

Solved

Department of Mathematics
University of Toronto

WEDNESDAY, OCTOBER 29, 2003 6:10-8:00 PM

MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce

Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.

For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. **ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.**

TOTAL MARKS: 100

FAMILY NAME:

GIVEN NAME:

STUDENT NO:

SIGNATURE:

TUTORIAL TIME and ROOM:

REGCODE and TIMECODE:

T.A.'S NAME:

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS2130	T0501C	W3C	SS2111
T0101B	M9B	BF 215	T0601A	R4A	SS1088
T0101C	M9C	LM 157	T0601B	R4B	RW 142
T0201A	M3A	SS2111	T0601C	R4C	LM 157
T0201B	M3B	LM 123	T0701A	F2A	BF 323
T0201C	M3C	RW 143	T0701B	F2B	LM 157
T0201D	M3D	MP 134	T0701C	F2C	SS2111
T0301A	T3A	RW 142	T0801A	F3A	RW 142
T0301B	T3B	SS2111	T0801B	F3B	LM 157
T0301C	T3C	MP 134	T5101A	R5A	SS1088
T0401A	W9A	BF 215	T5201A	R6A	SS1088
T0401B	W9B	LM 157			
T0501A	W3A	LM 157			
T0501B	W3B	LM 123			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

PART A. Multiple Choice

1. [4 marks]

For the first six months of the year, the nominal annual rate is 5% compounded quarterly. For the last six months, it is 4% compounded monthly. To the nearest two decimal places, the effective annual rate is

- A. 4.50%
- B. 4.58%
- C. 9.20%
- D. 4.75%
- E. 4.63%

$$1 + r_e = \left(1 + \frac{.05}{4}\right)^2 \left(1 + \frac{.04}{12}\right)^6$$

$r_e = .04583$

2. [4 marks]

Merle buys a car for \$24,000 with no down-payment and agrees to make payments at the end of each month, beginning one month after the purchase. If the nominal rate of interest is 4.8% compounded monthly, and the payments are \$400 per month, (except for the last one, which is less), then Merle will make a total of

- A. 60 payments
- B. 83 payments
- C. 84 payments
- D. 68 payments
- E. 69 payments

$$24,000 = 400 a_{\overline{n}|.004}$$

$$60 = \frac{1 - (1.004)^{-n}}{.004}$$

$$(1.004)^{-n} = 1 - 60 \times .004$$

$$n = \frac{-\ln [1 - 60 \times .004]}{\ln 1.004}$$

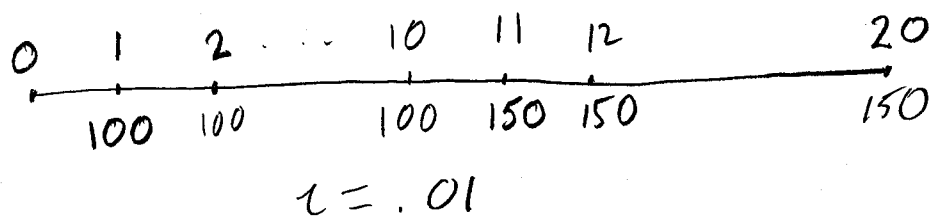
$n \approx 68.7$ So it takes

69 payments

3. [4 marks]

If the nominal annual rate of interest is 12% compounded monthly, then, one month before the first payment, an annuity consisting of 10 payments of \$100 each at the end of each month, followed by 10 more payments of \$150 each at the end of each month, has a present value (to the nearest \$100) of

- A. \$2,200
- B. \$2,300
- C. \$2,400
- D. \$2,500
- E. \$2,600



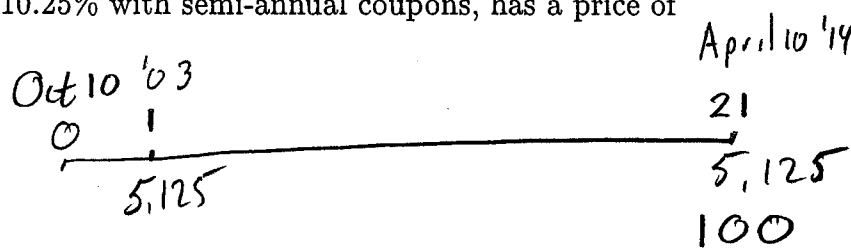
$$P.V. = 100 a_{\overline{10}|.01} + 150 a_{\overline{10}|.01} (1.01)^{-10}$$

$$\approx \boxed{\$2233}$$

4. [4 marks]

On October 10th 2003, a bond maturing on April 10, 2014, with annual yield to maturity 5.12%, and an annual coupon rate of 10.25% with semi-annual coupons, has a price of

- A. \$95.34 per \$100 of face value
- B. \$107.62 per \$100 of face value
- C. \$125.49 per \$100 of face value
- D. \$141.27 per \$100 of face value
- E. \$160.08 per \$100 of face value



$$P = 100 (1.0256)^{-21} + 5.125 a_{\overline{21}|.0256}$$

$$P \approx \boxed{141.269}$$

5. [4 marks]

The owner of a \$100 bond has just redeemed a coupon so that 8 semiannual coupons remain, worth \$3 each. If the market price of the bond is \$100.70, then its semiannual yield rate is closed to

- A. 2.8%
- B. 3.1%
- C. 2.9%
- D. 2.7%
- E. 3.0%

$$100.70 = 100(1+i)^{-8} + 3a_{\overline{8}|i}$$

Since $P > 100$, $i < 3$ but not much.
Try $i = 2.8\%$

$$100(1.028)^{-8} + 3a_{\overline{8}|0.028} \approx 101.416$$

Price too high, so yield too low.
Raise yield. But only 2.9% is possible.

Calculation: $100(1.029)^{-8} + 3a_{\overline{8}|0.029} \approx 100.7049$

6. [4 marks]

$$\begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}^2 - 5 \begin{bmatrix} 4 & 2 \\ 1 & 1 \end{bmatrix}^T \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$

- A. [-4]
- B. [0]
- C. [1]
- D. [2]
- E. [-1]

NOTE: If A is a matrix, A^T denotes the transpose of A .

$$\begin{aligned} & \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} - 5 \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \left(\begin{bmatrix} 22 & 6 \\ 9 & 7 \end{bmatrix} - \begin{bmatrix} 20 & 5 \\ 10 & 5 \end{bmatrix} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \boxed{(-1)} \end{aligned}$$

NAME: _____

STUDENT NO: _____

7. [4 marks]

The coefficient matrix of the system $\begin{matrix} x & - & y & = & 2 \\ -2x & + & 3y & = & 1 \end{matrix}$ is

- A. $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- B. $\begin{bmatrix} 1 & -1 \\ -2 & 3 \end{bmatrix}$
- C. $\begin{bmatrix} 1 & -1 & | & 2 \\ 0 & 1 & | & 5 \end{bmatrix}$
- D. $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$
- E. $\begin{bmatrix} 1 & -1 & | & 2 \\ -1 & 3 & | & 1 \end{bmatrix}$

by inspection

8. [4 marks]

$$\begin{vmatrix} 1 & -1 & 1 & 2 \\ 1 & 3 & 2 & 1 \\ -1 & 1 & -2 & 0 \\ 1 & -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 & 2 \\ 0 & 4 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 4 & 1 & -1 \\ 0 & -1 & 2 \\ 0 & -1 & -1 \end{vmatrix} = 4 \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} -1 & 2 \\ 0 & -3 \end{vmatrix} = \boxed{12}$$

- A. 6
- B. -4
- C. 12
- D. 4
- E. -12

NAME: _____

STUDENT NO: _____

9. [4 marks]

The system given by

$$\begin{aligned} w - 2x + y + z &= 0 \\ 2w &\quad - 2y + 2z = 1 \\ w + x - y + 2z &= -1 \end{aligned}$$

has

A. a unique solution

B. no solution

C. an infinite number of solutions with one parameter

D. an infinite number of solutions with two parameters

E. an infinite number of solutions with three parameters

$$\left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 0 \\ 2 & 0 & -2 & 2 & 1 \\ 1 & 1 & -1 & 2 & -1 \end{array} \right) \rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 0 \\ 0 & 4 & -4 & 0 & 1 \\ 0 & 3 & -2 & 1 & -1 \end{array} \right)$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & -2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 1 & -\frac{7}{4} \end{array} \right)$$

3 non-zero rows and 4 variables

\Rightarrow ∞-many solns with one parameter

10. [4 marks]

Which of the following matrices is its own inverse, i.e., which matrix M has the property

$$M = M^{-1}?$$

A. $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ not square so no M^{-1}

B. $M = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ not square so no M^{-1}

C. $M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & ? \\ ? & ? \end{pmatrix} \neq I$

D. $M = \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix}$ $\begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} 2 & \frac{1}{2} \\ \frac{1}{2} & 2 \end{pmatrix} = \begin{pmatrix} 4\frac{1}{4} & ? \\ ? & ? \end{pmatrix} \neq I$

E. $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

PART B. Written-Answer Questions

1. [15 marks]

On the day of his daughter's birth, Mr. Dodd deposited \$10,000 in a trust fund at 5% compounded quarterly until the end of eighteen years. At the end of each month after her eighteenth birthday, and ending on her twenty-second birthday, the daughter is to receive equal payments while she is at university.

[6] (a) If interest is to be 6% compounded monthly after the daughter's eighteenth birthday, how much will she receive every month?

$$\begin{aligned} \text{After 18 years, there is } & 10,000(1.0125)^{72} \\ & = \$24,459.20 \text{ in the fund} \\ & = R a_{\overline{48}|.005} \end{aligned}$$

$$R = \frac{10,000(1.0125)^{72}}{a_{\overline{48}|.005}} = \boxed{574.43}$$

if R is the monthly payment.

[9] (b) On her tenth birthday, Mr. Dodd does the above calculation and decides that the monthly payments will not be big enough. The trust fund allows him to make further deposits at the same rate of interest (5% compounded quarterly) until his daughter is eighteen. If he makes equal quarterly deposits, beginning one quarter after her tenth birthday, until she is eighteen, how big must they be if her monthly income is to be \$1000?

Let X = size of the quarterly deposits.
There are 32 of these, so

$$\begin{aligned} X s_{\overline{32}|.0125} &= (1000 - R) a_{\overline{48}|.005} = 18,120.91 \\ X &= \frac{425.57 a_{\overline{48}|.005}}{s_{\overline{32}|.0125}} \end{aligned}$$

$$\boxed{X = \$464}$$

NAME: _____

STUDENT NO: _____

2. [15 marks]

A \$50,000 mortgage has equal payments at the end of each month for 6 years (72 months). Interest is 8% compounded semiannually. [For the part-questions below give answers to the nearest cent.]

[5] (a) Find the amount of each payment.

$$50,000 = R a_{\overline{72}|i} \quad \text{where } (1+i)^{12} = (1.04)^2$$

$$R = 50,000 \frac{[(1.04)^{\frac{1}{6}} - 1]}{1 - (1.04)^{-12}}$$

$$\boxed{R = \$873.49}$$

[5] (b) What is the principal outstanding at the beginning of the 48th month of the mortgage?

The beginning of the 48th month is just after the 47th payment, so 25 payments remain.

$$P.O. = R a_{\overline{25}|i}$$

$$= 873.49 \frac{[1 - (1.04)^{-\frac{50}{12}}]}{(1.04)^{\frac{1}{6}} - 1} = \boxed{\$20,080.51}$$

[5] (c) How much interest is included in the 48th payment?

$$i \times P.O. = [(1.04)^{\frac{1}{6}} - 1] \times 20,080.51$$

$$= \boxed{\$131.69}$$

NAME: _____

STUDENT NO: _____

3. [15 marks]

[9] (a) Find the inverse of the following matrix (if it exists).

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 2 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array} \right) \xrightarrow{R_2 \rightarrow R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array} \right)$$

$$\begin{array}{l} \xrightarrow{R_2 \rightarrow -R_2} \\ \xrightarrow{R_3 \rightarrow R_3 - R_2} \end{array} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & -1 & -2 & 1 & 1 \end{array} \right) \xrightarrow{R_3 \rightarrow -R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 2 & -1 & -1 \end{array} \right) \xrightarrow{R_1 \rightarrow R_1 - 2R_2 - R_3}$$

The inverse is

$$\begin{pmatrix} -5 & 3 & 1 \\ 2 & -1 & 0 \\ 2 & -1 & -1 \end{pmatrix}$$

[6] (b) Find the solution of the following system

$$\begin{array}{rcl} x + 2y + z & = & -1 \\ 2x + 3y + 2z & = & 1 \\ y - z & = & -3 \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -5 & 3 & 1 \\ 2 & -1 & 0 \\ 2 & -1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 0 \end{pmatrix}$$

$$\text{or } \boxed{x=5, y=-3, z=0}$$

4. [15 marks]

An accountant is evaluating the dollar value of the shares of three related companies. Company A has 4000 shares issued and has assets of \$40,000 plus 2000 shares of Company B and 1000 shares of Company C. Company B has 10,000 shares issued and has assets of \$100,000 plus 2000 shares of Company A and 1000 shares of Company C. Company C has 5000 shares issued and has assets of \$150,000 plus 1000 shares of Company A and 1000 shares of Company B.

The total dollar value of the shares issued by a company is equal to its assets plus the dollar value of all shares held from other companies.

[8] (a) If a , b and c are the dollar values of one share of each of companies A, B and C respectively, show that

$$\begin{aligned} -a - b + 5c &= 150 \\ -2a + 10b - c &= 100 \\ 4a - 2b - c &= 40 \end{aligned}$$

$$\text{Company A: } 4000a = 40,000 + 2000b + 1000c$$

$$4000a - 2000b - 1000c = 40,000$$

Dividing by 1000

$$\text{A: } \boxed{4a - 2b - c = 40} \text{ the 3rd eqn.}$$

$$\text{Company B: } 10,000b = 100,000 + 2000a + 1000c$$

$$-2000a + 10,000b - 1000c = 100,000$$

Dividing by 1000

$$\text{B: } \boxed{-2a + 10b - c = 100} \text{ the 2nd eqn.}$$

$$\text{Company C: } 5000c = 150,000 + 1000a + 1000b$$

$$-1000a - 1000b + 5000c = 150,000$$

$$\text{C: } \boxed{-a - b + 5c = 150} \text{ the first equation.}$$

Question continues on Page 11

[7] 4 (b) Solve the system in (a) in order to find the dollar value of one share of each company.

[You may use any method you like, but show your work.]

$$\left(\begin{array}{ccc|c} -1 & -1 & 5 & 150 \\ -2 & 10 & -1 & 100 \\ 4 & -2 & -1 & 40 \end{array} \right) \begin{array}{l} R_1 \rightarrow -R_1 \\ R_2 \rightarrow R_2 + 2R_1 \\ R_3 \rightarrow R_3 - 4R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -5 & -150 \\ 0 & 12 & -11 & -200 \\ 0 & -6 & 19 & 640 \end{array} \right) \begin{array}{l} R_2 \leftrightarrow R_3 \\ R_3 \rightarrow R_3 + 2R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -5 & -150 \\ 0 & -6 & 19 & 640 \\ 0 & 0 & 27 & 1080 \end{array} \right)$$

$$\begin{array}{l} R_3 \rightarrow \frac{1}{27}R_3 \\ R_2 \rightarrow -\frac{1}{6}R_2 \end{array} \left(\begin{array}{ccc|c} 1 & 1 & -5 & -150 \\ 0 & 1 & -\frac{19}{6} & -\frac{640}{6} \\ 0 & 0 & 1 & 40 \end{array} \right)$$

$$c = 40$$

$$b = -\frac{640}{6} + \frac{19}{6} \times c = -\frac{640}{6} + \frac{19}{6} \cdot 40 = \frac{120}{6}$$

$$b = 20$$

$$a = -150 - b + 5c = -150 - 20 + 5 \cdot 40$$

$$a = 30$$

So

$a = 30$
$b = 20$
$c = 40$