Solved

Department of Mathematics
University of Toronto

WEDNESDAY, OCTOBER 31, 2001, 6:10 - 8:00 PM
MAT 133Y TERM TEST #1

Calculus and Linear Algebra for Commerce
Duration: 1 hour 50 minutes

Aids Allowed: A non-graphing calculator, with empty memory, to be supplied by student.

Instructions: Fill in the information on this page, and make sure your test booklet contains 10 pages. In addition, you should have a **multiple-choice answer sheet**, on which you should fill in your name, number, tutorial time, tutorial room, and tutor's name.

This test consists of 10 multiple choice questions, and 4 written-answer questions.
For the **multiple choice questions** you can do your rough work in the test booklet, but you must record your answer by circling the appropriate letter **on the answer sheet** with your pencil. Each correct answer is worth 4 marks; a question left blank, or an incorrect answer, or two answers for the same question is worth 0. For the **written-answer questions**, present your solutions in the space provided. The value of each written-answer question is indicated beside it. ENCLOSE YOUR FINAL ANSWER IN A BOX AND WRITE IT IN INK.

TOTAL MARKS: 100

FAMILY NAME: ______________________

GIVEN NAME: ______________________

STUDENT NO: ______________________

SIGNATURE: ______________________

TUTORIAL TIME: ______________________

TUTORIAL ROOM: ______________________

T.A.'S NAME: ______________________

Regcode	Timecode	Room	Regcode	Timecode	Room
T0101A	M9A	SS2111	T0501C	W3C	NF 4
T0101B	M9B	SS2128	T0501D	W3D	CR 103
T0101C	M9C	SS2130	T0601A	R4A	UC 52
T0201A	M3A	LA 341	T0601B	R4B	UC 85
T0201B	M3B	UC 163	T0601C	R4C	UC 328
T0201C	M3C	SS2130	T0701A	F2A	UC 87
T0201D	M3D	LA 240	T0701B	F2B	WI 523
T0301A	T3A	SS2128	T0701C	F2C	SS1086
T0301B	T3B	SS1069	T0801A	F3A	SS2130
T0301C	T3C	SS2106	T0801B	F3B	SS2111
P0301D	T3D	VC 206	T5101A	R5A	UC 52
T0401A	W9A	SS1074	T5101B	R5B	UC 85
T0401B	W9B	SS2111	T5101C	R5C	UC 244
T0401C	W9C	LM 123	T5201A	R6A	UC 144
T0501A	W3A	TF 201	T5201B	R6B	UC 244
T0501B	W3B	TF 200			

FOR MARKER ONLY	
Multiple Choice	
B1	
B2	
B3	
B4	
TOTAL	

PART A. Multiple Choice

1. *[4 marks]*
If interest is at the nominal rate of 12% compounded every 4 months then the effective quarterly rate is closest to

A. 2.874%

B. 4%

C. 2.985%

D. 3%

E. 2.242%

$$(1+i)^4 = \left(1 + \frac{.12}{3}\right)^3$$

$$i = (1.04)^{3/4} - 1 \approx .02985$$

2. *[4 marks]*
If $1,200 is invested at the effective annual rate of 5% for 5 years and then at the nominal rate of 5% compounded quarterly for 5 more years, then in 10 years the investment will be worth

A. $1,963.49

B. $1,629.68

C. $1,972.34

D. $1,954.67

E. $9,266.08

$$1200(1.05)^5\left(1+\frac{.05}{4}\right)^{20}$$

$$\approx 1963.49$$

3. *[4 marks]*
A \$10,000 loan is amortized by equal semi-annual payments over 5 years (the first payment due in 6 months). If interest is charged at 8% compounded semi-annually, then the principal repaid in the first payment is:

A. \$762.47

B. \$795.38

C. \$806.21

D. \$832.91

E. \$853.64

$$10{,}000 = R\,a_{\overline{10}|.04}$$

$$R = \frac{10{,}000 \times .04}{1-(1.04)^{-10}} \approx 1232.91$$

Interest in 1st period $= .04 \times 10{,}000$
$= \$400.00$

Principal repaid $= 1232.91 - 400$
$= \$\,832.91$

4. *[4 marks]*
If a \$300,000 mortgage amortized over 25 years at 6% compounded semi-annually has monthly payments, then each payment is closest to

A. \$2,500.00

B. \$1,919.42

C. \$2,209.71

D. \$1,932.90

E. \$2,216.45

$$(1.03)^2 = (1+i)^{12}$$

$$300{,}000 = R\,a_{\overline{300}|i}$$

$$R = \frac{300{,}000\, i}{1-(1+i)^{-300}}$$

$$= \frac{300{,}000\left[(1.03)^{\frac{1}{6}}-1\right]}{1-(1.03)^{-50}}$$

$$\approx \$1919.42$$

5. *[4 marks]*
The price of a bond with semi-annual coupons and exactly 18 years remaining to maturity, with a coupon rate of 8.2% and a yield to maturity of 4.6%, is closest to

A. $100.00

B. $119.10

C. $118.73

D. $143.43

E. $143.74

$$P = 100(1.023)^{-36} + 4.1\, a_{\overline{36}|.023}$$

$$\approx \$143.74$$

6. *[4 marks]*
Just after the 120th payment, the principal outstanding on a $100,000 mortgage amortized over 20 years at 7.2% compounded semi-annually with monthly payments is closest to

A. $50,000

B. $33,019

C. $66,981

D. $64,583

E. $68,762

$$100{,}000 = R\, a_{\overline{240}|i} \qquad R = \frac{100{,}000}{a_{\overline{240}|i}}$$

$$P.O = R\, a_{\overline{120}|i}$$

$$= 100{,}000 \frac{a_{\overline{120}|i}}{a_{\overline{240}|i}}$$

$$= 100{,}000 \frac{[1-(1+i)^{-120}]}{[1-(1+i)^{-240}]}$$

$$(1+i)^{12} = (1.036)^2$$
$$(1+i)^{120} = (1.036)^{20}$$

$$= \frac{100{,}000}{1+(1+i)^{-120}}$$

$$= \frac{100{,}000}{1+(1.036)^{-20}}$$

$$\approx 66{,}981.38$$

7. *[4 marks]*

Let $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{bmatrix}$, and suppose $C = ABB^T$. Find the entry in the second row and the first column of C, if C is defined.

A. 0

B. C is not defined

C. -1

D. 2

E. -2

$$C = \begin{pmatrix} -2 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & \\ 3 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} & & \\ 2 & & \end{pmatrix}$$

8. *[4 marks]*

Find the value of k, if any, for which the homogeneous system

$$\begin{aligned} 2x_1 - kx_2 &= 0 \\ -x_1 + x_2 &= 0 \\ 3x_1 - (k+1)x_2 &= 0 \end{aligned}$$

has solutions other than $x_1 = 0, x_2 = 0, x_3 = 0$.

A. for all k

B. there is no such value of k

C. 0

D. 1

E. 2

$$\left(\begin{array}{cc|c} 2 & -k & 0 \\ -1 & 1 & 0 \\ 3 & -(k+1) & 0 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2-k & 0 \\ 0 & 2-k & 0 \end{array}\right) \quad \text{so } k = 2$$

9. *[4 marks]*

The determinant of $\begin{bmatrix} 5 & -9 & -2 \\ 7 & 3 & 1 \\ 16 & 11 & 3 \end{bmatrix}$ is

A. -23

B. -37

C. 11

D. 29

E. -7

$$\begin{vmatrix} 19 & -3 & 0 \\ 7 & 3 & 1 \\ -5 & 2 & 0 \end{vmatrix} = (-1)^{2+3} \begin{vmatrix} 19 & -3 \\ -5 & 2 \end{vmatrix} = -(38-15)$$
$$= -23$$

10. *[4 marks]*
Let c be a real number. Cramer's rule can be used to solve the system

$$4cx + y = 5$$
$$cx + c^2 y = 9$$

if and only if

A. $c \neq 0$

B. $c \neq \frac{1}{2}$ and $c \neq -\frac{1}{2}$

C. $c \neq 2$ and $c \neq -2$

D. $c \neq 0$ and $c \neq 4$

E. $c \neq 0, c \neq \frac{1}{2}$, and $c \neq -\frac{1}{2}$

$$0 \neq \begin{vmatrix} 4c & 1 \\ c & c^2 \end{vmatrix} = 4c^3 - c$$
$$= c(4c^2 - 1)$$

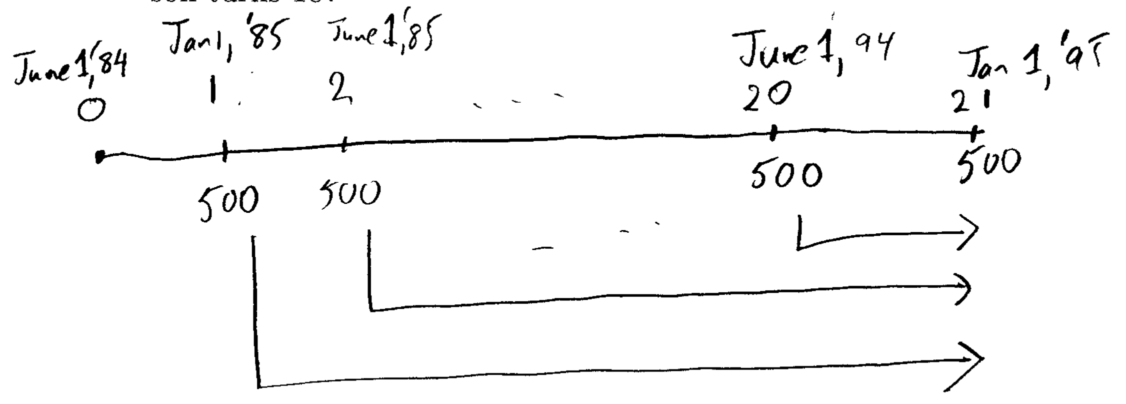
So $c \neq 0$ and $c \neq \pm\frac{1}{2}$

PART B. Written-Answer Questions

1. *[13 marks]*

   On January 1, 1985, at the time of his son's birth, a father deposited \$500 into a savings account paying an effective annual rate of 10% and continued to make similar deposits every 6 months thereafter. After January 1, 1995, the account earned 6% compounded semi-annually.

   How much will there be in the account just before the deposit on January 1, 2003 when his son turns 18?

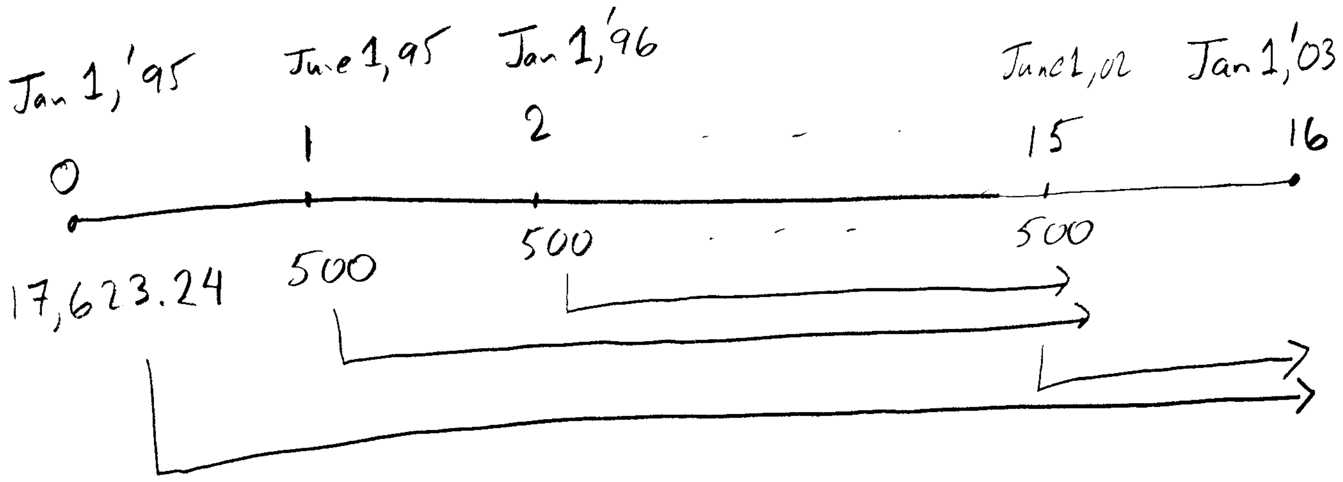


On Jan 1, '95, there was $500\, s_{\overline{21}|i}$ in the acct.

where $(1+i)^2 = 1.10$ $\quad 1+i = 1.10^{\frac{1}{2}}$

$$= 500\,\frac{\left[(1+i)^{21}-1\right]}{i}$$

$$= 500\,\frac{\left[(1.10)^{\frac{21}{2}}-1\right]}{(1.10)^{\frac{1}{2}}-1} \approx \$17,623.24$$



$$S = 17,623.24(1.03)^{16} + 500\, s_{\overline{15}|.03}(1.03)$$

$$\left(\text{or } + 500\, s_{\overline{16}|.03} - 500\right)$$

$$= \boxed{\$37,858.57}$$

2. (a) *[9 marks]*

Just after January 1, 2001 the amount of money needed to buy all the outstanding bonds of XYZ Inc. maturing on June 1, 2010 was $53,000,000. The annual coupon rate was 7%; the coupons were semi-annual; the yield to maturity was 5.4%. Assuming that on the date of issue, whenever that was, the yield to maturity rate was 7%, what was the face value of all these bonds? [To the nearest $100]

(b) *[8 marks]*

Exactly one year later, just after January 1, 2002, the total value of these bonds falls to $45,000,000. What is the yield to maturity then? [You may stop when the total price is within $1,000,000 of the actual price.]

a) $n = 19 \quad r = .035 \quad i = .027 \quad P = 53{,}000{,}000$

$V = ?$

$$53{,}000{,}000 = V(1.027)^{-19} + .035 V a_{\overline{19}|.027}$$

$$V = \frac{53{,}000{,}000}{(1.027)^{-19} + .035\, a_{\overline{19}|.027}}$$

$\approx 47{,}419{,}074$

$V = \boxed{\$47{,}419{,}100 \text{ to the nearest } \$100}$

b) $n = 17 \quad r = .035 \quad P = 45{,}000{,}000$

$V = 47{,}419{,}100$

$P < V \Rightarrow i > r$. Try $i = .04$ say

$$P = 47{,}419{,}100(1.04)^{-17} + .035 \times 47{,}419{,}100\, a_{\overline{17}|.04}$$

$P \approx \$44{,}534{,}675$ already good enough (within \$1M of \$45M)

so $i = .04$

Yield to maturity $\approx \boxed{8\%}$. good enough (actually 7.8% is closer, but not necessary)

3. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 0 & x \\ 3 & 2 & y \end{bmatrix}$ where x, y are arbitrary numbers.

[5] (a) Find x and y such that the matrix A is symmetric (i.e. $A^T = A$) and invertible.

[5] (b) Find the inverse of A in case $x = 1$, $y = 4$.

[5] (c) Use the result of part (b) to find the matrix X from the matrix equation $XA = A^T$ in case $x = 1$, $y = 4$.

a) $A^T = A$ if $x = 2$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 3 & 2 & y & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -1 & -1 & 1 & 0 \\ 0 & -1 & y-9 & -3 & 0 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 \to R_3 + R_2]{R_2 \to -R_2} \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & y-8 & -2 & -1 & 0 \end{array}\right)$$

A invertible as long as $y - 8 \neq 0$

so $\boxed{x = 2 \text{ and } y \neq 8}$

or: $\det \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & y \end{vmatrix} = -\begin{vmatrix} 1 & 2 \\ 3 & y \end{vmatrix} - 2\begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}$

$= -(y-6) - 2(2-3) = -y + 8$

A invertible iff $\det A \neq 0$, so $-y + 8 \neq 0$; i.e. $y \neq 8$

so $\boxed{x = 2 \text{ and } y \neq 8}$

b) $$\left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 3 & 1 & 0 & 0 \\ 0 & -1 & -2 & -1 & 1 & 0 \\ 0 & -1 & -5 & -3 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & -1 & 0 \\ 0 & 0 & -3 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{3} & -\frac{5}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \end{array}\right)$$

$$\boxed{A^{-1} = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix}}$$

c) $XA = A^T \Rightarrow X = A^T A^{-1} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 1 & 4 \end{pmatrix}\begin{pmatrix} -2 & 2 & 1 \\ -1 & -5 & 2 \\ 2 & 1 & -1 \end{pmatrix}$

$$\boxed{= \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 \\ 2 & 4 & -1 \\ 1 & 5 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{3} & \frac{4}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{5}{3} & \frac{1}{3} \end{pmatrix}}$$

4. Woody Tobias, Jr. has a 1 litre Ehrlenmeyer flask filled with a mixture of three liquids (red, green, and blue) which he has made. The red liquid weighs 1 kg per litre, the green weighs 0.8 kg per litre, and the blue weighs 1.5 kg per litre. Moreover, the flask contains exactly twice as much blue liquid as red, by volume.

[8] (a) Set up a system of linear equations which will determine the volume of each liquid Woody used to make the mixture, if the total weight of liquid in the flask is 1.3 kg.

[7] (b) Use Cramer's rule on your system from part (a) to find the volume of red liquid used to make the mixture.

Note: No marks will be assigned for a different method of solution.

a) Let R, G and B be the no. of litres of red, green and blue liquids respectively.

$$\left.\begin{aligned} R + G + B &= 1 \\ R + .8G + 1.5B &= 1.3 \\ 2R \qquad - B &= 0 \end{aligned}\right\} \quad \text{or} \quad \left.\begin{aligned} R + G + B &= 1 \\ R + .8G + 1.5B &= 1.3 \\ B &= 2R \end{aligned}\right\}$$

b)
$$R = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 1.3 & .8 & 1.5 \\ 0 & 0 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 1 & .8 & 1.5 \\ 2 & 0 & -1 \end{vmatrix}} = \frac{-\begin{vmatrix} 1 & 1 \\ 1.3 & .8 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ 0 & -.2 & .5 \\ 0 & -2 & -3 \end{vmatrix}} = \frac{-(.8 - 1.3)}{\begin{vmatrix} -.2 & .5 \\ -2 & -3 \end{vmatrix}}$$

$$R = \frac{.5}{.6+1} = \frac{.5}{1.6} = \boxed{\frac{5}{16} \text{ litres}} = \boxed{.3125}$$