MAT 136 Calculus I Lecture Notes

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Contents

1 How to Use These Notes

First let me state the two most important questions we want to answer in this class

How to integrate a function $f(x)$

The majority of the contents in this note is geared to answer this questions.

In these notes, I will explicitly indicate which are the important materials. Although one should be comfortable with all the materials in this note, but certain critical concept must be emphasized to illuminate the essence of the course. First, definitions are in **bold**. I will use a scale of stars to measure the importance of the material. The importance of the concept decreases as the number of star decreases. Whenever you see a 5-star tag [$\star\star\star\star\star$], this means you must know this inside out. And [$\star$] is just a bit more important than the regular text you see in the notes. Finally, I will *italicize* item that 1) you need to be careful with as they might be confusing, or 2) to leave a comment.

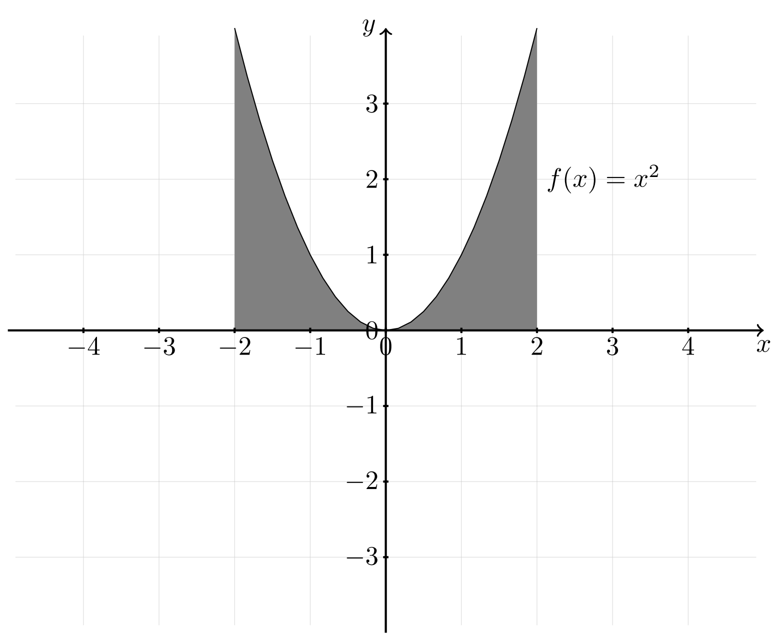
The corresponding section numbers in [Stewart] are listed in bracket in the heading of each section. *And I will NOT write the integration constant $+C$ in the notes to save some ink, but you should always write it!!*

2 Introduction to the Integral

2.1 (5.1) Area and Distances

2.1.1 Area

A mathematical illustrative example of the integral is area under a curve. Let $f(x)$ be a *non-negative* continuous function. We will say the **area under** $y = f(x)$ **from** $x = a$ **to** $x = b$ to be the area bounded between the lines $x = a$, $x = b$, the x-axis, and the graph of $f(x)$.



In this example, the shaded region represents the area under the curve $y = f(x) = x^2$ from $x = -2$ to $x = 2$. In general, to find the area under the curve $y = f(x)$ from $x = a$ to $x = b$, we divide the interval $[a, b]$ into segments

$$[x_0, x_1],\ [x_1, x_2],\ \cdots,\ [x_{n-1}, x_n] \tag{2.1.1}$$

of width Δx. That is,

$$\Delta x = x_i - x_{i-1} = \frac{b-a}{n} \tag{2.1.2}$$

and $x_i = a + i\Delta x$ for all $i = 1, ..., n$. On each interval, the funtion $f(x) \approx f(x_i)$ since the interval is very small so that $f(x)$ is about constant. So that the area under $f(x)$ is approximately $f(x_i)\Delta x$. Hence, we can then approximate the area by

$$f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x. \tag{2.1.3}$$

In fact the choice of **sample point** does not matter. In fact, we can choose any sample point x_i^* in the interval $[x_{i-1}, x_i]$ and compute the sum

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x. \tag{2.1.4}$$

As we choose the interval $[x_{i-1}, x_i]$ to be small, equivalently Δx to be smaller. As we choose the the width Δx to smaller by taking the number of segments $n \to \infty$, the **area under $f(x)$ from $x = a$ to $x = b$ is**

$$A = \lim_{n\to\infty} f(x_1^*)\Delta x + f(x_2^*)\Delta x + \cdots + f(x_n^*)\Delta x. \tag{2.1.5}$$

2.1.2 Displacement/Distance

A physically motivating example for the integral is the displacement traveled by a car with velocity $f(t)$ at time t. Suppose that from time $t = a$ to $t = b$ a car travels at a velocity $f(t)$. If $f(t) = v$ is a constant. Then the displacement traveled in Δt units of time is simply

$$d = v\Delta t = f(t)\Delta t. \tag{2.1.6}$$

Now suppose that $f(t)$ is variable. If it is continuous, then its value varies slightly if t changes by a small amount. Hence we may approximate the distance traveled by dividing the time interval $[a, b]$ into small segments

$$[t_0, t_1],\ [t_1, t_2],\ \cdots,\ [t_{n-1}, t_n] \tag{2.1.7}$$

of width $\Delta t = \frac{b-a}{n}$. On each time segment $[t_{i-1}, t_i]$, the car travels approximately

$$f(t_i^*)\Delta t \tag{2.1.8}$$

units of length, where t_i^* is any sample point in the interval $[t_{i-1}, i_i]$. The total displacement, d, is, therefore, approximately,

$$d \approx f(t_1^*)\Delta t + f(t_2^*)\Delta + \cdots + f(t_n^*)\Delta t. \tag{2.1.9}$$

Taking the segments $[t_{i-1}, t_i]$ smaller for better approximation, we see that

$$d = \lim_{n\to\infty} f(t_1^*)\Delta t + f(t_2^*)\Delta + \cdots + f(t_n^*)\Delta t. \tag{2.1.10}$$

2.1.3 Sigma Notation

We will henceforth denote use the **sigma notation**

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n. \tag{2.1.11}$$

Then the area under the curve $f(x)$ from $x = a$ to $x = b$ is

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \quad [\star\star\star] \tag{2.1.12}$$

and the displacement traveled in the time interval $[a, b]$ by a car of velocity $f(t)$ is

$$d = \lim_{n\to\infty} \sum_{i=1}^{n} f(t_i^*)\Delta t \quad [\star\star\star] \tag{2.1.13}$$

2.2 (5.2) The Definite Integral

2.2.1 Definition of the Definite Integral

Let $f(x)$ be a function defined on $[a, b]$. We partition $[a, b]$ in to n intervals of equal width $\Delta x = \frac{b-a}{n}$:

$$[x_0, x_1],\ [x_1, x_2],\ \cdots,\ [x_{n-1}, x_n] \tag{2.2.1}$$

where $a = x_0$ and $b = x_n$. We define the **integral of $f(x)$ from a to b** to be

$$\boxed{\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x} \tag{2.2.2}$$

provided this limit exists and is independent of the choice of sample points x_i^*. If the limist exists, we say that f **is integrable**.

The following names are given to the parts of the integral

$$\underbrace{\int_a^b}_{\text{integral sign}} \underbrace{f(x)}_{\text{integrand}} \underbrace{dx}_{\text{integrate with respect to } x} \tag{2.2.3}$$

We remark that the symbol dx *only* indicate that we are integrating with respect to x, should there be any more variables appearing in the integrand.

The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x \tag{2.2.4}$$

is called a **Riemann sum**.

2.2.2 Properties of the Sigma Sum

The following list contains properties of the sigma sum. The first three properies are the most important. The rest are useful when we compute integrals explicitly from its definition. (*The first three are important. Do not memorize the last 4 properties, they can be readily searched on Google and will be provided in a formula sheet for any tests if a question demands it*)

1. $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$.
2. $\sum_{i=1}^{n}(a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.
3. If $a_i \leq b_i$ for all i, then $\sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} b_i$.
4. $\sum_{i=1}^{n} 1 = n$.
5. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
6. $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
7. $\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

2.2.3 Properties of the Integral

The first three properties of the sigma sum translates, through the Riemann sum, into properties for the integral. The last two properties listed below does not come from a property listed in the previous subsection.

1. $[\star\star\star]$ $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ (linearity)
2. $[\star\star\star]$ $\int_a^b f(x) + g(x)dx = \int_a^b f(x) + \int_a^b g(x)dx$ (linearity)
3. $\int_a^b 1\, dx = b - a$
4. If $f(x) \leq g(x)$, then $\int_a^b f(x)dx \leq \int_a^b g(x)$
5. $[\star\star]$ If $a \leq b \leq c$, then $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$.
6. $[\star\star]$ $\int_a^b f(x)dx = -\int_b^a f(x)dx$.

2.3 (4.9) Antiderivatives

First, I want to dispel any possible confusion in notation. For our purpose, a real function is a map from an interval I to the real line. I will use *both* f and $f(x)$ to mean such a function. However, confusion may arise when we want to talk about a function evaluated at a point $x = x_0$. In this case the symbol $f(x_0)$ would mean the function f evaluated at the point $x = x_0$. In the former case, the argument x in $f(x)$ is generic, while in the latter case the point x_0 is given and fixed. In what follows, I will use f to denote a function. But the meaning of $f(x)$, either as a function or as the evaluation of f at the point x, depends on the context. If needed, I will specify its meaning.

Given any function $f(x)$ on $[a, b]$, an **antiderivatives** is any function $F(x)$ on $[a, b]$ such that

$$F'(x) = f(x). \tag{2.3.1}$$

If $F(x)$ is an antiderivative of $f(x)$, so is $F(x)+C$ for any constant C and these are all the possible antiderivatives for $f(x)$. For notation, we will use both

$$\frac{d}{dx}F(x), \quad F'(x) \tag{2.3.2}$$

to mean the derivative of $F(x)$.

We remark that the symbol $\frac{d}{dx}F(x)$ *may have ambiguous meaning.* It could mean 1) $\frac{d}{dx}(F(x))$ – the derivative, with respect to x, of $F(x)$ or 2) $\left(\frac{d}{dx}F\right)(x)$ – the evaluation of the derivative of F at x. However, with either interpretation, $\frac{d}{dx}(F(x)) = \left(\frac{d}{dx}F\right)(x)$. So there is in fact no ambiguity. To clarify notation even more, the symbol

$$\left(\frac{d}{dx}F(x)\right)(a) = F'(a) \tag{2.3.3}$$

means the derivative, with respect to x, of F (or equivalently, the function $F(x)$) evaluated at the point $x = a$. However,

$$\left(\frac{d}{dx}F(a)\right)(x) = 0 \tag{2.3.4}$$

In conclusion, the order or taking derivative of a function and evaluating the function at a point matters!

In Figure 1, I have listed for your a table of important derivatives. The left column you should know off by heart [$\star\star\star$].

2.4 (5.3) Fundamental Theorem of Calculus (FTC)

Theorem 2.4.1 (FTC1 [$\star\star\star\star\star$])**.** *Assume that f is continuous on the interval $[a, b]$, then the function*

$$F(x) = \int_a^x f(t)dt, \ a \le x \le b \tag{2.4.1}$$

Function	Particular antiderivative	Function	Particular antiderivative
$cf(x)$	$cF(x)$	$\sin x$	$-\cos x$
$f(x) + g(x)$	$F(x) + G(x)$	$\sec^2 x$	$\tan x$
$x^n \ (n \neq -1)$	$\dfrac{x^{n+1}}{n+1}$	$\sec x \tan x$	$\sec x$
$\dfrac{1}{x}$	$\ln \lvert x \rvert$	$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
e^x	e^x	$\dfrac{1}{1+x^2}$	$\tan^{-1} x$
b^x	$\dfrac{b^x}{\ln b}$	$\cosh x$	$\sinh x$
$\cos x$	$\sin x$	$\sinh x$	$\cosh x$

Figure 1: Table of Antiderivatives [Stewart]

is differentiable on (a,b) *and* $F'(x) = f(x)$. *In another word,* $F(x)$ *is an antiderivative of* $f(x)$.

Theorem 2.4.2 (FTC2 [⋆⋆⋆⋆⋆])**.** *Assume that* f *is continuous on the interval* $[a,b]$. *Let* $F(x)$ *be an antiderivative of* $f(x)$. *Then*

$$\int_a^b f(t)dt = F(b) - F(a). \tag{2.4.2}$$

The way the theorem are written may not be best suited for our purpose. What you should remember, for the purpose of this course and computing integrals, is the following formuli [⋆⋆⋆⋆⋆]

$$\boxed{\frac{d}{dx}\int_a^x f(t)dt = f(x), \qquad \int_a^b F'(t)dt = F(b) - F(a)}. \tag{2.4.3}$$

What you should remember is that integrals and derivatives are almost inverses of each other! We make a final note that the equation on the right in the box is also called the net change theorem.

2.5 (5.4) Indefinite Integrals

The **indefinite integral** of f is an (hence any) antiderivative of f. (*For motivation of why we can't just call it the antiderivative and make extra names, review FTC*). It is

$$\boxed{\int f(x)dx = F(x) + C} \tag{2.5.1}$$

where $F'(x) = f(x)$. *Don't forget your constant C! We also remark that the definite integral $\int_a^b f(x)dx$ is a number while the indefinite integral $\int f(x)dx$ is a function.*

Figure 2 is table of indefinite integrals. *You are expected to know all the materials here, especially the first ten identities (all left column and the first one from the right column).*

$$\int cf(x)\,dx = c\int f(x)\,dx \qquad \int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int k\,dx = kx + C$$

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1) \qquad \int \frac{1}{x}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C \qquad \int b^x\,dx = \frac{b^x}{\ln b} + C$$

$$\int \sin x\,dx = -\cos x + C \qquad \int \cos x\,dx = \sin x + C$$

$$\int \sec^2 x\,dx = \tan x + C \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C \qquad \int \csc x \cot x\,dx = -\csc x + C$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C \qquad \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\int \sinh x\,dx = \cosh x + C \qquad \int \cosh x\,dx = \sinh x + C$$

Figure 2: Table of Indefinite Integrals [Stewart]

3 Techniques of Integration

Always remember:

> [$\star\star\star\star\star$] *The general philosophy is that techniques should give you a tool to convert unknown integrand to known integrands.*

As the sentence implies, this philosophy consists two parts

1. Have a good reservoir of known integrals, and
2. Be fluent with integration techniques.

The first item is something you accumulate by experience. As a starting point you should memorize all the integrals in Figure 2. For the second item, we summarize all major techniques for computing integrals explicitly. Standard techniques we include in this sections include

1. Linearity
2. Substitution Rule,
3. Integration by Parts,
4. Trigonometric Substitution, and
5. Partial Fraction.

The last two subsections are strategy for integration and (possibly non-standard but useful) special techniques.

3.1 (5.2) Linearity [$\star\star\star\star\star$]

Here are the most useful formula for integration. If I could I will put 6 stars, but that'd be against my own principle.

$$\boxed{\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx} \tag{3.1.1}$$

$$\boxed{\int cf(x)dx = c\int f(x)dx} \tag{3.1.2}$$

where c is a constant.

Here is an example. Integrate $\int (1+x^2)^2 + 9e^x + \frac{\pi}{x}dx$. We see that by linearity

$$\int (1+x^2)^2 + 9e^x + \frac{\pi}{x}dx = \int (1+x^2)^2dx + 9\int e^x dx + \pi\int \frac{1}{x}dx. \tag{3.1.3}$$

Now we integrate term by term.

$$\int (1+x^2)^2 dx = \int 1 + 2x^2 + x^4 dx = \int 1dx + 2\int x^2 dx + \int x^4 dx \tag{3.1.4}$$

$$= x + \frac{2}{3}x^3 + \frac{1}{5}x^5, \tag{3.1.5}$$

and $\int e^x dx = e^x$, and $\int \frac{1}{x} dx = \ln x$.

3.2 (5.5) The Substitution Rule

The substitution rules are given in the following theorem.

Theorem 3.2.1 ((Substitution Rule))**.** *Suppose that $u = g(x)$ is differentiable and its range is in the interval I and f is continuous on I, then*

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{3.2.1}$$

Moreover, if g' is continuous on $[a,b]$, then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du. \tag{3.2.2}$$

Just like the FTC, as stated, the rules are not as helpful as they are in practice. In practice, the most important part of this technique is to identify the substitution $u = g(x)$. Below I included a (non-exhaustive) list of representative samples of techniques.

Identify $g(x)$**.** [$\star\star\star\star\star$] This techniques is simple to explain and very hard to use in practice: try to find f and g such that the integrand looks like $\int f(g(x))g'(x)dx$ where you know how to integrate f (with antiderivative F). Then we do the following steps.

1. Write the substitution $u = g(x)$. The integral $\int f(g(x))g'(x)dx$ becomes

$$\int f(u)g'(x)dx. \tag{3.2.3}$$

2. We must change dx to du. To do this, we use the following intuitive identity,

$$\boxed{du = \frac{du}{dx}dx = \left(\frac{d}{dx}g(x)\right)dx = g'(x)dx}. \tag{3.2.4}$$

This tells us

$$\boxed{\int f(g(x))g'(x)dx = \int f(u)du} \tag{3.2.5}$$

Scaling. [⋆⋆⋆⋆⋆] Here is the idea:

1. Regard the integrand as $f(ax)$ for some constant a such that you know an antiderivative of f, say F.
2. Use the substitution $u = ax$ with $du = adx$. Then

$$\int f(ax)dx = \int f(u)\frac{1}{a}du = \frac{1}{a}F(u) = \frac{1}{a}F(ax). \tag{3.2.6}$$

Compute the integral $\int 3\cos(9x)dx$.

1. Use the substitution is $u = 9x$.
2. Compute the differential

$$du = \left(\frac{d}{dx}(9x)\right)dx = 9dx. \tag{3.2.7}$$

   Equivalently,

$$dx = \frac{1}{9}du. \tag{3.2.8}$$

3. Put everything back into the integral. We get

$$\begin{aligned} \int 3\cos(9x)dx &= 3\int \cos(9x)dx \quad \text{(by linearity)} && (3.2.9)\\ &= 3\int \cos(u)\frac{1}{9}du \quad \text{(by substitution)}. && (3.2.10) \end{aligned}$$

   Now one can integrate this easily.

Translation. [⋆⋆⋆⋆⋆] Here is the idea:

1. Regard the integrand as $f(x+a)$ for some constant a such that you know an antiderivative of f, say F.
2. Use the substitution $u = x + a$ with $du = dx$. Then

$$\int f(x+a)dx = \int f(u)du = F(u) = F(x+a). \tag{3.2.11}$$

Compute the integral $\int \frac{1}{1+x}dx$.

1. We know that $(\ln(x))' = \frac{1}{x}$. So, use the substitution is $u = x + 1$.
2. Then we see that $du = dx$ and

$$\int \frac{1}{1+x}dx = \int \frac{1}{u}du = \ln u = \ln(1+x) \tag{3.2.12}$$

Substitute for the Bad Part. [$\star\star\star$] Here is the idea:

1. Suppose that we can write the integrand as $f_1(g(x))f_2(x)$ where $f_1(g(x))$ is hard to integrate explicitly or very ugly, but $f_1(x)$ and $f_2(x)$ are simple and nice.

2. Use the substitution $u = g(x)$.

3. Then we get $du = g'(x)dx$ So that

$$\int f_1(g(x))f_2(x)dx = \int f_1(u)\frac{f_2(x)}{g'(x)}du \tag{3.2.13}$$

4. If $\frac{f_2(x)}{g'(x)}$ is a function of $g(x)$, then we are in business.

Compute $\int x^3\sqrt{x^2+1}dx$.

1. The square root here is causing headache here. But $f_1(x) = \sqrt{x}$ is nice.

2. So we set $u = x^2 + 1$.

3. We get $g'(x) = 2x$ and $du = 2xdx$. And we get

$$\int x^3\sqrt{x^2+1}dx = \int x^3\sqrt{u}dx \tag{3.2.14}$$

$$= \frac{1}{2}\int x^2\sqrt{u}du \tag{3.2.15}$$

$$= \frac{1}{2}\int (u-1)\sqrt{u}du \tag{3.2.16}$$

$$= \frac{1}{2}\int u^{3/2} - u^{1/2}du \tag{3.2.17}$$

Adding Zero [$\star\star\star$] The philosophy of this technique is as follows

1. Very often it is easier to identify the integrand as $f(g(x))$ than $f(g(x))g'(x)$. So, we regard the integrand as $f(g(x))$ for some f and g such that you know an antiderivative, F, of f.

2. Add and subtract $f(g(x))g'(x)$ from $f(g(x))$ to get

$$\int f(g(x))dx = \int f(g(x))(1-g'(x))dx + \int f(g(x))g'(x)dx \tag{3.2.18}$$

$$= \int f(g(x))(1-g'(x))dx + \int f(u)du \tag{3.2.19}$$

$$= \int f(g(x))(1-g'(x))dx + F(u) \tag{3.2.20}$$

where $u = g(x)$.

3. We hope that the term $f(g(x))(1-g'(x))$ is less complicated than $f(g(x))$ and can be integrated.

We illustrate this with and example. Compute the example $\int \frac{1}{1+e^x}dx$.

1. We regard $\frac{1}{1+e^x} = f(e^x)$ where $f(x) = \frac{1}{1+x}$. We know that f has antiderivative $\ln(1+x)$. We set $u = e^x$.
2. Subtract and add $e^x \cdot \frac{1}{1+e^x}$ to get

$$\int \frac{1}{1+e^x}dx = \int \frac{1+e^x}{1+e^x}dx - \int \frac{e^x}{1+e^x}dx \tag{3.2.21}$$
$$= \int 1\,dx - \int \frac{e^x}{1+e^x}dx \tag{3.2.22}$$
$$= x - \int \frac{u}{1+u}\frac{1}{u}du \tag{3.2.23}$$
$$= x + \int \frac{1}{1+u}du \tag{3.2.24}$$

Multiplying by One. [$\star\star$] The philosophy of this technique is as follows

1. Regard the integrand as $\frac{f_1(x)}{f_2(x)}$.
2. Try to multiply a function $h(x)$ in the numerator and in the denominator such that

$$\frac{d}{dx}(f_2(x)h(x)) = f_1(x)h(x) \tag{3.2.25}$$

3. use the substitution $u = f_2(x)h(x)$, then $du = \frac{d}{dx}(f_2(x)h(x))dx = f_1(x)h(x)dx$ so that

$$\int \frac{f_1(x)}{f_2(x)}dx = \int \frac{f_1(x)h(x)}{f_2(x)h(x)}dx = \int \frac{1}{u}du = \ln u = \ln(f_2(x)h(x)) \tag{3.2.26}$$

Compute the example $\int \sec(x)dx$.

1. The integrand is $\frac{\sec(x)}{1}$.
2. Look for $h(x)$ such that $(h(x)\cdot 1)' = h(x)\sec(x)$. To do this, first, we look for functions such that when you differentiate, you pick up a factor of sec. We observe that $(\tan(x))' = \sec^2(x)$ works. But certainly $(\tan(x))' = \sec^2(x) \neq \tan(x)\sec(x)$. But luckily, $\tan(x)\sec(x) = (\sec(x))'$. So we simply add them and note that

$$(\tan(x) + \sec(x))' = \sec^2(x) + \tan(x)\sec(x) = \sec(x)(\tan(x) + \sec(x)) \tag{3.2.27}$$

So we use the substitution $u = \tan(x) + \sec(x)$ and conclude that

$$\int \sec(x)dx = \ln(\tan(x) + \sec(x)) \tag{3.2.28}$$

by equation (3.2.26).

Implicit Function [$\star\star$] Here is the idea

1. Suppose that the integrand can be written as $f(x)g(x)$
2. Assume that $g(x)$ has a (simple) antiderivative $G(x)$ and there is a function h such that $h(G(x)) = f(x)$ and h is easy to integrate. Very often $h(x) = f(G^{-1}(x))$ if the function $G(x)$ has an inverse. *This condition is guaranteed if $g(x) > 0$ for all x or $g(x) < 0$ for all x.*
3. Use the substitution $u = G(x)$. We see that $du = G'(x)dx = g(x)dx$ and

$$\int f(x)g(x)dx = \int f(x)G'(x)dx = \int h(G(x))G'(x)dx = \int h(u)du \tag{3.2.29}$$

We compute $\int \frac{1}{2(x+3)\sqrt{x+2}}dx$.

1. We have the integrand is the product of $\frac{1}{x+3}$ and $\frac{1}{2\sqrt{x+2}}$.
2. We can integrate either one. But we don't want to be left with an ugly square root in the denominator at the end. So we pick $g(x) = \frac{1}{\sqrt{x+2}}$. Note that $g(x) > 0$ for all x in the proper domain so its antiderivative has an inverse. We see that an antiderivative of g is $G(x) = \sqrt{x+2}$.
3. So we substitute $u = \sqrt{x+2}$ and we see that $x = u^2 - 2$ and $du = \frac{1}{2\sqrt{x+2}}dx$ and

$$\int \frac{1}{2(x+3)\sqrt{x+2}}dx = \int \frac{1}{(u^2-2+3)}du = \int \frac{1}{u^2+1}du = \tan^{-1}(u) \tag{3.2.30}$$

Repeated use of the Substitution Rule. [$\star\star\star\star$] Sometimes it is necessary to use the substitution rule several times to convert the integrand into some simple function which you can integrate explicitly!

3.3 (7.1) Integration by Parts

The integration by part formula is

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int f'(x)g(x)dx} \tag{3.3.1}$$

Here is the idea of the technique and some strategies.
Identify u and dv. [$\star\star\star\star\star$]

1. The integrand looks like a product of two function $f(x)g'(x)$, one of them $g'(x)$ you know how to integrate and the other $f(x)$ you know how to differentiate.

2. You that if you can shift one derivative from g to f, then you can integrate $\int f'(x)g(x)dx$ instead of $\int f(x)g'(x)dx$.

3. Then we use formula (3.3.1). In practice, we use the pattern

$$u = \quad \square \quad dv = \quad \square \tag{3.3.2}$$

$$du = \quad \square \quad v = \quad \square \tag{3.3.3}$$

   Then the rule is written as

$$\boxed{\int udv = uv - \int vdu.} \tag{3.3.4}$$

For example, integrate $\int xe^x$. Here we have

$$u = x \quad dv = e^x \tag{3.3.5}$$

$$du = ? \quad v = ? \tag{3.3.6}$$

Then $du = dx$ and we can compute v by computing $\int e^x dx = e^x$. In the end we get

$$u = x \quad dv = e^x \tag{3.3.7}$$

$$du = dx \quad v = e^x. \tag{3.3.8}$$

So,

$$\int xe^x = xe^x - \int e^x = xe^x - e^x. \tag{3.3.9}$$

Recognize the coefficient 1 [$\star\star\star\star\star$]**.** Here is the main idea

1. Sometimes the integrand simply looks like a function $f(x)$ which you can't factor. But hopefully $xf'(x)$ is nicer.

2. Write the integral $\int f(x)dx = \int 1 \cdot f(x)dx$.

3. Set $u = f(x)$ and $dv = 1 \cdot dx$ and perform integration by part to get

$$\int f(x)dx = xf(x) - \int xf'(x)dx \tag{3.3.10}$$

Here is an example. Integrate $\int \tan^{-1}(x)dx$.

1. Set $u = \tan^{-1}(x)$ and $dv = 1dx$. We get

$$u = \tan^{-1}(x) \quad dv = dx \tag{3.3.11}$$

$$du = \frac{1}{1+x^2} \quad v = x. \tag{3.3.12}$$

2. The integral becomes

$$\int \tan^{-1}(x)dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}dx \tag{3.3.13}$$

The second term on the RHS can be integrated using the substitution $u = x^2$.

Reduction of Power: Polynomials [$\star\star\star\star\star$] Here is the idea:

1. The integrand looks like $x^n f(x)$ for some $f(x)$ that you know how to integrate (as many times as possible). We want to reduce the power of x^n by as much as we can since we know how to integrate $f(x)$ repeatedly.

2. Use $u = x^n$ and $dv = f(x)dx$.

3. We get

$$u = x^n \quad dv = f(x)dx \tag{3.3.14}$$

$$du = nx^{n-1} \quad v = F(x) \tag{3.3.15}$$

where $F(x)$ is an antiderivative of $f(x)$.

4. The integral becomes

$$\int x^n f(x)dx = x^n f(x) - \int nx^{n-1}F(x)dx \tag{3.3.16}$$

Here is an example: $\int x^5 e^x dx$.

1. We can integrate e^x for as many times as we need.

2. Use $u = x^5$ and $dv = e^x dx$.

3. We get

$$u = x^5 \quad dv = e^x dx \tag{3.3.17}$$

$$du = 5x^4 \quad v = e^x \tag{3.3.18}$$

4. The integral becomes

$$\int x^5 e^x dx = x^5 e^x - 5\int x^4 e^x dx. \tag{3.3.19}$$

Using the same idea, one can repeat integration by part to integrate, by getting rid of all powers of x.

Reduction of Power: General Setting [$\star\star\star\star\star$] Here is the idea:

1. The integrand looks like $(f(x))^n g'(x)$. You can integrate $g'(x)$ and you want to reduce the power n.

2. Use $u = (f(x))^n$ and $dv = g'(x)dx$.

Here is an example: $\int (\ln x)^4 dx$.

1. It's not so obvious how to choose factors. But we can write the integrand as $1 \cdot (\ln x)^4$.

2. Use $u = (\ln x)^4$ and $dv = dx$.

3. We get

$$u = (\ln x)^5 \quad dv = dx \tag{3.3.20}$$

$$du = 5(\ln x)^4 \frac{1}{x} \quad v = xdx \tag{3.3.21}$$

4. The integral becomes

$$\int (\ln x)^5 dx = x(\ln x)^5 - 5 \int (\ln x)^4 dx \tag{3.3.22}$$

and we repeat.

Recursive Integration This one is best illustrated by example. The main idea is that integration by part does not make the integrand look simpler, however after some iterations of integration by parts, you get back to where you started, but with a different coefficient!

Compute $\int e^x \cos(x)dx$. We use

$$u = \cos(x) \quad dv = e^x dx \tag{3.3.23}$$

$$du = -\sin(x)dx \quad v = e^x \tag{3.3.24}$$

to get

$$\int e^x \cos(x)dx = \cos(x)e^x + \int \sin(x)e^x dx. \tag{3.3.25}$$

This is truly usely, as the integrand does not become any simpler. But let's not give up. We do integration by parts again. This time we use

$$u = \sin(x) \quad dv = e^x dx \tag{3.3.26}$$

$$du = \cos(x)dx \quad v = e^x \tag{3.3.27}$$

to get

$$\int e^x \cos(x)dx = \cos(x)e^x + \int \sin(x)e^x dx \tag{3.3.28}$$

$$= \cos(x)e^x + \sin(x)e^x - \int \cos(x)e^x. \tag{3.3.29}$$

We note that we get $\int \cos(x)e^x$ back as we started. You might say this is utterly useless. But note that the sign of $\int \cos(x)e^x$ on the RHS is *negative*. So we may solve for it and move it to the LHS to get

$$2 \int e^x \cos(x)dx = \cos(x)e^x + \sin(x)e^x. \tag{3.3.30}$$

3.4 (7.1) Trig Integrals

In order for us to compute trig integrals. We need 2 things

1. Integral of simple trig functions. You need to memorize all the integrals from Fig. 2.
2. Basic trig identities. I listed them in Fig. 3. You need to know [$\star\star\star$]: 1) Pythagorean identities, 2) double angle, 3) half angle, and 4) product formuli. It is useful to know the addition and subtraction formuli.

Trigonometric Identities

Reciprocal

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$
$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$
$$\cot x = \frac{1}{\tan x} \qquad \tan x = \frac{1}{\cot x}$$

Pythagorean

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Negative Angle

$$\sin(-x) = -\sin x$$
$$\cos(-x) = \cos x$$
$$\tan(-x) = -\tan x$$

Addition and Subtraction

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$
$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Periodicity

$$\sin(x+2\pi) = \sin x \qquad \cos(x+2\pi) = \cos x$$
$$\csc(x+2\pi) = \csc x \qquad \sec(x+2\pi) = \sec x$$
$$\tan(x+\pi) = \tan x \qquad \cot(x+\pi) = \cot x$$

Cofunction

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \cos x = \sin\left(\frac{\pi}{2} - x\right)$$
$$\tan x = \cot\left(\frac{\pi}{2} - x\right) \qquad \cot x = \tan\left(\frac{\pi}{2} - x\right)$$
$$\sec x = \csc\left(\frac{\pi}{2} - x\right) \qquad \csc x = \sec\left(\frac{\pi}{2} - x\right)$$

Product

$$\sin x \cos y = \frac{1}{2}\left(\sin(x+y) + \sin(x-y)\right)$$
$$\sin x \sin y = \frac{1}{2}\left(\cos(x-y) - \cos(x+y)\right)$$
$$\cos x \cos y = \frac{1}{2}\left(\cos(x+y) + \cos(x-y)\right)$$
$$\cos x \sin y = \frac{1}{2}\left(\sin(x+y) - \sin(x-y)\right)$$

Factoring

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\sin x - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$
$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$
$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Double Angle

$$\sin(2x) = 2\sin x \cos x \qquad \cos(2x) = 1 - 2\sin^2 x$$
$$\cos(2x) = \cos^2 x - \sin^2 x \qquad \tan(2x) = \frac{2\tan x}{1 - \tan^2 x}$$
$$\cos(2x) = 2\cos^2 x - 1$$

Half-Angle

$$\sin\frac{x}{2} = \pm\sqrt{\frac{1-\cos x}{2}} \qquad \tan\frac{x}{2} = \frac{1-\cos x}{\sin x}$$
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}} \qquad \tan\frac{x}{2} = \frac{\sin x}{1+\cos x}$$

Figure 3: Trig Identities, source: Google Image

Trig integrals that we concern ourselves with are of the follow three forms:

Product of $\sin^n(x)$ **and** $\cos^m(x)$ We give a summary of the strategy for computing this kind of integrals in Fig. 4.

Product of $\tan^n$ **and** $\sec^m$. We give a summary of the strategy for computing this kind of integrals in Fig. 5.

Product of $\sin(ax)$ **and** $\cos(bx)$. We give a summary of the strategy for computing this kind of integrals in Fig. 6.

4 Trig Substitution

5 Other Techniques

$I + J$ and $I - J$ try $\sqrt{\tan(x)}$.

References

[Stewart] J. Stewart, *Single Variable Calculus Early Transcendentals*, 8th edition. ISBN-13: 978-1-305-27033-6

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x \, dx &= \int \sin^m x \, (\cos^2 x)^k \cos x \, dx \\ &= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx \end{aligned}$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x \, dx &= \int (\sin^2 x)^k \cos^n x \, \sin x \, dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx \end{aligned}$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Figure 4: Product of $\sin^n(x)$ and $\cos^m(x)$, [Stewart]

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx$$

Then substitute $u = \sec x$.

Figure 5: Product of $\tan^m(x)$ and $\sec^m(x)$, [Stewart]

2 To evaluate the integrals (a) $\int \sin mx \cos nx \, dx$, (b) $\int \sin mx \sin nx \, dx$, or (c) $\int \cos mx \cos nx \, dx$, use the corresponding identity:

(a) $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$

(b) $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$

(c) $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Figure 6: Product of $\sin(ax)$ and $\cos(bx)$, [Stewart]