128	T0701B	F2B	MP 118
T0201C	M3C	WI 524	T0701C	F2C	SS1084
T0201D	M3D	UC 52	T0801A	F3A	MP 118
T0301A	T3A	UC 87	T0801B	F3B	WI 523
T0301B	T3B	UC 256	T5101A	M5A	LM 155
T0401A	W9A	LM 123	T5101B	M5B	WI 523
T0401B	W9B	LM 157	T5201A	M6A	LM 123
T0501A	W3A	UC 244			
T0501B	W3B	UC 328			
T0501C	W3C	UC 52			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

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PART A. Multiple Choice

1. *[4 marks]*
How many years will it take an investment to triple in value at an effective rate of 7%?

   A. 16.5 years
   B. 21 years
   C. 42.857 years
   D. 16.238 years
   E. 1.031 years

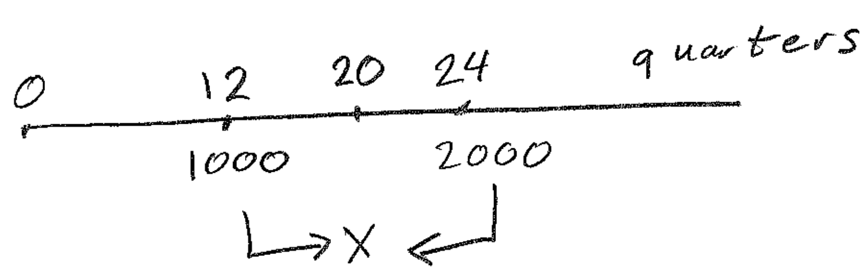
$3 = (1.07)^n$

$\ln 3 = n \ln(1.07)$

$n = \dfrac{\ln 3}{\ln(1.07)} \approx \boxed{16.2376}$

2. *[4 marks]*
A person owes $1,000 in 3 years and $2,000 in 6 years. If the interest rate is 8% compounded quarterly, then to pay off both debts in 5 years, he must then pay

   A. $3,155.03
   B. $3,469.33
   C. $3,000.00
   D. $4,457.84
   E. $3,019.35



$X = 1000(1.02)^8 + 2000(1.02)^{-4}$

$\approx \boxed{3019.35}$

3. *[4 marks]*
A loan of \$25,000 is amortized over 10 years at an interest rate of 6% compounded monthly. If payments are made at the end of each month, then the principal repaid in the first payment is:

A. \$277.55

B. \$158.05

C. \$152.55

D. \$83.33

E. \$125.00

$i = \frac{.06}{12} = .005$

$25{,}000 = R\, a_{\overline{120|}.005}$

$R = \frac{.005 \times 25{,}000}{1 - (1.005)^{-120}}$

$R \approx 277.55$

Since in the 1st period, principal outstanding is \$25,000, interest in the first pymt is $.005 \times 25000 = 125$, So principal paid is $277.55 - 125 = \boxed{\$152.55}$

Alternatively, principal outstanding after 1st payment $\approx 277.55\, a_{\overline{119|}.005} \approx 24{,}847.37$

So $25{,}000 - 24{,}847.37 = 152.63$ has been repaid. ≈ 152.55 [277.55 is an approximation]

4. *[4 marks]*
A person makes 40 quarterly deposits of \$200 into an account earning 4% compounded quarterly. If the first deposit is made **right away**, then at the end of 10 years, the amount in the account will be closest to:

A. \$9,875.05

B. \$9,777.27

C. \$9,604.89

D. \$9,989.08

E. \$19,078.05

$-1 \quad 0 \quad 39 \quad 40$

$200 \quad 200$

$200\, s_{\overline{40|}.01}(1.01) = 200\left[\frac{1.01^{40} - 1}{.01}\right] \times 1.01$

$\approx \boxed{9875.05}$

5. *[4 marks]*
If a \$50,000 mortgage has monthly payments at the end of each month for 5 years and interest is 8% compounded semiannually then the amount of each payment is

A. \$1,035.63

B. \$1,028.49

C. \$1,057.28

D. \$1,042.05

E. \$1,010.71

$$(1.04)^2 = (1+i)^{12}$$

$$50{,}000 = R\, a_{\overline{60}|i}$$

$$R = \frac{50{,}000\, i}{1-(1+i)^{-60}} = \frac{50{,}000\left[(1.04)^{\frac{1}{6}}-1\right]}{\left[1-(1.04)^{-10}\right]}$$

$$\approx \boxed{\$1010.71}$$

6. *[4 marks]*
If a \$100 bond has 8 years to maturity, semiannual coupons worth \$3 each, and an annual yield of 7%, then its market price is

A. \$93.95

B. \$90.03

C. \$94.65

D. \$91.27

E. \$92.29

$$P = 100(1.035)^{-16} + 3\, a_{\overline{16}|.035}$$

$$\approx \boxed{\$93.95}$$

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7. *[4 marks]*
If a \$100 bond has 5 years until maturity, an annual yield rate of 8%, and sells for \$90, then its semiannual coupon rate is closest to

A. 3.0%

B. 2.8%

C. 4.0%

D. 2.1%

E. 3.3%

$$\$90 = 100(1.04)^{-10} + 100r\, a_{\overline{10}|.04}$$

where r is the semi-annual coupon rate.

$$100r = \frac{90 - 100(1.04)^{-10}}{a_{\overline{10}|.04}}$$

$$= \frac{.04\left[90 - 100(1.04)^{-10}\right]}{\left[1 - (1.04)^{-10}\right]}$$

$$\approx 2.767\ldots$$

$\boxed{2.8}$ is closest

8. *[4 marks]*
The system of equations

$$\begin{cases} x + 2y = 3 \\ 2x + ay = 4 \end{cases}$$

has no solution if the constant a is equal to:

A. 1

B. 2

C. 3

D. 4

E. 5

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 2 & a & 4 \end{array}\right) \xrightarrow{R_2 \to R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & a-4 & -2 \end{array}\right)$$

If $a \neq 4$, unique soln.
If $\boxed{a=4}$, 2nd row is $(0\ 0 \mid -2)$
no soln.

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9. *[4 marks]*
If $\begin{bmatrix} 0 & 1 \\ 2 & -2 \end{bmatrix} X = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ then X is:

$$X = \begin{pmatrix} 0 & 1 \\ 2 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \frac{\begin{pmatrix} -2 & -1 \\ -2 & 0 \end{pmatrix}}{-2} \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = \boxed{\begin{pmatrix} \frac{5}{2} & \frac{1}{2} \\ 2 & 0 \end{pmatrix}}$$

A. $\begin{bmatrix} \frac{5}{2} & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix}$

C. $\begin{bmatrix} 2 & 1 \\ 2 & \frac{1}{2} \end{bmatrix}$

D. $\begin{bmatrix} -\frac{5}{2} & -\frac{1}{2} \\ -2 & 0 \end{bmatrix}$

E. $\begin{bmatrix} \frac{5}{2} & 2 \\ \frac{1}{2} & 0 \end{bmatrix}$

Note: $\left(\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 2 & -2 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 2 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1 & \frac{1}{2} \\ 0 & 1 & 1 & 0 \end{array}\right)$

then continue as before.

10. *[4 marks]*
The system of equations:

$$\begin{cases} x_1 \quad\quad +2x_3 - x_4 = 1 \\ \quad x_2 + x_3 + 2x_4 = 0 \\ -x_1 \quad\quad - x_3 \quad\quad = 2 \\ \quad -2x_2 - 3x_3 - 3x_4 = -3 \end{cases}$$

has

A. no solutions

B. a unique solution

C. a 1-parameter family of infinitely many solutions

D. a 2-parameter family of infinitely many solutions

E. a 3-parameter family of infinitely many solutions

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ -1 & 0 & -1 & 0 & 2 \\ 0 & -2 & -3 & -3 & -3 \end{array}\right) \xrightarrow{R_3 \to R_3 + R_1} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & -2 & -3 & -3 & -3 \end{array}\right)$$

$$\xrightarrow{R_4 \to R_4 + 2R_2} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & -1 & 1 & -3 \end{array}\right) \xrightarrow{R_4 \to R_4 + R_3} \left(\begin{array}{cccc|c} 1 & 0 & 2 & -1 & 1 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

4 variables − 3 non-zero rows = $\boxed{\text{1 parameter}}$

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PART B. Written-Answer Questions

1. *[15 marks]*

On January 1, 2005 a retiree had two ordinary annuities as follows:

i) \$3000 payable on January 1st each year, the final payment being on January 1, 2020.

ii) \$500 payable at the beginning of each month, the final payment being on January 1, 2020.

Immediately after receiving the payments on January 1, 2005, he requests that these two annuities be combined into a single annuity, payable on January 1st and July 1st each year, the final payment being on January 1, 2020. If all the annuities are based on an **effective annual rate** of 5%, then find:

[3] (a) the rate per payment period for each of the above three annuities.

i) .05 or $\boxed{5\%}$ per year

ii) $1.05 = (1+i)^{12}$; $i = 1.05^{\frac{1}{12}} - 1 \approx .004074 = \boxed{.4074\%}$ per month

Combo: $1.05 = (1+r)^2$; $r \approx 1.05^{\frac{1}{2}} - 1 \approx .024695 = \boxed{2.4695\%}$ per half-year

[6] (b) the values of each of the two old annuities on January 1, 2005 (right after the payments were received).

i) 0 — Jan 1, 06: 1 — 2 — … — Jan 1,'20: 15

3000 3000 … 3000

$3000\, a_{\overline{15}|.05} \approx \boxed{\$31{,}138.97}$

ii) 0 — Jan 1, 06: 12 — … — Jan 1, '20: 180

… 500 … 500

$500\, a_{\overline{180}|i} = 500\dfrac{[1-(1+i)^{-180}]}{i} = 500\dfrac{[1-(1.05)^{-15}]}{[1.05^{\frac{1}{12}}-1]} \approx \boxed{\$63{,}692.58}$

[6] (c) the value of the semi-annual payment for the new annuity.

Together, i) and ii) are worth

$94{,}831.55 = X a_{\overline{30}|r}$

0 — Jan 1,'06: 2 — … — Jan 1, '20: 30

… X … X

$X = \dfrac{94{,}831.55\, r}{1-(1+r)^{-30}} = 94{,}831.55\dfrac{[(1.05)^{\frac{1}{2}}-1]}{[1-(1.05)^{-15}]} \approx \boxed{\$4512.43}$

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2. *[18 marks]*

A \$90,000 mortgage has monthly payments for 10 years at the **effective monthly rate** of 0.4%.

[4] (a) Find the amount of each payment.

$$90{,}000 = R\, a_{\overline{120}|.004}$$

$$R = \frac{90{,}000 \times .004}{1-(1.004)^{-120}} \approx \boxed{\$945.82}$$

[6] (b) Find the principal outstanding just after the 40th payment. ∴ 80 payments left.

$$P.O. = R\, a_{\overline{80}|.004} = \frac{90{,}000}{a_{\overline{120}|.004}} \cdot a_{\overline{80}|.004}$$

$$P.O. = 90{,}000 \frac{[1-1.004^{-80}]}{[1-1.004^{-120}]} \approx \boxed{\$64{,}643.50}$$

or $P.O. = 945.82\, a_{\overline{80}|.004} \approx \$64{,}643.80$ also OK though not accurate.

[8] (c) How many payments need to be made to repay \$60,000 of principal?
[More precisely, but confusingly, what is the smallest number of payments after which at least \$60,000 has been repaid?]

Principal outstanding must be $\leq 30{,}000$

If n payments remain,

$$30{,}000 = 945.82\, a_{\overline{n}|.004}$$

$$30{,}000 = 945.82 \frac{1-(1.004)^{-n}}{.004}$$

$$\frac{.004 \times 30{,}000}{945.82} = 1-(1.004)^{-n}$$

$$(1.004)^{-n} = 1 - \frac{.004 \times 30{,}000}{945.82}$$

$$-n \ln(1.004) = \ln\left[1 - \frac{.004 \times 30{,}000}{945.82}\right]$$

$$n \approx 33.9866\ldots$$

If there were 34 payments left, P.O. would be more than \$30,000, so principal repaid would be less than \$60,000. So must have 33 payments left and $\boxed{\text{87 have been made}}$.
But 34 is very close, so we accept 86 as an answer as well.

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3. *[13 marks]*

[7] (a) Find the inverse of the following matrix (if it exists):

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 3 & 0 & 0 & 1 & 0 \\ 1 & 1 & -2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -1 & -6 & -2 & 1 & 0 \\ 0 & -1 & -5 & -1 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 + R_2]{R_2 \to -R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 6 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \xrightarrow[R_1 \to R_1 - 3R_3]{R_2 \to R_2 - 6R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -2 & 3 & -3 \\ 0 & 1 & 0 & -4 & 5 & -6 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 6 & -7 & 9 \\ 0 & 1 & 0 & -4 & 5 & -6 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

The inverse is $\boxed{\begin{pmatrix} 6 & -7 & 9 \\ -4 & 5 & -6 \\ 1 & -1 & 1 \end{pmatrix}}$ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

Check only

[6] (b) Find the solution(s) of the following system (if there are any):

$$\begin{cases} x + 2y + 3z = 2 \\ 2x + 3y = 1 \\ x + y - 2z = 0 \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 0 \\ 1 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & -7 & 9 \\ -4 & 5 & -6 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix}}$$ unique

Can check to see if soln. It is.

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4. *[14 marks]*

A person is selling hot dogs, hamburgers and bottles of water. The prices are \$1 for a bottle of water, \$2 for a hot dog and \$3 for a hamburger. At the end of the day she has made a total of \$330. She also knows that she sold a total of 200 items and that the number of bottles of water sold was the same as the numbers of hot dogs and hamburgers sold added together (everybody bought an item of food and a drink).
How many hot dogs did she sell?

Let D = no. of hot-dogs, H of hamburgers, W of bottles of water

$$\left\{\begin{array}{l} W+3H+2D=330 \\ W+H+D=200 \\ W-H-D=0 \end{array}\right. \quad \text{and } W=D+H \text{ or}$$

$$\begin{array}{c} \begin{array}{ccc} W & H & D \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 2 & 330 \\ 1 & 1 & 1 & 200 \\ 1 & -1 & -1 & 0 \end{array}\right) \end{array} \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 330 \\ 0 & -2 & -1 & -130 \\ 0 & -4 & -3 & -330 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + 4R_2]{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 330 \\ 0 & 1 & \frac{1}{2} & 65 \\ 0 & 0 & -1 & -70 \end{array}\right) \xrightarrow{R_3 \to -R_3} \begin{array}{c} \begin{array}{ccc} W & H & D \end{array} \\ \left(\begin{array}{ccc|c} 1 & 3 & 2 & 330 \\ 0 & 1 & \frac{1}{2} & 65 \\ 0 & 0 & 1 & 70 \end{array}\right) \end{array}$$

$$\boxed{D = 70}$$

To check: $H = 65 - \frac{1}{2}D = 65 - 35 = 30$

$W = 330 - 2D - 3H = 330 - 140 - 90 = 100$

i.e. $W = 100$, $H = 30$, $D = 70$

$W + 3H + 2D = 100 + 90 + 140 = 330$ ✓

$W + H + D = 100 + 30 + 70 = 200$ ✓

$W = D + H$ i.e. $100 = 70 + 30$ ✓

Of course, many other orders for the variables and the equations are possible. The order of the variables was chosen so that D would be solved for first (the last variable is obtained first in row reduction).