University of Toronto
Faculty of Arts and Science

MAT136H1Y Final Exam

Instructor: Li Chen
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Duration - 180 minutes
No Aids Permitted

Surname: ____________________

Given Name: ____________________

Student Number: ____________________

This exam contains 12 pages (including this cover page) and 5 problems (one problem is bonus). Once the exam begins, check to see if any pages are missing and ensure that all required information at the top of this page has been filled in.

No aids are permitted on this examination. Examples of illegal aids include but are not limited to textbooks, notes, calculators, cellphones, or any electronic device.

Unless otherwise indicated, you are required to **show your work** on each problem on this exam. The following rules apply:

- **Total points available is 110; but the test is out of 100.** The last problem is a bonus problem.
- **Unreadable answers will receive no mark.**
- **Organize your work** in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work, will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- **If you need more space**, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	40	
2	15	
3	15	
4	15	
5	15	
Total:	100	

Part 1: Short Answers (40 marks)

No justification is necessary and no mark will be awarded for them

1. For each of the following questions, write your final answer in the box on the right hand side. Only your final answer will be graded.

   (a) (5 points) True of False? Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ are two sequences of real numbers. If $\sum_{n=1}^{\infty} a_n$ is divergent and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n + b_n$ convergent.

   Solution: The Series $\sum_{n=1}^{\infty} 1$ is clearly divergent while $\sum_{n=1}^{\infty} 0$ is convergent. But $\sum_{n=1}^{\infty} 1 + 0$ is divergent.

   Final Answer

   False

   (b) (5 points) True or False? Suppose that we have a sequence real numbers $a_1, a_2, a_3, \ldots$ such that $a_n < 1/n^2$ if $n \geq 1000$. Then $\sum_{n=1}^{\infty}$ is convergent.

   Solution: The first 999 terms adds to a finite number, i.e. $a_1 + a_2 + \cdots + a_{999}$ is a finite number. The rest $a_{1000} + a_{1001} + \cdots \leq 1/1000^2 + 1/1001^2 + \cdots < \infty$ by the p-test (in this case $p = 2 > 1$).

   Final Answer

   True

   (c) (5 points) Assume that $\sum_{n=1}^{\infty} a_n = 2$. Compute $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - a_n$

   Solution: First, we need to compute $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$. We recall that

   $$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^n}{n}.$$

   Since the series is convergent at $x = 1$ (by the alternating series test), we conclude that

   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln(1+1) = \ln(2).$$

   So we have

   $$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} - a_n = \ln(2) - 2$$

   Final Answer

   $\ln(2) - 2$

(d) (5 points) Let

$$f(x) := e^{-x}$$

What is the 100-th Taylor coefficient of the Taylor Series of $f(x)$ centered at $x = 0$?

Solution: We know that

$$e^{-x} = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}$$

So the 100-th Taylor coefficient is $(-1)^{100}/100! = 1/100!$.

Final Answer

$$\boxed{\frac{1}{100!}}$$

All of the following series are convergent, compute the value of each series explicitly.

(e) (5 points)

$$\sum_{n=1}^{\infty} 2\frac{\cos(\pi + n\pi)}{9^n}$$

Solution: we first note that $\cos(\pi n + \pi) = (-1)^{n+1}$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. Hence

$$\sum_{n=0}^{\infty} 2\frac{\cos(\pi + n\pi)}{9^n} = 2(-1)\sum_{n=0}^{\infty}\left(\frac{-1}{9}\right)^n = \frac{-2}{1+1/9} = -\frac{18}{11}$$

Final Answer

$$\boxed{\frac{-18}{11}}$$

(f) (5 points)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6}$$

Solution: We note that $n^2+5n+6=(n+2)(n+3)$ factors. We use partial fraction method. We look for A, B such that

$$\frac{1}{(n+2)(n+3)} = \frac{A}{n+2} - \frac{B}{n+3}$$

Multiply the equation by $(n+2)(n+3)$ on both sides, we arrive at

$$1 = A(n+3) - B(n+2) = (A-B)n + (3A-2B).$$

This tells us that $A = B$ and $3A - 2B = 1$. It follows that $A = B = 1$. Hence, we get

$$\sum_{n=1}^{\infty} \frac{1}{n^2+5n+6} = \sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+3}$$

is a telescoping series. Moreover, the N-th partial sum is

$$\sum_{n=1}^{N} \frac{1}{n+2} - \frac{1}{n+3} = \frac{1}{3} - \frac{1}{N+3}$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n+2} - \frac{1}{n+3} = \lim_{N\to\infty} \frac{1}{3} - \frac{1}{N+3} = \frac{1}{3}$$

Final Answer

$$\boxed{\frac{1}{3}}$$

(g) (5 points)

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{(2n+2)!}$$

Solution: We recognize that

$$\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}.$$

We re-write

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{(2n+2)!} = \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{\pi})^{2n}}{(2n+2)!} = \frac{1}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{\pi})^{2n+2}}{(2n+2)!} = \frac{-1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n (\sqrt{\pi})^{2n}}{(2n)!}$$

Writing out the power series for $\cos(x)$ explicitly for the 0-th term, we see that

$$\cos(\sqrt{\pi}) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n (\sqrt{\pi})^{2n}}{(2n)!}.$$

It follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^n}{(2n+2)!} = \frac{-1}{\pi} (\cos(\sqrt{\pi}) - 1)$$

Final Answer

$$\boxed{\tfrac{-1}{\pi} (\cos(\sqrt{\pi}) - 1)}$$

(h) (5 points)

$$\sum_{n=0}^{\infty} \ln[\sec^2(n) - \tan^2(n)]$$

Solution: We note that $\sec^2(n) - \tan^2(n) = 1$ and $\ln(1) = 0$. So every term of the series is 0.

Final Answer

$$\boxed{0}$$

Part 1: Long Answers (60 marks)

Show your work for full marks

2. Determine if the following series are absolutely convergent, conditionally convergent, or divergent.

   (a) (5 points)

$$\sum_{n=2}^{\infty} \frac{n^n(-1)^n}{n!(n+1)}$$

Solution: The principle of dominance shows that $n^2 >> n! >> (n+1)$. So we expect the series to diverge. The only part that might make the series converge is the oscillating $(-1)^n$. But it turns out this oscillating term does not help us to get a convergent series.

To prove our guess, we employ the divergence test. We consider

$$\lim_{n\to\infty} \frac{n^n(-1)^n}{n!(n+1)}.$$

If n is even and n is large, we see that

$$\frac{n^n(-1)^n}{n!(n+1)} = \frac{n^n}{n!(n+1)} = \frac{n \cdot n \cdot n \cdots n}{n(n-1)(n-2)\cdots(1)(n+1)} = \frac{n}{n} \cdot \frac{n}{n-1} \cdot \frac{n}{n-2} \cdots \frac{n}{2} \cdot \frac{1}{n+1} \geq 1$$

where there are exactly n-factors on the top and $n+1$ factors on the bottom after the second equality. Likewise, if n is odd, then

$$\frac{n^n(-1)^n}{n!(n+1)} \leq -1.$$

This shows that the limit $\lim_{n\to\infty} \frac{n^n(-1)^n}{n!(n+1)}$ does not exits. Hence the series is divergent.

(b) (10 points)

$$\sum_{n=3}^{\infty} \frac{(-1)^n}{10\ln(n) - 1/2^n}$$

Solution: Do not let the $1/2^n$ mislead you. The 2^n is on the bottom and so $1/2^n$ is small for n large. In particular $10\ln(n)$ dominates $1/2^n$. So the series behaves roughly like $\sum_{n=3}^{\infty} \frac{(-1)^n}{10\ln(n)}$, which we know is conditionally convergent. We consider absolute convergence first. Taking absolute value, we obtain

$$\sum_{n=3}^{\infty} \frac{1}{\ln(n) - 1/2^n}.$$

We note that

$$\frac{1}{10\ln(n) - 1/2^n} \geq \frac{1}{10\ln(n)}.$$

Since $\sum_{n=3}^{\infty} \frac{1}{10\ln(n)}$ diverges, the comparison test shows that the series is not absolutely convergent.

Now we consider conditional convergence. We note that the term $\frac{1}{10\ln(n)-1/2^n}$ is decreasing. So the alternating series test and the fact that

$$\lim_{n\to\infty} \frac{1}{10\ln(n) - 1/2^n} = 0$$

show that the series is conditionally convergent.

3. (15 points) For each integer $k > 0$, find the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

Solution: We apply the ratio test. We see that

$$\lim_{n\to\infty} \frac{((n+1)!)^k}{(kn+k)!}|x|^{n+1}\frac{(kn)!}{(n!)^k|x|^n} = \lim_{n\to\infty} \frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+1)}|x| = \frac{|x|}{k^k}$$

The series converges if $|x|/k^k < 1$ and diverges if > 1. Equivalently, the radius of convergence is k^k.

The difficulty is in the computation of the limit. We note that we can write

$$\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+1)} = \frac{n+1}{kn+k}\cdot\frac{n+1}{kn+k-1}\cdot\frac{n+1}{kn+k-2}\cdots\frac{n+1}{kn+1}$$

where there are k-factors above. Each factor admits a limit, as $n \to \infty$ of $1/k$ (Note we are taking the limit as $n \to \infty$, but k is fixed. If it makes it easier, pretend $k = 5$ and work out the details first). Hence, together, they contribute $1/k^k$.

4. (a) (5 points) Find a closed form of

$$\sum_{n=0}^{\infty} \frac{x^n}{n}$$

Solution: We recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.$$

Integrating term-by-term we get

$$-\ln(1-x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(b) (5 points) Let $f(x) = \ln(1-x)/x$ if $x > 0$ and $f(0) = 1$ if $x = 0$. Show that f is continuous on $[0, 1)$

Solution: Since $\ln(1-x)$ is continuous on $[0, 1]$, we only need to show that the function is continuous at $x = 0$. We note that, by L'Hopital's rule

$$\lim_{x \to 0} \ln(1-x)/x = \lim_{x \to 0} \frac{-1/(1-x)}{1} = 1 = f(0).$$

This shows that $f(x)$ is continuous on $[0, 1)$

(c) (5 points) Assume that $\int_0^1 \frac{\ln(1-x)}{x} = -\pi^2/6$. Use parts (a) and (b) to compute

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Solution: By part (a), we see that

$$-\frac{\ln(1-x)}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}.$$

By (b) shows that this function is integrable and

$$\frac{\pi^2}{6} = \int_0^1 -\frac{\ln(1-x)}{x} dx = \sum_{n=1}^{\infty} \int_0^1 \frac{x^{n-1}}{n} dx = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

5. (a) (8 points) Suppose that a sequence of real numbers, $\{a_n\}_{n=0}^{\infty}$, satisfies

$$a_{n+1} = -\frac{a_n}{(n+1)}$$

with $a_0 = 1$. Find a closed form formula for a_n in terms of n only. (Use the notation $n! = n(n-1)(n-2)\cdots(2)(1)$ and define $0! = 1$)

Solution: Using the relation $a_{n+1} = -\frac{a_n}{(n+1)}$, we see that

$$a_n = \frac{(-1)a_{n-1}}{n} = \frac{(-1)^2 a_{n-2}}{n-1} = \frac{(-1)^3 a_{n-3}}{n-2} = \cdots = \frac{(-1)^n a_0}{n!} = \frac{(-1)^n}{n!}$$

(b) (7 points) Solve the differential equation

$$f' = -f \text{ and } f(0) = 1$$

by writing f as a series representation $\sum_{n=0}^{\infty} a_n x^n$. Then determine the radius of convergence of this series.

Solution: Write $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Then

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

Since $f' = -f$, we must have, after matching coefficients of x^n,

$$(n+1)a_{n+1} = -a_n.$$

and $a_0 = 1$. We find from part (a) that

$$f(x) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} = e^{-x}.$$

Hence the radius of convergence is ∞.

This page is for additional work and will not be marked.