Solved

Department of Mathematics
University of Toronto

WEDNESDAY, November 3, 2004 6:10-8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	BF 323	T0501C	W3C	RW 229
T0101B	M9B	LM 123	T0501D	W3D	BA1240
T0101C	M9C	LM 157	T0601A	R4A	LM 157
T0201A	M3A	UCA101	T0701A	F2A	BF 323
T0201B	M3B	SS2106	T0701B	F2B	LM 157
T0201C	M3C	WB 119	T0701C	F2C	MP 118
T0201D	M3D	LM 123	T0801A	F3A	WA 142
T0301A	T3A	MP 137	T0801B	F3B	WI 523
T0301B	T3B	UC 328	T5101A	R5A	LM 155
T0301C	T3C	MP 134	T5101B	R5B	LM 157
T0401A	W9A	BF 323	T5201A	R6A	SS2111
T0401B	W9B	LM 123			
T0501A	W3A	LM 123			
T0501B	W3B	UC 328			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

NAME: ______________________________ STUDENT NO: ____________________

PART A. Multiple Choice

1. *[4 marks]*
At what nominal annual rate compounded monthly will \$400 grow to \$1,000 in 5 years?

   A. 18.47%

   B. 1.54%

   C. 23.80%

   D. 1.68%

   E. 20.11%

Let r = nominal annual rate

$$400\left(1+\frac{r}{12}\right)^{60} = 1000$$

$$\left(1+\frac{r}{12}\right)^{60} = 2.5$$

$$r = 12\left[2.5^{\frac{1}{60}} - 1\right] = .18466\ldots$$

$$\boxed{18.47\%}$$

2. *[4 marks]*
A person owes x now and x in 5 years. He plans to pay off both these debts in 3 years by making a single payment of \$2,000. If interest is 7% compounded annually then $x =$

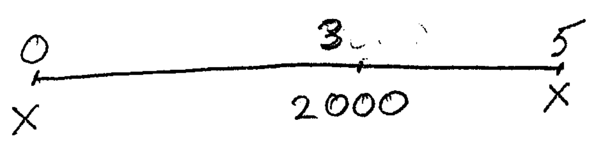
   A. \$1,000

   B. \$1,225.04

   C. \$816.30

   D. \$1,167.55

   E. \$953.07



$$x + x(1.07)^{-5} = 2000(1.07)^{-3}$$

$$x\left[1 + (1.07)^{-5}\right] = 2000(1.07)^{-3}$$

$$x = \frac{2000(1.07)^{-3}}{1+(1.07)^{-5}}$$

$$\boxed{x = \$953.07}$$

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3. *[4 marks]*
If \$350 is deposited in an account which earns interest at 4% compounded quarterly for the first 3 years and then earns 6% compounded semi-annually for the next 3 years, how much will there be in the account at the end of 6 years?

A. \$469.03

B. \$812.31

C. \$470.92

D. \$810.56

E. \$468.91

$$350(1.01)^{12}(1.03)^{6} = \boxed{\$470.92}$$

4. *[4 marks]*
Today Mr. Smith has \$50,000 saved in an account which is earning 4.2% compounded monthly. He plans to withdraw \$2,000 per month, the first withdrawal to be made one month from today. Assuming the interest will never change, how many full \$2,000 withdrawals will Mr. Smith be able to make?

A. 27

B. 30

C. 28

D. 26

E. 29

$$50{,}000 = 2000\, a_{\overline{n|}, \frac{.042}{12}}$$

$$25 \times \frac{.042}{12} = 1 - \left(1 + \frac{.042}{12}\right)^{-n}$$

$$\left(1 + \frac{.042}{12}\right)^{-n} = \left(1 - \frac{25 \times .042}{12}\right)$$

$$n = \frac{-\ln\left(1 - \frac{25 \times .042}{12}\right)}{\ln\left(1 + \frac{.042}{12}\right)}$$

$$n = 26.207\ldots$$

$\boxed{\text{26 full withdrawals}}$

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5. *[4 marks]*
A person will pay a $1,000 debt by making 12 equal monthly payments, the first payment to be made **now**. If interest is 6% compounded monthly, then the amount of each payment is

A. $87.19

B. $85.64

C. $84.25

D. $86.07

E. $83.33

0 1 . . . 11 12

R R . . . R

Annuity Due

$$R + R\,a_{\overline{11}|\frac{.06}{12}} = 1000$$

$$R = \frac{1000}{1 + a_{\overline{11}|\frac{.06}{12}}}$$

$$R = 85.638\ldots$$

$$\boxed{R = \$85.64}$$

6. *[4 marks]*
Just after a coupon has been clipped, a $1,000 bond has 13 semiannual coupons, which are worth $24 each. If the bond's annual yield is 5%, then its market price is

A. $989.02

B. $976.37

C. $992.35

D. $1,000.00

E. $981.75

$$P = 1000(1.025)^{-13} + 24\,a_{\overline{13}|.025}$$

$$P = 989.0168\ldots$$

$$\boxed{P = \$989.02}$$

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7. *[4 marks]*

We only need row-echelon form not to include a row of zeros:

Let $P = \begin{bmatrix} 10 & 1 & 0 \\ 2 & x & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$\xrightarrow[R_2 \to -2R_1+R_2]{R_1 \to \frac{1}{10}R_1} \begin{pmatrix} 1 & \frac{1}{10} & 0 \\ 0 & x-\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Then P^{-1} exists as long as

A. $x \neq 0$

B. $x \neq 2$

C. $x \neq -10$

D. $x \neq \frac{1}{5}$

E. x is any real number

P^{-1} exists as long as $x - \frac{1}{5} \neq 0$ i.e. $\boxed{x \neq \frac{1}{5}}$

8. *[4 marks]*

If X is a 2×2 matrix and $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} X = \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}$ then $X =$

A. $\begin{bmatrix} 3 & 6 \\ 1 & -\frac{1}{2} \end{bmatrix}$

B. $\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$

C. $\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ 4 & -1 \end{bmatrix}$

D. $\begin{bmatrix} -1 & 2 \\ 1 & -\frac{1}{2} \end{bmatrix}$

E. $\begin{bmatrix} -\frac{1}{3} & \frac{4}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix}$

Either directly $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

and $X = -\frac{1}{2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} 2 & -4 \\ -2 & 1 \end{pmatrix}$

$\boxed{X = \begin{pmatrix} -1 & 2 \\ 1 & -\frac{1}{2} \end{pmatrix}}$

or by row reduction: $\left(\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 3 & 4 & 1 & 4 \end{array}\right) \xrightarrow{R_2 \to -3R_1 + R_2} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 1 \\ 0 & -2 & -2 & 1 \end{array}\right)$

$\xrightarrow[R_1 \to -2R_2+R_1]{R_2 \to -\frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -\frac{1}{2} \end{array}\right)$

$\boxed{X = \begin{pmatrix} -1 & 2 \\ 1 & -\frac{1}{2} \end{pmatrix}}$

9. *[4 marks]*
$U, V,$ and W are matrices. U is 1×3, V is 3×5, and W is 3×1. The matrix product that does not make sense is

A. UV

B. WV

C. WU

D. UW

E. $(UV)^T W^T$

$_1U_3 \; _3V_5$ OK

$\boxed{_3W_1 \; _3V_5 \quad \text{BAD}}$

$_3W_1 \; _1U_3$ OK

$_1U_3 \; _3W_1$ OK

$_1(UV)_5$ from A.

$_5(UV)^T_1 \; _1W^T_3$ OK

10. *[4 marks]*
If the augmented matrix for a system of linear equations is

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 1 & 3 & 1 & 2 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

then the system of equations has

A. no solution

B. a unique solution

C. a 1-parameter family of infinitely many solutions

D. a 2-parameter family of infinitely many solutions

E. a 3-parameter family of infinitely many solutions

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

4 unknowns, 3 equations

so a $\boxed{\text{1-parameter family of } \infty\text{-many solns}}$

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PART B. Written-Answer Questions

1. *[16 marks]*
A $200,000 mortgage is to be repaid over 25 years by equal monthly payments, the first payment one month after the loan is made. Interest is 5% compounded semi-annually.

[5] (a) Find the amount of each payment. (accuracy counts a lot)

$$200{,}000 = R\, a_{\overline{300}|i} \qquad (1.025)^2 = (1+i)^{12}$$

$$R = \frac{200{,}000\, i}{1-(1+i)^{-300}} = \frac{200{,}000\left[1.025^{\frac{1}{6}}-1\right]}{1-(1.025)^{-50}}$$

$$\boxed{R = \$1163.21}$$

(For those who used it, $i = .0041239155$)

[5] (b) How much principal is outstanding just after the 120th payment? (accuracy again!)

180 payments remain:

$$P.O = R\, a_{\overline{180}|i} = 1163.21\frac{\left[1-(1.025)^{-30}\right]}{(1.025)^{\frac{1}{6}}-1} = \boxed{\$147{,}592.29}$$

or:

$$= \frac{200{,}000}{a_{\overline{300}|i}} \cdot a_{\overline{180}|i} = 200{,}000\frac{\left[1-(1.025)^{-30}\right]}{\left[1-(1.025)^{-50}\right]} = \boxed{\text{same answer}}$$

[6] (c) After which payment will the outstanding principal first be less than one half the original amount of the loan?

Let n = number of payments remaining.

$R\, a_{\overline{n}|i} = 100{,}000$. We need $\underline{300-n}$.

$$\frac{200{,}000}{a_{\overline{300}|i}}\, a_{\overline{n}|i} = 100{,}000$$

$$\frac{1-(1+i)^{-n}}{1-(1+i)^{-300}} = \frac{1}{2}$$

$$1-(1.025)^{-\frac{n}{6}} = \frac{1-(1.025)^{-50}}{2}$$

$$\frac{1+(1.025)^{-50}}{2} = (1.025)^{-\frac{n}{6}}$$

$$n = \frac{-6\ln\left[\frac{1+1.025^{-50}}{2}\right]}{\ln(1.025)} = 106.37\ldots$$

$300-n = 193.6\ldots$ So after 193 payments we're not there yet.

The answer is $\boxed{194}$

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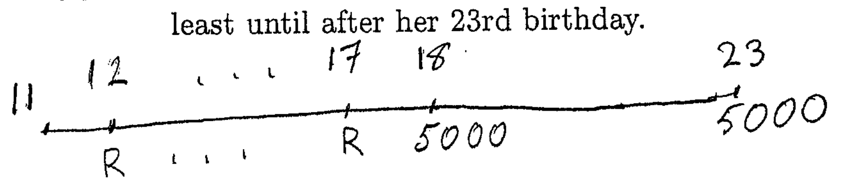
2. *[17 marks]*

Joe plans to give his daughter, Ann, \$5,000 on each of her birthdays from her 18th to her 23rd inclusive. To save the necessary money, Joe plans to make equal deposits on each of Ann's birthdays from her 12th to her 17th inclusive, into a fund which earns 7% effective annual interest; the \$5,000 gifts are to be made from the same fund. In the following, give answers to the nearest cent.

[8] (a) What should be the amount of each deposit if the 7% interest rate is expected to last at least until after her 23rd birthday.



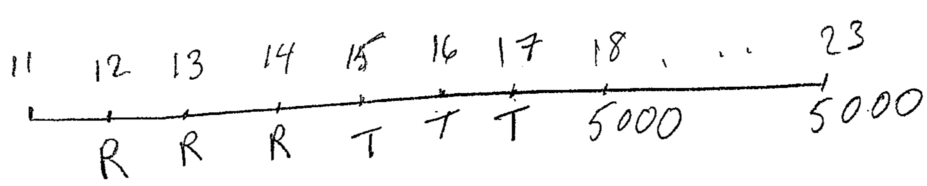
$$R = \frac{5000\, a_{\overline{6}|.07}}{s_{\overline{6}|.07}}$$

$$R s_{\overline{6}|.07} = 5000\, a_{\overline{6}|.07}$$

$$R = 5000 \frac{[1 - 1.07^{-6}]}{[1.07^{6} - 1]} = \boxed{\$3331.71}$$

or: Since payments and deposits match: $\$3331.71 = R = 5000(1.07)^{-6}$

[9] (b) Just after Ann's 14th birthday, the interest rate changes to 6% effective annual and is expected to remain 6% until after her 23rd birthday. What should be the amount of each of the remaining deposits (which are all to be equal), if Joe is to achieve his original goal? Let T be the new payment.



If we a accumulate at 14, say

$$R s_{\overline{3}|.07} + T a_{\overline{3}|.06} = 5000\, a_{\overline{6}|.06} (1.06)^{-3}$$

$$\frac{-3331.71\, s_{\overline{3}|.07} + 5000\, a_{\overline{6}|.06} (1.06)^{-3}}{a_{\overline{3}|.06}} = T$$

$$\boxed{T = \$3715.77.}$$

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3. *[12 marks]*

Mary-Ann is on a diet. Her restrictions are: 1,500 calories per day, and no more than 100 grams of sugar and 20 grams of fat a day. However, she refuses to eat anything but marshmallows, coke and meat. How much of each of these items can she have, if she keeps at the limit of her restrictions, given that

100 grams of meat contains no sugar, 400 calories and 8 grams of fat,

marshmallows are 1/10 fat, half sugar, and contain 500 calories per 100 grams,

and

coke has no fat, 10 grams of sugar and 50 calories per 100 grams.

Let X = no. of 100 grams of meat

Y = no. of 100 grams of marshmallows

Z = no. of 100 grams of coke

Sugar: $0 \cdot X + \frac{1}{2} 100Y + 10Z = 100$

Calories $400X + 500Y + 50Z = 1500$

Fat $8X + \frac{1}{10} 100Y + 0 \cdot Z = 20$

$\frac{\text{Calories}}{50}$: $8X + 10Y + Z = 30$

Fat $8X + 10Y = 20$

$\frac{\text{Sugar}}{10}$ $5Y + Z = 10$

First 2 eqns say $\boxed{Z = 10}$ (subtraction)

3rd equation says, then, that $5Y = 0$ so $\boxed{Y = 0}$

Then $X = \frac{20}{8} = 2.5$

$\boxed{\begin{array}{l}\text{250 g. meat} \\ \text{No Marshmallows} \\ \text{1000 g. coke}\end{array}}$

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4. *[15 marks]*

Given the input/output matrix below for industries A and B

$$
\begin{array}{c}
\\
\text{Industry A} \\
\text{Industry B} \\
\\
\text{Other Production Factors}
\end{array}
\begin{array}{c}
\begin{array}{cccc} \text{Industry A} & \text{Industry B} & & \text{Final Demand} \end{array} \\
\left(\begin{array}{cc:c}
25 & 80 & 145 \\
75 & 40 & 85 \\
\hdashline
150 & 80 & -
\end{array}\right)
\end{array}
$$

if the final demand changes to 90 for industry A and 150 for industry B then find

[10] (a) the new total outputs for industries A and B.

Total production of A = 250 , of B = 200

$$\text{Coeff. matrix} = \begin{pmatrix} \frac{25}{250} & \frac{80}{200} \\ \frac{75}{250} & \frac{40}{200} \\ \frac{150}{250} & \frac{80}{200} \end{pmatrix} = \begin{pmatrix} \frac{1}{10} & \frac{2}{5} \\ \frac{3}{10} & \frac{1}{5} \\ \hdashline \frac{3}{5} & \frac{2}{5} \end{pmatrix}$$

$$I-A = \begin{pmatrix} \frac{9}{10} & -\frac{2}{5} \\ -\frac{3}{10} & \frac{4}{5} \end{pmatrix} \qquad (I-A)^{-1} = \frac{\begin{pmatrix} \frac{4}{5} & \frac{2}{5} \\ \frac{3}{10} & \frac{9}{10} \end{pmatrix}}{\frac{36}{50} - \frac{6}{50}} = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

$$X = \begin{pmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix} \begin{pmatrix} 90 \\ 150 \end{pmatrix} = \begin{pmatrix} 120+100 \\ 45+225 \end{pmatrix}$$

[5] (b) the new other production factors for industries A and B.

(a) ctd: $\boxed{\begin{array}{l} \text{220 units of output of A} \\ \text{270 units of output of B} \end{array}}$

b) $\frac{3}{5} \times 220 = $ 132 other prod. factors for A

$\frac{2}{5} \times 270 = $ 108 " " " " B