University of Toronto
Faculty of Arts and Science

MAT136H1Y Practice Midterm Solution

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Duration - 110 minutes
No Aids Permitted

Surname: ______________________________

Given Name: ______________________________

Student Number: ______________________________

This exam contains 9 pages (including this cover page) and 5 problems (one problem is bonus). Once the exam begins, check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include but are not limited to textbooks, notes, calculators, cellphones, or any electronic device.

Unless otherwise indicated, you are required to **show your work** on each problem on this exam. The following rules apply:

- **Total points available is 110; but the test is out of 100.** The last problem is a bonus problem.
- **Unreadable answers will be receive no mark.**
- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work, will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	40	
2	15	
3	15	
4	15	
5	15	
Total:	100	

Part 1: Short Answers (50 marks)

No justification is necessary and no mark will be awarded for them

1. For each of the following questions, write your final answer in the box on the righthand side. Only your final answer will be graded.

(a) (5 points) True of False? Suppose that f is integrable, then f is differentiable.
Solution: Consider the function $f(x) = |x|$. It is integrable since it is continuous. But it is not differentiable at $x = 0$ since it has a cusps.

Final Answer

False

(b) (5 points) Compute

$$\int_0^1 e^x + \sin(x) + x^5 \, dx$$

Solution:

$$\int_0^1 e^x + \sin(x) + x^5 \, dx = e^x - \cos(x) + \frac{1}{6}x^6 \Big|_0^1 = e - \cos(1) + \frac{1}{6}$$

Final Answer

$e - \cos(1) + \frac{1}{6}$

(c) (5 points) Compute $\int_0^1 \frac{d}{dt}(e^{-t^2}\ln(1+t))\, dt$
Soluion: By FTC, we see that this is $e^{-t^2}\ln(1+t)\Big|_0^1 = e^{-1}\ln(2)$

Final Answer

$e^{-1}\ln(2)$

(d) (5 points) Compute

$$\frac{d}{dx}\int_0^{x^2} \frac{t+1}{t^2+1}\, dt$$

.

Solution: Let $F(u) = \int_0^u \frac{t+1}{t^2+1}\, dt$. The question asks us to compute $\frac{d}{dx}F(x^2)$. By the chain rule, this is $F'(x^2)(2x)$. The FTC, therefore, tells us that

$$F'(x^2)(2x) = \frac{x^2+1}{(x^2)^2+1} \cdot (2x) = \frac{2x(x^2+1)}{x^4+1}$$

Final Answer

$\frac{2x(x^2+1)}{x^4+1}$

For the following problems, evaluate each definite *integral.*

(e) (5 points)

$$\int_1^e \frac{\ln(x)}{x}\,dx$$

Soluion: We note that $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$. So we perform the substitution $u = \ln(x)$. We see that $du = \frac{1}{x}dx$. It follows that

$$\int_1^e \frac{\ln(x)}{x}\,dx = \int_0^1 u du = \frac{1}{2}u^2\bigg|_0^1 = \frac{1}{2}$$

Final Answer

$\boxed{\frac{1}{2}}$

(f) (5 points)

$$\int_0^\pi \sin^3(x)\,dx$$

Soluion: We note that $\sin^2(x) = 1 - \cos^2(x)$. So we write our integral as

$$\int_0^\pi \sin^3(x)\,dx = \int_0^\pi (1-\cos^2(x))\sin(x)dx$$

Now, we perform a substitution $u = \cos(x)$ with $du = -\sin(x)dx$. It follows that

$$\int_0^\pi (1-\cos^2(x))\sin(x)dx = -\int_1^{-1}(1-u^2)du = u - \frac{1}{3}u^3\bigg|_{-1}^1 = \frac{4}{3}$$

Final Answer

$\boxed{\frac{4}{3}}$

(g) (5 points)

$$\int \frac{1}{\csc(2x)\sec(5x)}\,dx$$

Solution: We note that $\frac{1}{\csc(2x)\sec(5x)} = \sin(2x)\cos(5x)$. Hence, using the product formula we see that

$$\sin(2x)\cos(5x) = \frac{1}{2}(\sin(2x+5x)+\sin(2x-5x)) = \frac{1}{2}(\sin(7x)-\sin(3x))$$

We can easily integrate this to obtain $\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x)$

Final Answer

$\boxed{\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C}$

(h) (5 points)

$$\int \frac{x}{\sqrt{1+x^2}}\,dx$$

Solution: The radical $\sqrt{1+x^2}$ prompts us to use the substitution $x = \tan(\theta)$. Moreover, $dx = \sec^2(\theta)d\theta$. It follows that

$$\int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{\tan(\theta)}{\sec(\theta)}\sec^2(\theta)d\theta = \int \tan(\theta)\sec(\theta) = \sec(\theta)$$

Since $\tan(\theta) = x$, we have that $\cos(\theta) = \frac{1}{\sqrt{1+x^2}}$. Hence $\sec(\theta) = \sqrt{1+x^2}$.

Final Answer

$$\boxed{\sqrt{1+x^2} + C}$$

Part 1: Long Answers (50 marks)

Show your work for full marks

2. (15 points)

$$\int \frac{x^4 + 9x^2 + x + 2}{x^2 + 9}\, dx$$

Solution: We note that the numerator has higher degree than the denominator. So we perform long division first.

$$\begin{array}{rcl} & & x^2 \\ \hline x^2 + 9 &) & x^4 + 9x^2 + x + 2 \\ & & x^4 + 9x^2 \\ \hline & & x + 2 \end{array}$$

Hence, we obtain

$$\frac{x^4 + 9x^2 + x + 2}{x^2 + 9} = x^2 + \frac{x+2}{x^2+9}.$$

We find first that

$$\int x^2\, dx = \frac{1}{3}x^3.$$

Likewise, splitting the numerator, we see that

$$\int \frac{x+2}{x^2+9}\, dx = \int \frac{x}{x^2+9} dx + \int \frac{2}{x^2+9}\, dx.$$

We use substitution $u = x^2$. We get

$$\int \frac{x}{x^2+9}\, dx = \int \frac{1}{2}\frac{1}{u+9}\, du = \frac{1}{2}\ln(u+9) = \frac{1}{2}\ln(x^2+9)$$

Using $\int \frac{1}{1+x^2}\, dx = \arctan(x)$, we see that

$$\int \frac{2}{x^2+9}\, dx = \frac{2}{3}\arctan(x/3).$$

Combining all the terms, we arrive at

$$\int \frac{x^4 + 9x^2 + x + 2}{x^2 + 9}\, dx = \frac{1}{3}x^3 + \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan(x/3) + C.$$

3. Compute the following integrals.

(a) (5 points)

$$\int \csc(x)\,dx$$

Solution: We multiply and divide by $(\csc(x)+\cot(x))$ to get

$$\int \csc(x)\,dx = \int \frac{\csc^2(x)+\csc\cot(x)}{\csc(x)+\cot(x)}\,dx$$

where we note that the numerator is the negative of the derivative of the denominator. Hence, we perform the substitution $u=\csc(x)+\cot(x)$ with $du=-(\csc^2(x)+\csc\cot(x))dx$. Our integral becomes

$$\int \csc(x)\,dx = \int -\frac{1}{u}du = -\ln(u)+C = -\ln(\csc(x)+\cot(x))+C$$

(b) (5 points)

$$\int \csc^3(x)\,dx$$

Soluion: We observe that $\csc^3(x)=\csc^2(x)\csc(x)$, where we know how to integrate the first factor. So, we use integration by parts with

$$\begin{array}{ll} u=\csc(x) & dv=\csc^2(x)dx \\ du=-\cot(x)\csc(x)dx & v=-\cot(x). \end{array}$$

This gives us

$$\int \csc^3(x)\,dx = -\csc(x)\cot(x) - \int \cot^2(x)\csc(x)dx.$$

Now, we attempt to recover the original integral, with an opposite sign. To this end, we use the identity $1+\cot^2(x)=\csc^2(x)$. We get

$$\int \csc^3(x)\,dx = -\csc(x)\cot(x) - \int \cot^3(x)dx + \int \csc(x)dx = -\csc(x)\cot(x) - \ln(\csc(x)+\cot(x)) - \int \cot^3(x)dx$$

It follows that, after solve for $\int \csc^3(x)\,dx$ as an unknow in the above linear equation, we obtain

$$\int \csc^3(x)\,dx = -\frac{1}{2}\left(\csc(x)\cot(x)+\ln(\csc(x)+\cot(x)\right)+C$$

(c) (5 points) Use parts (a) and (b) to compute the following integral.

$$\int \cot^2(x)\csc(x)\,dx$$

Solution: We recall from part (b), after integration by parts, that we have obtained

$$\int \csc^3(x)\,dx = -\csc(x)\cot(x) - \int \cot^2(x)\csc(x)dx.$$

It follows that

$$\int \cot^2(x)\csc(x)dx = -\int \csc^3(x)\,dx - \csc(x)\cot(x).$$

Using the result from part (b) again, we obtain

$$\int \cot^2(x)\csc(x)dx = \frac{1}{2}\left(\csc(x)\cot(x)+\ln(\csc(x)+\cot(x)\right) - \csc(x)\cot(x)+C$$

4. (15 points)

$$\int (x+1)^2 e^x \, dx$$

Solution: We may write the integrand as $x^2e^x + 2xe^x + e^x$. This prompts us to use reduction of power (via integration by parts). The simplest term is

$$\int e^x dx = e^x.$$

The next simple term is $2\int xe^x dx$. We use

$$\begin{array}{ll} u = x & dv = e^x dx \\ du = dx & v = e^x \end{array}$$

to obtain

$$2\int xe^x dx = 2xe^x - 2\int e^x dx = 2xe^x - 2e^x.$$

Likewise, for the term $\int x^2 e^x \, dx$, we use

$$\begin{array}{ll} u = x^2 & dv = e^x dx \\ du = 2xdx & v = e^x \end{array}$$

to get

$$\int x^2 e^x dx = x^2 e^x - 2\int xe^x dx = x^2e^x - 2xe^x + 2e^x.$$

Combining all three terms, we get

$$\int (x+1)^2 e^x \, dx = x^2e^x - 2xe^x + 2e^x + 2xe^x - 2e^x + e^x = x^2e^x + e^x$$

5. (15 points) Find the volume of the solid given by the region enclosed by the curve

$$x^2 + (y-1)^2 = 1$$

and rotated about the y-axis.

Solution: We note that the region describe is a circle with center at $(0, 1)$ and radius 1. Hence, after rotating about the y-axis, we obtain a ball of radius 1. Standard formula from high school tells us its volume is $\frac{4}{3}\pi$. However, we can compute this using an integral. Namely, we use the disc method. We note that $x = \sqrt{1-(y-1)^2}$ for $y = 0$ to $y = 2$. Hence the volume is given by

$$\pi \int_0^2 1-(y-1)^2 dy = \pi \left. (y - \frac{1}{3}(y-1)^3) \right|_0^2 = \pi(2 - \frac{1}{3} - 0 - \frac{1}{3}) = \frac{4}{3}\pi$$

This page is for additional work and will not be marked.