Solved

Department of Mathematics
University of Toronto

WEDNESDAY, OCTOBER 30, 2002, 6:10 - 8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 11 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions** , present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME and ROOM: ______________________

REGCODE and TIMECODE: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	**Regcode**	Timecode	Room
T0101A	M9A	RW142	T0501C	W3C	LA341
T0101B	M9B	LM157	T0501D	W3D	NF 6
T0101C	M9C	LM123	T0601A	R4A	RW142
T0201A	M3A	LM157	T0601B	R4B	UC244
T0201B	M3B	UC85	T0601C	R4C	SS2130
T0201C	M3C	LA240	T0701A	F2A	MP118
T0201D	M3D	LA204	T0701B	F2B	SS2130
T0301A	T3A	VC212	T0701C	F2C	RW229
T0301B	T3B	NF113	T0801A	F3A	RW142
T0301C	T3C	CR403	T0801B	F3B	SS2111
T0301D	T3D	NF 7	T5101A	R5A	MP118
T0401A	W9A	LM157	T5101C	R5C	UC244
T0401B	W9B	MP118	T5201B	R6B	LM157
T0401C	W9C	LM155			
T0501A	W3A	RW143			
T0501B	W3B	RW229			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

PART A. Multiple Choice

1. *[4 marks]*
The nominal annual rate of interest at which $2500 will grow to $7000 in eight years compounded quarterly is closest to

A. 13.08%

B. 13.74%

C. 13.15%

D. 13.29%

E. 12.94%

$$7000 = 2500\left(1+\frac{r}{4}\right)^{32}$$

$$r = \left[\left(\frac{7000}{2500}\right)^{\frac{1}{32}} - 1\right] \times 4 = .13079\ldots$$

13.08 %

2. *[4 marks]*
If $2000 is deposited at the end of each quarter into a fund paying 8% compounded quarterly, then the fund will first contain at least $50,000 after

A. 6 years and 3 months

B. 5 years and 9 months

C. 5 years and 3 months

D. 4 years and 9 months

E. 4 years and 3 months

$$50{,}000 = 2000\, s_{\overline{n}|.02}$$

where n = no. of quarters

$$25 = \frac{(1.02)^n - 1}{.02}$$

$$(1.02)^n = \frac{3}{2}$$

$$n = \frac{\ln 1.5}{\ln 1.02} \approx 20.48 \text{ quarters}$$

$$\approx 5\text{ yrs} + \tfrac{1}{2}\text{ quarter}$$

5 yrs is not enough, but 5 yrs + 1 quarter will do

$\boxed{\text{5 years and 3 months}}$

3. *[4 marks]*
A \$15,000 car sells for \$2,000 down with the remainder to be paid by equal payments made at the end of each month for 3 years. If interest is 8.4% compounded monthly, then each payment is

A. \$423.36

B. \$409.78

C. \$417.44

D. \$395.21

E. \$429.03

$$13{,}000 = R\,a_{\overline{36}|.007}$$

$$R = \frac{13{,}000 \times .007}{1-(1.007)^{-36}} \approx \boxed{409.78}$$

4. *[4 marks]*
The price on September 5, 2002 of the following bond

Issuer	Coupon Rate	Maturity Date	Yield to Maturity
Manitoba	10.500	March 5, 2031	5.90

is closest to

A. 133.19

B. 148.14

C. 155.62

D. 162.75

E. 163.10

$$P = 100(1.0295)^{-57} + 5.25\,a_{\overline{57}|.0295}$$

$$\approx 163.0997$$

$$\approx \boxed{163.10}$$

5. *[4 marks]*

$$\begin{bmatrix} 0 & -1 & 3 \\ 2 & 5 & -1 \\ 1 & 1 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 2 & 4 & 4 \\ -6 & 0 & 0 \\ 2 & 0 & -4 \end{bmatrix} =$$

A. $\begin{bmatrix} 0 & -1 \\ 2 & 5 \end{bmatrix}$

B. $\begin{bmatrix} 0 & -1 & 3 & 0 & 2 & 1 \\ 2 & 5 & -1 & -3 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & -2 \end{bmatrix}$

C. $\begin{bmatrix} 1 & 1 & 5 \\ -1 & 5 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1 & 4 \\ -1 & 5 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

E. $\begin{bmatrix} 0.5 & 3.5 & 5.5 \\ -4.5 & 5.5 & -0.5 \\ 2 & 1.5 & -2.5 \end{bmatrix}$

$$\begin{pmatrix} 0 & -1 & 3 \\ 2 & 5 & -1 \\ 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 2 & 2 \\ -3 & 0 & 0 \\ 1 & 0 & -2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 & 1 & 5 \\ -1 & 5 & -1 \\ 2 & 1 & -2 \end{pmatrix}}$$

6. *[4 marks]*
The system

$$\begin{cases} 7x + 2y - z &= 5 \\ x + y + z &= 2 \\ -3x - y &= 0 \end{cases}$$

has

A. no solution.

B. a unique solution.

C. infinitely many solutions depending on one parameter.

D. infinitely many solutions depending on two parameters.

E. infinitely many solutions depending on three parameters.

$$\left(\begin{array}{ccc|c} 7 & 2 & -1 & 5 \\ 1 & 1 & 1 & 2 \\ -3 & -1 & 0 & 0 \end{array}\right) \xrightarrow[\substack{R_2 \to R_2 - 7R_1 \\ R_3 \to R_3 + 3R_1}]{R_1 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -5 & -8 & -9 \\ 0 & 2 & 3 & 6 \end{array}\right) \xrightarrow[\substack{R_2 \to \frac{1}{2}R_2 \\ R_5 \to R_3 + 5R_2}]{R_3 \leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & \frac{3}{2} & 3 \\ 0 & 0 & -\frac{1}{2} & 6 \end{array}\right)$$

$\Rightarrow$ unique soln

$z = -12$
$y = 3 - \frac{3}{2} z = 21$
$x = 2 - z - y = -7$

or compute $\det \neq 0$
$\Rightarrow$ unique soln.

7. *[4 marks]*
Let

$$F = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then $F^{-1} =$

A. $\begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

B. $\begin{bmatrix} 1 & 0 & 1/2 \\ 1/2 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

C. $\begin{bmatrix} 0 & 1/4 & -1/4 \\ 1/2 & 0 & 1 \\ 1 & 0 & 1/2 \end{bmatrix}$

D. $\begin{bmatrix} 0 & 1/2 & 1/2 \\ 0 & 0 & 1 \\ 1/2 & -1/4 & -1/4 \end{bmatrix}$

E. $\begin{bmatrix} 1/2 & 0 & 1 \\ 0 & 0 & 1 \\ 1/2 & -1/4 & -1 \end{bmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & -1 & -4 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 4 & 2 & -1 & 0 \\ 0 & 0 & -4 & -2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{array}\right)$$

$$\boxed{F^{-1} = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}}$$

8. *[4 marks]*
In the solution to the following system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 3 \\ 2x_1 - x_2 &= 1 \\ -x_1 + 2x_3 &= 0 \end{aligned}$$

the value of x_3 is

A. $\frac{-4}{7}$

B. $\frac{4}{5}$

C. $\frac{-2}{7}$

D. $\frac{4}{7}$

E. -4

e.g.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 2 & -1 & 0 & 1 \\ -1 & 0 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -3 & -2 & -5 \\ 0 & 1 & 3 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 7 & 4 \end{array}\right)$$

$7x_3 = 4$

$\boxed{x_3 = \frac{4}{7}}$

9. *[4 marks]*
Let

$$H = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & -1 & 5 \\ 0 & 0 & 7 \end{bmatrix}.$$

Then the determinant of H is

A. 0

B. 28

C. -5

D. 7

E. -14

$$|H| = \begin{vmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & -2 & 0 \end{vmatrix} \begin{vmatrix} 1 & 0 & \frac{1}{2} \\ 0 & -1 & 5 \\ 0 & 0 & 7 \end{vmatrix}$$

expanding the first det by 3rd col
and the 2nd det by 1st col

$$|H| = \begin{vmatrix} 2 & -1 \\ 0 & -2 \end{vmatrix} \begin{vmatrix} -1 & 5 \\ 0 & 7 \end{vmatrix}$$

$$= (-4)(-7) = \boxed{28}$$

10. *[4 marks]*
If A is a 3×3 matrix such that $|A| \neq 0$, then which of the following statements is false?

A. $|-A| = -|A|$ $\quad |-A| = (-1)^3 |A| = -|A|$ so true

B. $|A^T| = |A|$ $\quad$ true

C. $|2A| = 8|A|$ $\quad |2A| = 2^3 |A| = 8|A|$ so true

D. $|A^{-1}| = \dfrac{1}{|A|}$ $\quad$ true

E. There is a 3×1 non-zero matrix X such that $AX = 0$.

If $|A| \neq 0$, A is invertible; but then

$$X = A^{-1}AX = A^{-1}(0) = 0$$

$\boxed{X \text{ cannot be non-zero}}$

PART B. Written-Answer Questions

1. A person wishes to save \$50,000 by making equal deposits at the end of each month for 12 years. Interest is 6.6% per year compounded monthly.

[5] (a) What should be the amount of each deposit?

[10] (b) Immediately after the 84^{th} deposit, the interest rate changes to 5.4% compounded monthly. Accordingly, the remaining deposits (which are to be monthly and equal to each other) must be adjusted to still achieve the goal of \$50,000 in a total of 12 years. What should be the amount of each of the remaining deposits?

$i = \frac{.066}{12} = .0055$ at the beginning

a) $50{,}000 = R\, s_{\overline{144}|.0055}$

$$R = \frac{50{,}000}{s_{\overline{144}|.0055}} = \frac{50{,}000 \times .0055}{(1.0055)^{144} - 1} \approx \boxed{\$228.59}$$

b) Immediately afte the 84th deposit he has

$$R\, s_{\overline{84}|.0055} \approx 228.59 \frac{[(1.0055)^{84} - 1]}{.0055}$$

$$\approx \$\,24{,}323.53$$

which will accumulate to

$R\, s_{\overline{84}|.0055} (1.0045)^{60} \approx \$31{,}843.66$

by the end of 12 years at the new inteest rate,

leaving him \$18,156.34 short of his goal.

Let X be the size of the remaining deposit.

$$X\, s_{\overline{60}|.0045} = 18{,}156.34$$

$$X = \frac{18{,}156.34 \times .0045}{(1.0045)^{60} - 1}$$

$$\approx \boxed{\$\,264.27}$$

There are other ways to get to this answer.

2. An \$80,000 mortgage is to be repaid by equal payment made at the end of each month for 10 years. Interest is 7% compounded semi-annually.

[5] (a) What is the amount of each payment?

[5] (b) How much principal is outstanding at the beginning of the 9^{th} year of the repayment schedule?

[5] (c) How much interest is included in the 97^{th} mortgage payment?

$(1.035)^2 = (1+i)^{12}$

a) $80,000 = R\, a_{\overline{120}|i}$

$R = \dfrac{80,000\, i}{1-(1+i)^{-120}} = 80,000 \dfrac{\left[(1.035)^{\frac{1}{6}} - 1\right]}{1-(1.035)^{-20}}$

$\boxed{R \approx \$924.75}$

b) At the beginning of the 9th year, 8 full years of, i.e., 96 monthly, payments have been made. 24 remain

Principal outstanding $= R\, a_{\overline{24}|i} = R\dfrac{\left[1-(1+i)^{-24}\right]}{i}$

$= R\dfrac{\left[1-(1.035)^{-4}\right]}{(1.035)^{\frac{1}{6}}-1} \approx \boxed{\$20,675.30}$

c) The interest in the 97 th payment is $i \times$ principal outstanding between the 96th and 97 th payments

$\approx \left[(1.035)^{\frac{1}{6}} - 1\right] \times 20,675.30$

$\approx \boxed{\$118.88}$

[13] 3. A rental office has two types of apartments for rent. To prepare an apartment of type A, 2 labor hours of painting and 3 labor hours of cleaning are needed. For an apartment of type B, 4 labor hours of painting and 3 labor hours of cleaning are needed. There are 20 labor hours of painting and 18 labor hours of cleaning available. Use inverse matrices to solve a system of equations that determines how many apartments of each type can be prepared if all the available labor hours are to be utilized.

Let x_A = no. of apts. of type A
x_B = " " " " B

painting $2x_A + 4x_B = 20$
cleaning $3x_A + 3x_B = 18$

$$\begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} 20 \\ 18 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & 4 \\ 3 & 3 \end{pmatrix} \qquad A^{-1} = \frac{1}{6-12}\begin{pmatrix} 3 & -4 \\ -3 & 2 \end{pmatrix}$$

$$\boxed{A^{-1} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix}}$$

$$\begin{pmatrix} x_A \\ x_B \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & -\frac{1}{3} \end{pmatrix}\begin{pmatrix} 20 \\ 18 \end{pmatrix} = \begin{pmatrix} -10+12 \\ 10-6 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$$

$$\boxed{\begin{matrix} x_A = 2 \\ x_B = 4 \end{matrix}}$$

4. Let $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & a & 1 & 2 \\ a & 0 & -1 & 3 \\ 1 & -a & 0 & 1 \end{pmatrix}$

[5] (a) Find the determinant of A in terms of a. [Suggestion: Use some row and/or column reduction]

[3] (b) For what value(s) of a will A not have an inverse?

[3] (c) If $AX = B$ where

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix},$$

then find x_3 when A has an inverse. [Suggestion: Use Cramer's Rule]

[6] (d) For each value of a for which A does not have an inverse, find all possible values of x_3. Why can't you use Cramer's Rule?

a) Many different ways, e.g.:

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & a & 1 & 2 \\ a & 0 & -1 & 3 \\ 1 & -a & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & a & 1 & 2 \\ 0 & 0 & a-1 & 3 \\ 0 & -a & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & 1 & 2 \\ 0 & a-1 & 3 \\ -a & 1 & 1 \end{vmatrix}$$ after expanding by 1st col.

$$= \begin{vmatrix} a & 1 & 2 \\ 0 & a-1 & 3 \\ 0 & 2 & 3 \end{vmatrix} = a \begin{vmatrix} a-1 & 3 \\ 2 & 3 \end{vmatrix}$$ after expanding by 1st col

$$= a[3(a-1)-6] = \boxed{3a(a-3)}$$

b) A has no inverse when $\det A = 0$, i.e., $\boxed{a=0 \text{ or } a=3}$

c) for $a \neq 0, 3$

$$x_3 = \frac{\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & a & 1 & 2 \\ a & 0 & -1 & 3 \\ 1 & -a & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & a & 1 & 2 \\ a & 0 & -1 & 3 \\ 1 & -a & 0 & 1 \end{vmatrix}} = \boxed{1}$$ since numerator and denominator are the same.

d) Two cases: i) $a=0$ ii) $a=3$

Cramer's Rule cannot be used because in both these cases the denominator in Cramer's Rule is 0.

(Extra page if necessary)

d i) $a=0$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 3 & -1 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_4 \to R_4 - R_1} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 3 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow[R_4 \to R_4 - R_2]{R_3 \to R_3 + R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{array}\right)$$

so. $x_4 = 0$

$x_3 = 1 - 2x_4 = 1$

Hence $\boxed{x_3 = 1 \text{ only}}$

$x_1 = -1 + x_3 = 0$

$x_2 =$ anything

ii) $a = 3$

$$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 3 & 1 & 2 & 1 \\ 3 & 0 & -1 & 3 & -1 \\ 1 & -3 & 0 & 1 & 0 \end{array}\right) \xrightarrow[R_4 \to R_4 - R_1]{R_3 \to R_3 - 3R_1} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & -3 & 1 & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_4 \to R_4 + R_2} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 2 & 3 & 2 \end{array}\right) \xrightarrow{R_4 \to R_4 - R_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 3 & 1 & 2 & 1 \\ 0 & 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_2 \to \frac{1}{3}R_2]{R_3 \to \frac{1}{2}R_3} \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & \frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & 1 & \frac{3}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$\boxed{x_3 = 1 - \frac{3}{2}x_4}$

$x_2 = \frac{1}{3} - \frac{2}{3}x_4 - \frac{1}{3}x_3$

$= \frac{1}{3} - \frac{2}{3}x_4 - \frac{1}{3}\left(1 - \frac{3}{2}x_4\right)$

$= -\frac{1}{6}x_4$

$x_1 = -1 + x_3 = -1 + \left(1 - \frac{3}{2}x_4\right)$

$= \frac{3}{2}x_4$

$x_1 = -\frac{3}{2}x_4$

$x_2 = -\frac{1}{6}x_4$

$x_3 = 1 - \frac{3}{2}x_4$ but x_4 can be anything so x_3 can be any number in this case

Including $x_3 = 1$ when $x_4 = 0$.