THE EFFECTIVE DYNAMICS OF THE VOLUME PRESERVING MEAN CURVATURE FLOW

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To Jürg, Tom and Herbert with friendship and admiration.

Abstract. We consider the dynamics of small closed submanifolds ('bubbles') under the volume preserving mean curvature flow. We construct a map from (n+1)-dimensional Euclidean space into a given (n+1)-dimensional Riemannian manifold which characterizes the existence, stability and dynamics of constant mean curvature submanifolds. This is done in terms of a reduced area function on the Euclidean space, which is given constructively and can be computed perturbatively. This allows us to derive adiabatic and effective dynamics of the bubbles. The results can be mapped by rescaling to the dynamics of fixed size bubbles in almost Euclidean Riemannian manifolds.

Keywords: volume preserving mean curvature flow, constant mean curvature surface, submanifold, mean curvature flow, adiabatic dynamic, effective dynamic.

1. Background and main results

The goal of this note is to introduce the Lyapunov-Schmidt map extending the Feshbach-Schur one ([8], see also [7, 19])[1] to a nonlinear setting and to use it to derive the adiabatic (and effective) dynamics of submanifolds under the volume preserving mean curvature flow.

Let (M, g) be an $n+1$-dim oriented Riemannian manifold. Recall that a family of immersions $\psi(t,\cdot) : \Sigma \to M$, $\psi(t,\Sigma) := S_t, t \geq 0$, of compact, connected, orientable manifolds[2] is called the volume preserving mean curvature flow (VPF), iff it satisfies the equation

$$\partial_t \psi^N = \bar{H}(\psi) - H(\psi), \tag{1.1}$$

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[1]The Feshbach-Schur map was introduced in the context of quantum electrodynamics and was used in statistical mechanics and random Schrödinger operators in [25] and [10], respectively (see also [14, 15]).

[2]We choose their orientation to be compatible with that of M.

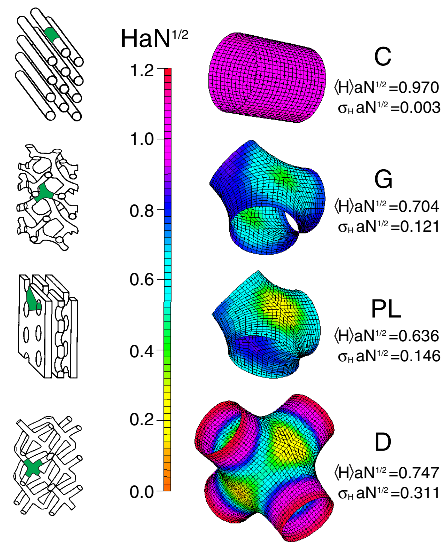

Figure 13. Interfacial curvature, H, distributions for the C, G, PL and D microstructures. The green patches on the schematic diagrams to the left indicate elementary interfacial units. The detailed shapes of these units are then displayed on the right by means of a SCFT calculation along the L/C phase boundary at $\chi N = 20$ and $f = 0.3378$. The curvature distribution over each unit is specified by the colour scale with the average and standard deviation quoted to the far right. Adapted from [69].

where $\partial_t \psi^N$ is the normal component of the velocity vector $\partial_t \psi$, defined as $\partial_t \psi^N = g\big(\partial_t \psi, \nu(\psi)\big)$, with $\nu(\psi)$, the outward unit normal to S_t, is the normal component of the velocity vector $\partial_t \psi$ Here $H(\psi)$ is the mean curvature of S_t and $\bar{H} = \bar{H}(t)$ its mean value over S_t,

$$\bar{H} := \frac{\int_{S_t} H d\sigma}{\int_{S_t} d\sigma}. \tag{1.2}$$

The VPF appeared in material sciences almost a century ago in modeling cell, grain and bubble growth, etc. It has been derived – as the sharp interface limit (non-rigorously) – from the Kawasaki dynamics in the Ising or Potts model, or from the phase field models of the Cahn-Hilliard type (see e.g. [11, 12]), which themselves can be considered as a (formal) hydrodynamic limit of a microscopic dynamics, say, the stochastic Ginzburg-Landau model (see Spohn [26]).

The static solutions to (1.1) are exactly the constant mean curvature immersions

$$H = \bar{H} \equiv h. \tag{1.3}$$

They appear in nature as (almost) stationary interfaces, e.g. in block copolymer melts ([23], see Fig. 1). (Because of close packing, surfaces deviate from CMC surfaces.)

As a consequence of (1.1), the enclosed volume of S_t is constant in time, while its area is decreasing (see below).

Let $A(\psi)$ and $V_{\text{enc}}(\psi)$ denote the area and enclosed volume of a closed surface given by an immersion ψ. In what follows we take $\Sigma = \mathbb{S}^n$, the standard unit $n-$dimensional sphere. Let

$$X_c^k := \{\psi \in H^k(\mathbb{S}^n, M) : V_{\text{enc}}(\psi) = c\}. \tag{1.4}$$

Given $c > 0$ small and $z \in \mathbb{M}^{n+1}$, there exists a unique geodesic sphere, $\theta_{\lambda,z} \in X_c^k$, of a radius $\lambda = \lambda(c)$, centred at z (λ depends on z as well), having enclosed volume c. The function $\lambda = \lambda(c)$ is invertible. We denote its inverse by $c(\lambda)$. The main result of this note is the following

Theorem 1.1. *For λ sufficiently small, there exists a family of maps $\Phi_\lambda : M \to X^k_{c(\lambda)}, r > n/2 + 2$, defined constructively in* (2.16) *below, such that*

(*i*) *For h large, there is an invertible function $\lambda = \lambda(h)$, s.t. $\psi_{\lambda,z} = \Phi_\lambda(z)$, with $\lambda = \lambda(h)$, solves (1.3), iff z is a critical point of $A(\Phi_\lambda(z))$ on $\{V_{\text{enc}}(\Phi_\lambda(z)) = c\}$.*

(*ii*) *Let z_* be a critical point of $A(\Phi_\lambda(z))$ on $\{V_{\text{enc}}(\Phi_\lambda(z)) = c\}$. Then the surface ψ_{λ,z_*} is a (non-degenerate) local minimum/(saddle or maximum) point of the area functional $A(\psi)$ on the set $\{V_{\text{enc}}(\psi) = c\}$ iff z_* is a (non-degenerate) local minimum/(saddle or local maximum) point of the function of $A(\Phi_\lambda(z))$ on $\{V_{\text{enc}}(\Phi_\lambda(z)) = c\}$.*

(*iii*) *With z_* as above, the surface ψ_{λ,z_*} is asymptotically stable, if z_* is a strict local minimum point of $A(\Phi_\lambda(z))$ on $\{V_{\text{enc}}(\Phi_\lambda(z)) = c\}$ and unstable if z_* is a saddle or local maximum point.*

We prove this theorem in Section 2. It can also be essentially deduced using the proofs and arguments in Alikakos and Freire ([1]). Its main value is a new viewpoint.

The map Φ_λ is constructed using the well-known Lyapunov - Schmidt argument. We call it the *Lyapunov - Schmidt* map.[3] Abstractly and informally, it is defined as follows. Let $Y \subset X$ be two Hilbert spaces. Given an orthogonal projection P on X and a point $u \in Y$, we say that a differentiable map $G : Y \to X$ is in the domain of Φ_{Pu} iff the operator $\bar{L} := \bar{P}dG(u)\bar{P}$, where $\bar{P} := \mathbf{1} - P$, is invertible on $\operatorname{Ran}\bar{P}$. For such a G, we define the new map $\Phi_{Pu}(G) : PY \to PX$ as

$$\Phi_{Pu}(G)(v) := PG(u + v + w(v)),$$

where $v \in PY$ and $w \equiv w(v)$ is a solution to the equation $\bar{P}G(u + v + w) = 0$ in $w \in \operatorname{Ran}\bar{P}Y$. (The requirement that the operator $\bar{L}$ is invertible on $\operatorname{Ran}\bar{P}$ and elementary smoothness properties guarantee that the equation $\bar{P}G(u+v+w) = 0$

[3]Like the Feshbach-Schur map, which comes from reconceptulazing of the well-known Feshbach-Schur perturbation theory, the Lyapunov-Schmidt map comes from rethinking the well-known Lyapunov-Schmidt reduction technique.

has a unique solution in an appropriate space.) We call Φ_{Pu} the *Lyapunov-Schmidt map*. One can compose such maps with varying P and u (possibly also with rescaling) to obtain a discrete dynamical system - a renormalization group.

Coming to the notion of stability, we will work with the following definitions. Recall (see Proposition 1.5 below) that the Gâteaux derivative dH of the mean curvature H is the hessian A'' of the area functional A.

Definition 1.2. *Linear stability* (or just stability in geometric analysis): We say that a (critical) hypersurface $\psi_* \in X_c^k$ is (linearly) stable, if $dH(\psi_*) = A''(\psi_*) \geq 0$ on $T_{\psi_*}X_c^k$. On the other hand, $\psi_* \in X_c^k$ will be called (linearly) unstable if $dH(\psi_*) = A''(\psi_*)$ on $T_{\psi_*}X_c^k$ has a negative spectrum.
Asymptotic stability: A CMC hypersurface $\psi_* \in X_c^k$ is asymptotically stable, iff there exists an $\varepsilon > 0$, such that for every initial hypersurface ψ_0 $\varepsilon-$close to ψ_* in the H^k norm, $k > \frac{n}{2}+2$, the solution converges (possibly after reparametrizations) to ψ_*, as $t \to +\infty$.

The theorem above, together with a refinement (due to [27, 24]) of beautiful results of [18] (see also [28]) about the area and enclosed volume expansions for small geodesic spheres, gives the following results due to Ye, Huisken and Yau, Alikakos and Freire, Pacard and Xu ([27, 22, 1, 24], the existence part) and Alikakos and Freire ([1], the stability part):

Theorem 1.3. *For each non-degenerate critical point z_* of the scalar curvature R of M^{n+1} and for each $c > 0$ sufficiently small:*
(i) *There exists a constant mean curvature hypersurface of enclosed volume c, enclosing z_*.*
(ii) *Varying c, these CMC hypersurfaces foliate a neighborhood of z_*.*
(iii) *The hypersurfaces corresponding to non-degenerate maximum of R are asymptotically stable and those corresponding to saddle points and minima are unstable.*

Let $S_z^n := \{\omega \in T_zM : g_z(\omega,\omega) = 1\}$, the unit sphere in the tangent space T_zM, $I_z : \mathbb{S}^n \to S_z^n$ is an identification of S_z^n with $\mathbb{S}^n$ and $\exp_z(v) : S_z^n \to M$ be the exponential map. For $\rho : \mathbb{S}^n \to \mathbb{R}_+$, we define the map

$$\theta_{\rho,z}(\omega) := \exp_z(\rho(\omega)I_z\omega). \tag{1.5}$$

Let $H^s \equiv H^s(\mathbb{S}^n, \mathbb{R})$. Consider the VPF (1.1) with initial configuration $\psi_0 = \theta_{\rho_0,z_0}$ close to the geodesic sphere, θ_{λ_*,z_*}, for some $z_* \in M$ and $\lambda_* > 0$, small, in the sense that $\|\rho_0 - \lambda_*\|_{H^k} + d_M(z_0, z_*)$, $k > n/2+1$, is sufficiently small, see Subsection 2.1. We compare the hypersurface $\psi(t,\cdot)$ with the *adiabatically* evolving hypersurface $\psi_{\lambda(t),z(t)} := \Phi_{\lambda(t)}(z(t))$ for some $z = z(t)$ and $\lambda = \lambda(t)$ to be determined. Let $a_\lambda(z) := A(\Phi_\lambda(z))$ and $v_{\text{enc}}(z) := V_{\text{enc}}(\Phi_\lambda(z))$ and

$$\chi_c = \{z \in M : v_{\text{enc}}(z) = c\}. \tag{1.6}$$

Statement (iii) is derived from the following (cf. [1])

Theorem 1.4. *With the definitions above and for c (or λ) sufficiently small, we have*

$$\psi(t,\omega) = \theta_{\xi(t),\psi_{\lambda(t),z(t)}(\omega)}(\omega), \tag{1.7}$$

$$\dot{z} = -\nabla a_\lambda(z) + O(\lambda^2 e^{-\delta t}), \tag{1.8}$$

$$\dot{\lambda} = O(\lambda^2 e^{-\delta t}), \tag{1.9}$$

where $z \in \chi_c$, $\delta = \frac{n+2}{2} - O(\lambda)$ and $\xi(t)$ satisfies the estimates

$$\|\xi\|_{H^k} \lesssim \lambda e^{-\delta t}. \tag{1.10}$$

We call evolution (1.8) - (1.9) the *adiabatic dynamics* of almost CMC surfaces. Eq (1.7) formulates the *nonlinear perturbation theory* (with $\xi(t)$ being a perturbation of $\psi_{\lambda(t),z(t)}$).

Foliations by CMC surfaces are of interest in general relativity (see e.g. [22]). For material sciences, the opposite scaling regime is of interest. In it, the ambient manifold is almost flat while the CMC surfaces are of size $O(1)$.

In the Euclidean case, $M = \mathbb{R}^{n+1}$, G. Huisken ([21]), in the general case, and M. Gage ([17]), for curves, proved that the solution to (1.1) exists globally and converges exponentially fast to a sphere, provided that the initial surface S_0 is uniformly convex and smooth. Athanassenas [4, 5] has shown neckpinching of certain class of rotationally symmetric surfaces under the volume preserving modification of the mean curvature flow. Later Escher and Simonett and Antonopoulou, Karali and Sigal ([16, 3]) proved the asymptotic stability of spheres (see also [20]). For related works, see [2, 6] and the references therein.

Finally, we present some standard relations (see e.g. [13]) which are helpful in understanding our results and which are used in the proofs.

Proposition 1.5. *The Gateaux derivatives of the area and enclosed volume of $S \hookrightarrow M$ equal*

$$dA(S)\eta = \int_S Hf d\sigma, \qquad dV_{\mathrm{enc}}(S)\eta = \int_S f d\sigma, \tag{1.11}$$

where $f := g(\eta, \nu(S))$.

Hence, from the variation formula (1.11) it follows that the tangent space to X_c is

$$T_\psi X_c^k = \{\xi : \mathbb{S}^n \to TM,\ \int_{\mathbb{S}^n} g(\xi, \nu(\psi)) d\sigma = 0\} \tag{1.12}$$

Proposition 1.6. *(i) Minimizers of the area functional $A(\psi)$ for a given enclosed volume $V_{\mathrm{enc}}(\psi)$ are critical points of $A(\psi)$ on X_c^k.*
(ii) The Euler-Lagrange equation for these critical points is exactly the CMC equation (1.3).

(iii) These critical points are critical points of the modified functional

$$A_h(\psi) = A(\psi) - hV_{\text{enc}}(\psi), \tag{1.13}$$

where h is determined by $c = V_{\text{enc}}(\psi)$ and vice versa.
(iv) The VPF (1.1) is a gradient flow for the area functional on closed surfaces with given enclosed volume.

Notation. In this paper we use the following notation:

- $A \lesssim B$ denotes an inequality of the form $A \leq CB$, where $C > 0$ is a uniform in λ and z constant.
- $\partial_z^\alpha := \prod_{i=1}^{n+1} \partial_{z_i}$, where $z = (z_1, \ldots, z_{n+1}) \in \mathbb{R}^{n+1}$ and $\alpha = (\alpha_1, \ldots, \alpha_{n+1})$, $|\alpha| = \sum_{i=1}^{n+1} \alpha_i$.
- $S_z^n := \{\omega \in T_z M : g_z(\omega, \omega) = 1\}$, the unit sphere in the tangent space $T_z M$ and $H^s \equiv H^s(S_z^n, \mathbb{R})$, the Sobolev space of the order s.

2. The Lyapunov-Schmidt map: Proof of Theorem 1.1

2.1. **Graphs over (geodesic) spheres.** We construct CMC hypersurfaces by deforming geodesic spheres $\theta_{\lambda,z}$ of radius λ centred at z. The geodesic spheres are the images of spheres, $S_z^n := \{\omega \in T_z M : g_z(\omega, \omega) = 1\}$, in the tangent space $T_z M$ nderu the exponential map $\exp_z(v)$ and can be parametrized by $\theta_{\lambda,z} : S_z^n \to M$, where

$$\theta_{\lambda,z}(\omega) = \exp_z(\lambda\omega), \qquad z \in \mathbb{R}^{n+1},\ \omega \in S_z^n \tag{2.1}$$

As usual, $\exp_z(0) = z$ and $\partial_\lambda \exp_z(0) = \omega$. Abusing notation, we define the graph over the sphere S_z^n (or the corresponding geodesic sphere) as (cf. (1.5))

$$\theta_{\rho,z}(\omega) = \exp_z(\rho(\omega)\omega), \qquad \rho(\omega) = \lambda(1 + \phi(\omega)) \tag{2.2}$$

We introduce the topology on the space of graphs over S_z^n as follows. We say the graph $\psi' = \theta_{\rho',z'}$ is close to $\psi = \theta_{\rho,z}$ iff it can be written as $\psi'(\omega) = \exp_{\psi(\omega)}(\xi(\omega)\omega)$, with ξ sufficiently small in an appropriate norm (H^k).

2.2. **The Lyapunov-Schmidt map.** Let dH denote the Gâuteaux derivative of a map H ($dH(\psi)\eta := \partial_s H(\psi_s)\big|_{s=0}$ for ψ_s with $\psi_{s=0} = \psi$ and $\partial_s \psi_s\big|_{s=0} = \eta$) and let

$$\hat{H}(\theta) = H(\theta) - \bar{H}(\theta) \tag{2.3}$$

Lemma 2.1. *Let $\rho(\omega) = \lambda(1 + \phi(\omega))$. Then the Gâteaux derivative, $L_{0,z} := d_\phi\big(\lambda\hat{H}(\theta_{\rho,z})\big)\big|_{\phi=0,\lambda=0}$, of $\hat{H}$ at $\theta_{\lambda,z}$ equals*

$$L_{0,z}u = -(\Delta_{\mathbb{S}^n} + n)u + \frac{n}{|\mathbb{S}^n|}\int_{\mathbb{S}^n} u \tag{2.4}$$

Moreover, $L_{0,z}$ is self-adjoint on $L^2(\mathbb{S}^n)$, its spectrum is discrete and non-negative and its kernel is spanned by constants and the coordinate functions ω^i for $\omega = (\omega^1, \dots, \omega^{n+1}) \in \mathbb{S}^n$.

Proof. The formula (2.4) can be read off from the expansion (A.3) of the mean curvature $H(\theta_{\rho,z})$, included in Appendix A. It also follows from Proposition 2.5 of [9].

The spectral properties of $L_{0,z}$ are part of the standard theory. □

We will look for a solution of (1.3) of the form (2.2) with

$$\rho(\omega) = \lambda\big(1 + \phi(\omega)\big), \qquad \phi \perp \omega^i, \ i = 0, \dots, n+1, \ \omega^0 := 1. \tag{2.5}$$

The conditions on ϕ imply $\int_{\mathbb{S}^n} \phi = 0$ and therefore

$$\lambda = \langle \rho \rangle = \frac{\int_{\mathbb{S}^n} \rho}{|\mathbb{S}^n|}.$$

(By Lemma B.2, with $\phi_{\lambda,z} = 0$, any $\psi = \theta_{\rho,z}$ sufficiently close to a geodesics sphere is of the form (2.2), with (2.5).)

Definition 2.2. Let P_0 be the $L^2(\mathbb{S}^n)$-orthogonal projection onto $\mathrm{Null}\, L_{0,z}$, which is the span of the eigenvectors $1, \omega^1, \dots, \omega^{n+1}$ of $-\Delta_{\mathbb{S}^n} - n$, with the eigenvalues $-n, 0$, and let $\bar{P}_0 := 1 - P_0$, the orthogonal projection onto $\big(\mathrm{Null}\, L_{0,z}\big)^{\perp}$.

Consider the equation for $\phi \in \bar{P}_0 H^k$, $k > \frac{n}{2} + 2$,

$$F(\phi, \lambda, z) = 0, \quad \text{where} \quad F(\phi, \lambda, z) := \bar{P}_0 \lambda \hat{H}(\theta_{\lambda(1+\phi),z}) \tag{2.6}$$

Remark 2.3. We observe that the function $\lambda \hat{H}(\theta_{\lambda(1+\phi),z})$ has by definition zero mean, i.e., it is L^2-orthogonal to 1. Hence, we need only project it onto the L^2-orthogonal complement of the span of $\omega^1, \dots, \omega^{n+1}$, but we will keep using the projection operator $\bar{P}_0$ for consistency with other parts of the proof.

Proposition 2.4. *For λ sufficiently small, equation (2.6) has a unique solution $\phi = \phi_{\lambda,z}$ in a small ball in $\bar{P}_0 H^k$, for $k > \frac{n}{2} + 2$, which satisfies the estimate*

$$\|\partial_\lambda^r \partial_z^\alpha \phi_{\lambda,z}\|_{H^k} \lesssim \lambda^{2-r}, \qquad r + |\alpha| \leq 2 \tag{2.7}$$

Proof. The proof is an application of the inverse function theorem. We have the following properties:
(i) The map $F : \bar{P}_0 H^k \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \bar{P}_0 H^{k-2}$ is smooth;
(ii) $F(0, 0, z) = 0$,
(iii) $d_\phi F(0, 0, z)$ is invertible.

(i) and (ii) are standard and follow from Lemma A.1 of Appendix A.

For (iii) we mention that, since $L_{0,z}$ is self-adjoint and has purely discrete spectrum and $\mathrm{Null} L_0$ =Range$(1 - \bar{P}_0)$, the operator $d_\phi F(0, 0, z) = \bar{P}_0 L_{0,z}$ is invertible on Ran $\bar{P}_0$. Hence, by the IFT there exists a unique solution ϕ to (2.6) for every λ sufficiently small.

To prove the estimates (2.7), we use the expansion

$$\lambda\hat{H}(\theta_{\lambda(1+\phi),z}) = \lambda\hat{H}(\theta_{\lambda,z}) + L_{\lambda,z}\phi + N_{\lambda,z}(\phi), \tag{2.8}$$

where $L_{\lambda,z} := d_\phi(\lambda\hat{H}(\theta_{\rho,z}))\big|_{\phi=0}$ and $N_{\lambda,z}(\phi)$ is defined by this expression.

Let $\bar{L}_{\lambda,z} = \bar{P}_0 L_{\lambda,z}\bar{P}_0$. We would like to use the above expansion and invert the operator $\bar{L}_{\lambda,z}$ and rewrite equation $F(\phi,\lambda,z) = 0$ as the fixed point problem, which we use to estimate ϕ. This is done with the help of the following lemma.

Lemma 2.5. *For λ sufficiently small, the operator $\bar{L}_{\lambda,z}$ is invertible and we have the following estimates*

$$\|\partial_\lambda^m\partial_z^\alpha \bar{L}_{\lambda,z}^{-1}\phi\|_{H^k} \lesssim \|\phi\|_{H^{k-2}}, \tag{2.9}$$

$$\|\partial_\lambda^m\partial_z^\alpha F(0,\lambda,z)\|_{H^k} \lesssim \lambda^{2-m}, \tag{2.10}$$

$$\|\partial_\lambda^m\partial_z^\alpha N_{\lambda,z}(\phi)\|_{H^{k-2}} \lesssim \|\phi\|_{H^k}^2 \tag{2.11}$$

and a similar estimate for $N_{\lambda,z}(\phi') - N_{\lambda,z}(\phi)$, for $m + |\alpha| \le 2$.

Proof. We write

$$L_{\lambda,z} = L_{0,z} + \lambda^2 M_{\lambda,z}, \tag{2.12}$$

where $M_{\lambda,z}$ is defined by this expression. By Lemma A.1, we have that

$$\|(L_{0,z}+1)^{-1}M_{\lambda,z}\|_{H^k\to H^k} \lesssim 1. \tag{2.13}$$

By (2.13), $\|\bar{L}_{0,z}^{-1}\lambda^2\bar{M}_{\lambda,z}\|_{H^k\to H^k} < 1$ for λ sufficiently small and therefore $\bar{L}_{\lambda,z}$ is invertible.

Next, let $\beta := (m,\alpha), w := (\lambda, z)$ and $\partial_w^\beta := \partial_\lambda^m\partial_z^\alpha$. Using the relation $\partial L^{-1} = -L^{-1}\partial L L^{-1}$, we see that we have that

$$\|\partial_w^\beta \bar{L}_{\lambda,z}^{-1}\|_{H^{k-2}\to H^k} = \sum\prod \|\partial_w^{\beta_i}(\lambda^2\bar{L}_{\lambda,z})\bar{L}_{\lambda,z}^{-1}\|_{H^k\to H^k}\|\bar{L}_{\lambda,z}^{-1}\|_{H^{k-2}\to H^k}^{r_i}$$

where the sum is taken over all partitions of $\beta := (m,\alpha)$ into smaller integers and the product over the integers in the given partition which give all the ways of distributing the partial ∂_w^β according to $\partial L^{-1} = -L^{-1}\partial L L^{-1}$. This gives (2.9).

Now, bounds (2.10) and (2.11) follow by comparing the expansion (2.8) with (A.3) of Appendix A. (2.10), follows from (A.3) with $\phi = 0$ and (2.11), from (A.6). □

Now, using (2.8), we can rewrite equation (2.6) as the fixed point problem

$$\phi = -L_{\lambda,z}^{-1}(F(0,\lambda,z) + \bar{P}_0 N_{\lambda,z}(\phi)). \tag{2.14}$$

Lemma 2.5 implies that this equation has a unique fixed point, $\phi = \phi_{\lambda,z}$, in a small ball in $\bar{P}_0 H^k$. Assuming that ϕ lies in a small ball in $\bar{P}_0 H^k$ of radius $\varepsilon > 0$, we see that the bounds in (2.7) follow from (2.14) and Lemma 2.5. Indeed, we derive

$$\begin{aligned} \|\phi\|_{H^k} &\leq \|\bar{L}_{\lambda,z}^{-1}(F(0,\lambda,z)\|_{H^k} + \|\bar{L}_{\lambda,z}^{-1}\bar{P}_0 N_{\lambda,z}(\phi)\|_{H^k} \\ &\lesssim \lambda^2 + \|N_{\lambda,z}(\phi)\|_{H^{k-2}} \\ &\lesssim \lambda^2 + \|\phi\|_{H^k}^2, \end{aligned} \tag{2.15}$$

which implies estimate (2.7), with $r = 0, \alpha = 0$, provided $\|\phi\|_{H^k}$ is sufficiently small. The estimates for the $\partial_\lambda^r \partial_z^\alpha$ derivatives of ϕ are obtained similarly using the fixed point equation and estimates in Lemma 2.5. Indeed, running through the same argument as equation (2.15), gives (2.7). □

Define the *Lyapunov-Schmidt* map, $\Phi_\lambda(z)$, as

$$\Phi_\lambda(z) = \theta_{\rho_{\lambda,z},z}, \quad \text{where} \quad \rho_{\lambda,z} := \lambda(1+\phi_{\lambda,z}), \ \phi_{\lambda,z} \in \bar{P}_0 H^k \ \text{ solves (2.6)}. \tag{2.16}$$

Note that $\Phi_\lambda(z)$ is a perturbation of the classical exponential map.

2.3. **Proof of Theorem 1.1.** We begin with part (i). Let $\psi_{\lambda,z} := \Phi_\lambda(z)$. If $\psi_{\lambda,z}$, with $\lambda = \lambda(h)$, solves (1.3), then, by Proposition 1.6, $dA_h(\psi_{\lambda,z}) = 0$, for $h = h(\lambda)$. Furthermore,

$$\partial_{z^i} A_h(\psi_{\lambda,z}) = dA_h(\psi_{\lambda,z})\partial_{z^i}\psi_{\lambda,z} \tag{2.17}$$

for every i. Hence, $\partial_{z^i} A_h(\psi_{\lambda,z}) = 0$, i.e. z is a critical point of $A_h(\psi_{\lambda,z})$.

Conversely, if $\partial_{z^i} A_h(\psi_{\lambda,z}) = 0$, we will argue that $dA_h(\psi_{\lambda,z}) = 0$. Define

$$\partial_{z^i}^N \psi_{\lambda,z} := g(\partial_{z^i}\psi_{\lambda,z}, \nu(\psi_{\lambda,z})). \tag{2.18}$$

Note that the superscript N in $\partial_z^N \psi_{\lambda,z}$ depends on $\psi_{\lambda,z}$.

In the case of the geodesic spheres we have $\partial_{z^i}\theta_{\lambda,z}\big|_{\lambda=0} = \partial_{z^i}\exp_z(\lambda\omega)\big|_{\lambda=0} = d\exp_z(0)(\partial_{z^i} z) = e_i$, where $\{e_i\}$ is the canonical basis in $\mathbb{R}^{n+1}$. Therefore $\partial_{z^i}^N\theta_{\lambda,z}\big|_{\lambda=0} := g(e_i,\omega) = \omega_i$, $i = 1,\dots n+1$, which span the null space of $L_{0,z}$ and P_0 can be written as

$$P_0 f = g_0^{ij}\langle f, \partial_{z^i}^N\theta_{\lambda,z}\big|_{\lambda=0}\rangle_{L^2}\partial_{z^j}^N\theta_{\lambda,z}\big|_{\lambda=0}, \tag{2.19}$$

where g_0^{ij} is the inverse matrix of $(g_0)_{ij} := \langle \partial_{z^i}^N\theta_{\lambda,z}\big|_{\lambda=0}, \partial_{z^j}^N\theta_{\lambda,z}\big|_{\lambda=0}\rangle_{L^2} = \big(\int_{\mathbb{S}^n}\omega_1^2\big)\delta_{ij}$.

We define the normal L^2 gradient, $\text{grad}^N G(\psi)$, of a functional $G(\psi)$ by

$$\int_{\mathbb{S}^n} \text{grad}^N G(\psi)\xi = dG(\psi)\xi,$$

for any normal variation ξ. Then $\text{grad}^N A_h(\psi) = H(\psi) - h$. By definition (2.16) and (2.6), $\bar{P}_0\text{grad}^N A_h(\psi_{\lambda,z}) = 0$. Hence

$$P_0\text{grad}^N A_h(\psi_{\lambda,z}) = \text{grad}^N A_h(\psi_{\lambda,z}) \tag{2.20}$$

Remembering (2.18), we define the projection on the span of the non L^2-orthogonal set $\{\partial^N_{z^i}\psi_{\lambda,z}\}$:

$$P_\lambda f := g^{ij}_\lambda \langle \partial^N_{z^i}\psi_{\lambda,z}, f\rangle_{L^2}\partial^N_{z^j}\psi_{\lambda,z}, \tag{2.21}$$

where g^{ij}_λ is the inverse matrix of $(g_\lambda)_{ij} := \langle \partial^N_{z^i}\psi_{\lambda,z}, \partial^N_{z^j}\psi_{\lambda,z}\rangle_{L^2}$. (For simplicity, we do not display the dependence on z.) Hence, by the assumption $\partial_{z^i} A_h(\psi_{\lambda,z}) = 0$, we have

$$\begin{aligned} P_\lambda \mathrm{grad}^N A_h(\psi_{\lambda,z}) &= g^{ij}_\lambda \langle \partial^N_{z^i}\psi_{\lambda,z}, \mathrm{grad}^N A_h(\psi_{\lambda,z})\rangle_{L^2}\partial^N_{z^j}\psi_{\lambda,z} \\ &= g^{ij}_\lambda \partial^N_{z^i} A_h(z)\partial^N_{z^j}\psi_{\lambda,z} = 0. \end{aligned} \tag{2.22}$$

Lemma 2.6. *The difference of the projections P_0, P_λ satisfies the estimate*

$$\|P_0 - P_\lambda\|_{L^2\to L^2} \lesssim \lambda(1 + \|\phi_{\lambda,k}\|_{H^k} + \lambda\|\partial_\lambda\phi_{\lambda,z}\|_{H^k}) \tag{2.23}$$

for $k > n/2 + 1$.

Proof. We show that the eigenfunctions of two projections are close. Let $f(\lambda) := \partial^N_{z^i}\theta_{\rho_{\lambda,z},z}$, with $\rho_{\lambda,z} := \lambda(1 + \phi_{\lambda,z})$. For ω fixed, we note that $f(\lambda)(\omega)$ is smooth in λ, $\phi_{\lambda,z}(\omega)$, and $\partial_z\phi_{\lambda,z}(\omega)$. Notice that $\partial_{z^i}\theta_{\rho_{\lambda,z},z}\mid_{\lambda=0} = \partial_{z^i}\theta_{\lambda,z}\mid_{\lambda=0}$. Then

$$\begin{aligned} &\|\partial^N_{z^i}\theta_{\lambda,z}\mid_{\lambda=0} - \partial^N_{z^i}\theta_{\rho_{\lambda,z},z}\|_{H^{k-1}} \\ &= \|f(\lambda) - f(0)\|_{H^{k-1}} \lesssim \lambda \int_0^1 \|f'(t\lambda)\|_{H^{k-1}}\, dt \\ &\lesssim \lambda(1 + \|\phi_{\lambda,k}\|_{H^k} + \lambda\|\partial_\lambda\phi_{\lambda,z}\|_{H^k}) \end{aligned}$$

The last inequality and definition (2.21) imply (2.23). □

Combining now (2.20), (2.22) and (2.23) and using

$$\mathrm{grad}^N A_h(\psi_{\lambda,z}) = P_0 \mathrm{grad}^N A_h(\psi_{\lambda,z}) = (P_0 - P_\lambda)\mathrm{grad}^N A_h(\psi_{\lambda,z}),$$

we derive

$$\begin{aligned} \|\mathrm{grad}^N A_h(\psi_{\lambda,z})\|_{L^2} \leq& \|P_0 - P_\lambda\|_{L^2} \cdot \|\mathrm{grad}^N A_h(\psi_{\lambda,z})\|_{L^2} \\ \lesssim& \lambda(1 + \|\phi_{\lambda,k}\|_{H^k} + \lambda\|\partial_\lambda\phi_{\lambda,z}\|_{H^k})\|\mathrm{grad}^N A_h(\psi_{\lambda,z})\|_{L^2} \end{aligned}$$

Thus, for λ sufficiently small it follows that $\mathrm{grad}^N A_h(\psi_{\lambda,z}) = 0$ and therefore $dA_h(\psi_{\lambda,z})\xi = 0$ for any normal variation ξ. Since $dA_h(\psi_{\lambda,z})\xi = 0$ for any tangential variation ξ, we have $dA_h(\psi_{\lambda,z}) = 0$.

Finally, we show that for λ sufficiently small, the map $\lambda \to h(\lambda) := H(\Phi_\lambda(z))$ is invertible. Using (2.8) and an expansion of $H(\theta_{\lambda,z})$ in λ (see also (A.3)), we find that

$$\lambda h = n + O(\lambda^2 + \|\phi\|_{H^k}) \tag{2.24}$$

$$= n + O(\lambda^2) \tag{2.25}$$

This and a similar estimate for the derivative in λ, together with the IFT, show that $\lambda \to h(\lambda)$ is invertible.

To prove part (ii) we let $A_h''(\psi) := d\,\mathrm{grad}\, A_h(\psi)$ (the hessian of $A_h(\psi)$) and use (2.17) to compute the second derivative of $A_h(\Phi_\lambda(z))$:

$$\begin{aligned}\partial_{z_i}\partial_{z_j} A_h(\Phi_\lambda(z)) =&\langle A_h''(\Phi_\lambda(z))\partial_{z_j}\Phi_\lambda(z), \partial_{z_i}\Phi_\lambda(z)\rangle_{L^2}\\ &+ \langle \mathrm{grad} A_h(\Phi_\lambda(z)), \partial_{z_i}\partial_{z_j}\Phi_\lambda(z)\rangle_{L^2}\end{aligned} \tag{2.26}$$

By (i), evaluating (2.26) at a critical point z_* of $A_h(\Phi_\lambda(z))$ we have

$$\partial_{z_i}\partial_{z_j} A_h(\Phi_\lambda(z_*)) = \langle A_h''(\Phi_\lambda(z_*))\partial_{z_j}\Phi_\lambda(z_*), \partial_{z_i}\Phi_\lambda(z_*)\rangle_{L^2} \tag{2.27}$$

As before the one direction is obvious: if $A_h''(\Phi_\lambda(z_*))$ is positive, then so is $\partial_z^2 A_h(\Phi_\lambda(z_*))$.

We show that $\partial_z^2 A_h(\Phi_\lambda(z_*)) \geq 0$ (> 0) implies $A_h''(\Phi_\lambda(z_*)) \geq 0$ (> 0). Abbreviate $A_h'' \equiv A_h''(\Phi_\lambda(z_*))$. By Proposition 1.5, $A_h'' = d_\psi \hat{H}(\psi)\big|_{\psi=\Phi_\lambda(z_*)}$ and therefore by the definition $L_{\lambda,z} := d_\phi(\lambda \hat{H}(\theta_{\rho,z}))\big|_{\phi=0}$, we have that $\lambda^2 A??_h = L_{\lambda,z_*} + O(\lambda^3)$ **(check!)** and by relation (2.12),

$$\lambda^2 A??_h = L_{0,z_*} + \lambda^2 M_{\lambda,z_*} + O(\lambda^3) \tag{2.28}$$

This relation and $\bar{P}_0 L_{0,z}\bar{P}_0 \gtrsim \bar{P}_0$ give $\bar{P}_0 A_h'' \bar{P}_0 \gtrsim \bar{P}_0$. This and (2.23) imply $\bar{P}_\lambda A_h'' \bar{P}_\lambda \gtrsim \bar{P}_\lambda$.

Due to (2.12) and (2.23), the cross-terms $\bar{P}_\lambda A_h'' P_\lambda$ and $P_\lambda A_h'' \bar{P}_\lambda$ are $O(\lambda)$. These estimates can be improved as follows. We recall from (2.20) that $\bar{P}_0 \mathrm{grad}^N A_h(\psi_{\lambda,z}) = 0$ for all z. Hence

$$\bar{P}_0 A_h'' P_\lambda = 0. \tag{2.29}$$

Using this identity, we see that

$$\begin{aligned}\bar{P}_\lambda A_h'' P_\lambda =&(\bar{P}_\lambda - \bar{P}_0)A_h'' P = (P_0 - P_\lambda)A_h'' P_\lambda\\ =&(P_0 - P_\lambda)P_\lambda A_h'' P_\lambda + (P_0 - P_\lambda)\bar{P}_\lambda A_h'' P_\lambda.\end{aligned}$$

Due to (2.23), this equation can be solved for $\bar{P}_\lambda A_h'' P_\lambda$, for λ sufficiently small, as $\bar{P}_\lambda A_h'' P_\lambda = S P_\lambda A_h'' P_\lambda$, where $S = [1 - (P_0 - P_\lambda)]^{-1}(P_0 - P_\lambda)$, which gives

$$\bar{P}_\lambda A_h'' P_\lambda = \bar{P}_\lambda S P_\lambda A_h'' P_\lambda =: \bar{P}_\lambda S B. \tag{2.30}$$

The definitions (2.21) and $\psi_{\lambda,z} := \Phi_\lambda(z)$ imply

$$\langle \varphi, P_\lambda A_h'' P_\lambda \varphi\rangle = \sum \bar{v}^i \langle \partial_{z^i}^N \psi_{\lambda,z}, A_h'' \partial_{z^k}^N \psi_{\lambda,z}\rangle v^k,$$

where $v^i := \sum_j g^{ij}\langle \partial_{z^j}^N \psi_{\lambda,z}, \varphi\rangle$. The latter relation at $z = z_*$, equality (2.27) (since the tangential derivatives are zero modes of A_h'' one can replace $\partial_{z^i}\psi_{\lambda,z}$ in (2.27) by $\partial_{z^i}^N \psi_{\lambda,z}$) and the assumption that $\partial_z^2 A_h(\Phi_\lambda(z_*)) \geq 0$ (> 0) imply that $B = P_\lambda A_h'' P_\lambda \geq 0$ (> 0). Using this, (2.30) and that $\|S\| \lesssim \lambda$, we find

$$\begin{aligned}|\langle \varphi, \bar{P}_\lambda S B \varphi\rangle| \leq&\|B^{1/2} S \bar{P}_\lambda \varphi\|\|B^{1/2}\varphi\|\\ \leq&\lambda C\|B^{1/2} Q\|\|\bar{P}_\lambda \varphi\|\|B^{1/2}\varphi\|,\end{aligned}$$

for some constant $C \lesssim 1$, where Q be the finite rank orthogonal projection onto $\mathrm{Ran}P_0 + \mathrm{Rran}P_\lambda$. This gives

$$\begin{aligned} 2|\langle\varphi, \bar{P}_\lambda A_h'' P_\lambda\varphi\rangle| \le& 2\lambda C\|\bar{P}_\lambda\varphi\|\|B^{1/2}\varphi\| \\ \le& \lambda^{2(1-\epsilon)}C\|\bar{P}_\lambda\varphi\|^2 + \lambda^{2\epsilon}\|B^{1/2}\varphi\|^2 \end{aligned}$$

which implies that

$$\begin{aligned} \langle\varphi, A_h''\varphi\rangle =& \langle\varphi, B\varphi\rangle + \langle\varphi, \bar{P}_\lambda A_h''\bar{P}_\lambda\varphi\rangle + 2\,\mathrm{Re}\langle\varphi, \bar{P}_\lambda A_h'' P_\lambda\varphi\rangle \\ \ge& \langle\varphi, B\varphi\rangle + c\|\bar{P}_\lambda\varphi\|^2 - \lambda^{2(1-\epsilon)}C\|\bar{P}_\lambda\varphi\|^2 - \lambda^{2\epsilon}\|B^{1/2}\varphi\|^2 \end{aligned}$$

for some $c \gtrsim 1$. Hence $\langle\varphi, A_h''\varphi\rangle \ge 0$ (> 0) provided λ is sufficiently small. This proves the assertion and with it part (ii).

Finally, we prove part (iii). By the definition, the unstable part of statement (iii) follows directly from statement (ii) and a standard argument. To prove the stability of minimizing CMC's, we use Theorem 1.4 as follows.

Proof of (iii)(stability). By (1.9), $\lambda \to \lambda_*$ for some λ_* as $t \to \infty$. If z_* is a strict local minimum point of $A(\Phi_\lambda(z))$ on $\{V_{\mathrm{enc}}(\Phi_\lambda(z)) = c\}$, then $z(t) \to z_*$ and therefore, by (1.7) and (1.10), $\psi(t) = \theta_{\xi(t),\psi_{\lambda(t),z(t)}} \to \theta_{0,\psi_{\lambda_*,z_*}} = \psi_{\lambda_*,z_*}$. This shows that the surface ψ_{λ,z_*} is asymptotically stable.

If z_* is a saddle or local maximum point, then the standard argument and (1.8) show that $z(t)$ stays away from z_* however close the initial condition z_0 is to z_*. This implies the same for $\psi(t)$ w.r.to ψ_{λ_*,z_*}. □

Note that if z_* is a non-degenerate local minimum point of $A(\Phi_\lambda(z))$ on $\{V_{\mathrm{enc}}(\Phi_\lambda(z)) = c\}$, then $z(t) \to z_*$ and therefore $\psi(t) \to \psi_{\lambda_*,z_*}$ exponentially fast. Indeed, the proof follows from (1.8) in a standard way. Indeed, let a_h'' denote the hessian of a_h. Passing to local coordinates, one expands

$$\nabla a_h(z) = a_h''(z_*)(z - z_*) + O(|z - z_*|^2) \tag{2.31}$$

around z_* and uses the Duhamel principle to rewrite (1.8) as the fixed point problem $\beta = \Psi(\beta)$, where $\beta(t) = z(t) - z_*$ and

$$\Psi(\beta)(t) := e^{-a_h''(z_*)t}\beta(0) + \int_0^{+\infty} e^{-a_h''(z_*)(t-s)}\big[O(\beta^2(s)) + O(\varepsilon^2 e^{-\delta s})\big]ds. \tag{2.32}$$

If $a_h''(z_*) \ge \alpha > 0$, then the standard fixed point argument in the space $X := C([0, +\infty), e^{-\alpha t}\mathbb{R}^{n+1})$ gives $\sup_t(e^{\alpha t}\|\beta(t)\|) \lesssim 1$ and therefore $|z(t) - z_*| \lesssim e^{-\alpha t}$, which implies that $\psi(t) = \theta_{\xi(t),\psi_{\lambda(t),z(t)}} \to \theta_{0,\psi_{\lambda_*,z_*}} = \psi_{\lambda_*,z_*}$ exponentially fast, giving the result.

3. Adiabatic dynamics: Proof of Theorem 1.4

Proof of Theorem 1.4. Let $\psi(\omega) : \mathbb{S}^n \to M$ be an immersion of the form $\psi(\omega) = \theta_{\rho,z'}(\omega) := \exp_{z'}(\rho(\omega)\omega)$ for some ρ and z'. We assume ψ close to some CMC surface $\psi_{\lambda',z'} := \Phi_{\lambda'}(z') = \theta_{\lambda'(1+\phi_{\lambda',z'}),z'}$ (in the sense of topology induced by ρ and z', see Subsection 2.1). We look for ρ in the form

$$\rho \equiv \rho(\xi) = \lambda(1 + \phi_{\lambda,z} + \xi), \tag{3.1}$$

for some λ and z close to λ' and z'. We note that ξ depends on λ, z, but we drop this dependence in our notation.

We denote $z^0 := \lambda$ and define

$$\sigma(\xi)(\omega) := g(\nu(\theta_{\rho,z}(\omega)), \partial_s\theta_{s,z}(\omega) \mid_{s=\rho}), \tag{3.2}$$

$$f_i(\xi)(\omega) := g(\nu(\theta_{\rho,z}(\omega)), \partial_{z^i}\theta_{\rho,z}(\omega)), \; i = 0, 1, ..., n+1. \tag{3.3}$$

For $\xi = 0$, we omit the argument 0, e.g. we denote $f_i(0) = f_i$, etc.

Having in mind that f_i and σ depend on λ and z, we define the *barycenter* of $\psi = \theta_{\rho,z'}$ to be (λ, z) that solve the equation

$$\langle f_i, \sigma\xi\rangle = 0 \text{ for } i = 0, ..., n+1. \tag{3.4}$$

By Lemma B.2, a unique solution exists provided ψ is sufficiently near some CMC surface $\psi_{\lambda',z'}$.

Suppose that $\psi(\omega, t) = \theta_{\rho(t),z(t)}(\omega)$ is a solution to VPF with some initial condition $\psi_0 = \theta_{\rho_0,z_0}$ near a CMC surface $\psi_{\lambda',z'}$ and let $\lambda(t), z(t)$ be its barycenter. Then we see that

$$\partial_t^N\psi = f_i(\xi)\dot z^i + \lambda\sigma(\xi)\dot\xi. \tag{3.5}$$

Our first task is to obtain the effective equation for λ and z. We begin with rewriting (1.1) as

$$\partial_t^N\psi = -\nabla^{N(\psi)}A(\psi)$$

on X_c^k and expand the r.h.s. in ξ and divide the resulting equation by $(\sigma(\xi)/\sigma) = 1 + O_{H^k}(\xi)$. Using (3.5), the definition $\psi_{\lambda,z} := \Phi_\lambda(z) = \theta_{\lambda(1+\phi_{\lambda,z}),z}$ and Lemma B.1, we can rewrite our equation $\partial_t^N\psi = -\nabla^{N(\psi)}A(\psi)$ on X_c^k as

$$\tilde f_j(\xi)\dot z^j + \lambda\sigma\dot\xi = -\nabla^{N(\psi_{\lambda,z})}A(\psi_{\lambda,z}) + \lambda L_{\lambda,z}\xi + \lambda^2 O_{H^k}(\xi^2) + \lambda O_{H^k}(|\dot\xi|\xi), \tag{3.6}$$

for $z \in \chi_c$, where $\tilde f_j(\xi) := f_j(\xi)\sigma/\sigma(\xi) = f_j + O_{H^k}(\xi)$ and $L_{\lambda,z} := \lambda d_\phi \nabla^{N(\theta_{\rho,z})}A(\theta_{\rho,z})|_{\rho=\rho_{\lambda,z}}$ $(\rho = \lambda(1+\phi))$. Now we multiply the equation by f_i, $i = 0, ..., n+1$, and divide the resulting equation by $(h(\xi)/h) = 1 + O_{H^k}(\xi)$, where the matrices h and $h(\xi)$ have the entries $h_{ij} = \langle f_i, f_j\rangle$ and $h_{ij}(\xi) = \langle f_i, \tilde f_j(\xi)\rangle$, with $i, j = 0, ..., n+1$. We have

$$\begin{aligned} h_{ij}\dot z^j = &- \langle f_i, \nabla^{N(\psi_{\lambda,z})}A(\psi_{\lambda,z})\rangle - \lambda\langle f_i, \sigma\dot\xi\rangle - \lambda\langle f_i, L_{\lambda,z}\xi\rangle \\ &+ \lambda^2 O_{H^k}(\xi^2) + \lambda O_{H^k}(|\dot\xi|\xi) + O_{H^k}(|\dot z|\xi). \end{aligned} \tag{3.7}$$

By Lemma B.1, we see that $h_{ij} \approx \mathrm{id}$ is invertible. Moreover, by the choice of the barycenter, we see that

$$\langle f_i, \sigma\dot\xi\rangle = -\langle\partial_t(f_i\sigma), \xi\rangle = O_{H^k}(|\dot z|\xi). \tag{3.8}$$

Set $v_i = \langle f_i, \nabla^{N(\psi_{\lambda,z})} A(\psi_{\lambda,z})\rangle$. Since $\partial_{z^j} A(\psi_{\lambda,z}) = dA(\psi_{\lambda,z})\partial_{z^j}\psi_{\lambda,z} = \int g(H\nu, \partial_{z^j}\psi_{\lambda,z}) = \int Hf_j + O_{H^k}(\xi)$ since $H = O(\lambda^{-1})$ (c.f. Lemma A.1), we have $v_j = \partial_{z^j} v$. This and the fact that f_j are almost the null eigenvectors of the linear part $L_{\lambda,z}$ imply (after solving for $\dot z$)

$$\dot z = -h^{ij} v_j + \lambda^2 O_{H^k}(\xi) = -\nabla v + \lambda^2 O_{H^k}(\xi) \tag{3.9}$$

where $-\nabla v = -h^{ij}\partial_{z^j} v$ for $z \in \chi_c$.

Now to get equation for ξ, let P denote the orthogonal projection onto $\mathrm{Span}\{f_i\}$ and let $\bar P = \mathbf{1} - P$. Applying $\bar P$ to (3.6) and using $\bar P f_i = 0$, we derive

$$\lambda \bar P(\sigma\dot\xi) = -\bar P \nabla^{N(\psi_{\lambda,z})} A(\psi_{\lambda,z}) - \lambda L_{\lambda,z}\xi - \bar P(\tilde f_i(\xi) - f_i)\dot z^i + \lambda^2 O_{H^k}(\xi^2) \tag{3.10}$$

By the choice of barycenter again (see (3.4)),

$$\begin{aligned} \bar P(\sigma\dot\xi) =& \sigma\dot\xi - P(\sigma\dot\xi) \\ =& \sigma\dot\xi + \partial_t(P\sigma)\xi \end{aligned} \tag{3.11}$$

So we get

$$\dot\xi = -\,\sigma^{-1} L_{\lambda,z}\xi + Rem, \tag{3.12}$$

$$\begin{aligned} Rem :=& -\lambda^{-1}\sigma^{-1}\big(\bar P \nabla^{N(\psi)} A(\psi) + \lambda\partial_t(P\sigma)\xi \\ &+ \bar P(f_i(\xi) - f_i)\dot z^i + \lambda^2 O_{H^k}(\xi^2)\big) \end{aligned} \tag{3.13}$$

Now, using equations (3.9) and (3.12), we estimate ξ. To this end, following [3], we use the Lyapunov functionals:

$$\Lambda_k(\xi) := \frac{1}{2}\langle \xi, (-\Delta - n)^k \xi\rangle. \tag{3.14}$$

These functionals satisfy the inequalities

$$\Lambda_k(\xi) \geq (n+2)\Lambda_{k-1}(\xi), \qquad C\|\xi\|^2_{H^k} \geq \Lambda_k(\xi) \geq c\|\xi\|^2_{H^k} \tag{3.15}$$

for some $C, c > 0$. As in [3], one uses the equation on ξ, (3.12), to obtain differential inequalities for $\Lambda_k(u)$, which together with (3.15) imply the bound (1.10) completing the proof of the theorem. □

4. Proof of Theorem 1.3: existence and linear stability/instability

Theorem 1.3(iii) follows readily from Theorem 1.1(iii). Thus we address only the first two statements.

4.1. **Existence of foliations.**

By Theorem 1.1(i), to prove existence of the CMC's, we have to prove existence of critical points of the function $A_h(\Phi_\lambda(a)) := A(\Phi_\lambda(a))-$

$hV_{enc}(\Phi_\lambda(a))$. To this end, we use the following expansion which follows from Corollary A.2 of Appendix A (see [24], Lemma 2.2):

$$A_h(\Phi_\lambda(z)) = \lambda^n \bigg(\frac{a_n}{n+1}\big[1+n-h\lambda+(\frac{h\lambda}{6(n+3)}-\frac{1}{6})R(z)\lambda^2\big] + \lambda^3 \mathrm{B}^{(0)}_{\lambda,z}(\phi) + \lambda^2 \mathrm{B}^{(2)}_{\lambda,z}(\phi)\bigg) \tag{4.1}$$

where a_n denotes the area of the Euclidean unit sphere $\mathbb{S}^n$, R is the scalar curvature of M and the remainders $\mathrm{B}^{(j)}_{\lambda,z}(\phi)$, $j = 0, 2$, satisfy the estimates

$$|\partial^i_\lambda \partial^\alpha_z \mathrm{B}^{(\mathrm{r})}_{\lambda,\mathrm{z}}(\phi)| \lesssim \sum_{\mathrm{j}\le \mathrm{i}, \beta\le\alpha} \|\partial^{\mathrm{j}}_\lambda \partial^\alpha_{\mathrm{z}} \phi\|^{\mathrm{r}}_{\mathrm{H}^{\mathrm{k}}} \tag{4.2}$$

provided $\|\phi\|_{H^k} \lesssim 1$ and $i + |\alpha| \le 2$. (If ϕ is independent of λ and z, then only the term with $j = |\beta| = 0$ survives on the r.h.s..)

Utilizing the estimate (2.7) and (4.1), the equations $\partial_{z_j} A_h(\Phi_\lambda(z)) = 0$, $j = 1, \dots, n+1$ take the form

$$\nabla_z R(z) + O(\lambda^2) = 0 \tag{4.3}$$

If λ is sufficiently small, the preceding equation can be solved for z near a non-degenerate critical point z_* of R by employing the implicit function theorem. The latter yields a solution z satisfying

$$z = z_* + O(\lambda^2) \tag{4.4}$$

Since $\Phi_\lambda(z)$ solves (2.6) and $\partial_z A_h(\Phi_\lambda(z)) = 0$, by Theorem 1.1 ($i$), it is also a solution of (1.3).

Equation (1.3) and expansion (A.3) imply the relation between h and λ:

$$h = \frac{n}{\lambda} + O(\lambda). \tag{4.5}$$

Now we show that the solutions $\Phi_\lambda(z_*)$ actually foliate the neighborhood of z_*. We exhibit a local homeomorphism from a neighborhood of z_* to in $\mathbb{R}^{n+1}$ that maps each CMC surface into a spherical shell centred at the origin. We simply use the map (2.16):

$$\begin{aligned}\Phi :&[0,\epsilon) \times \mathbb{S}^n \to M,\\ &(\lambda,\omega) \mapsto \Phi_\lambda(z_*).\end{aligned}$$

where, recall, $\Phi_\lambda(z) := \exp_z(\rho_{\lambda,z}\omega)$, $\rho_{\lambda,z} := \lambda(1+\phi_{\lambda,z})$. We note that $(r,\omega) \mapsto \exp_{z_*}(r\omega)$ is a local diffeomorphism, so it suffices to show that the map $(\lambda,\omega) \mapsto (\rho_{\lambda,z_*}),\omega)$ is a homeomorphism. Indeed, taking the Jacobian matrix, we get an upper diagonal matrix

$$\begin{pmatrix} \partial_\lambda \rho_{\lambda,z_*} & \cdots \\ 0 & \mathrm{id}_{n\times n}\end{pmatrix}.$$

Since $\|\partial_\lambda \rho_{\lambda,z_*} - 1\|_\infty = \|\phi_{\lambda,z_*} + \lambda\partial_\lambda\phi_{\lambda,z_*}\|_\infty \lesssim \lambda^2$ by Proposition 2.4, we see that the Jacobian matrix is invertible. It follows by the inverse function theorem that Φ is a homeomorphism.

The linear stability/instability statement of the theorem follows from the formula (4.1), the asymptotics (4.5) and Theorem 1.1(ii).

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Appendix A. Expansions of the induced metric and mean curvature

We Taylor expand the induced metric and mean curvature of $\theta_{\rho,z}$ and estimate the non-linear terms in ϕ in H^k, $k > \frac{n}{2}+2$. One may consult [18] for a thorough study on such classical expansions, we mostly follow the notation in [24]. We abuse slightly notation by denoting the pushforward of ω through $d\theta_{\rho,z}$ by the same symbol ω. For instance, given x^i,∂_i a set of coordinates and the associated vector fields, we write the corresponding basis of tangent vector fields on S^n_z: $\zeta_i := \lambda(1+\phi)\partial_i\omega + \lambda\partial_i\phi\,\omega \in TS^n_z$.

Lemma A.1 (See [24], Lemmas 2.1 and 2.4)**.** *The following expansions are valid:*

$$\lambda^{-2}(1+\phi)^{-2}g(\zeta_i,\zeta_j) = g(\partial_i\omega,\partial_j\omega) + \frac{1}{3}R(\omega,\partial_i\omega,\omega,\partial_j\omega)\cdot\lambda^2(1+\phi)^2 + \frac{\partial_i\phi\partial_j\phi}{(1+\phi)^2} + \lambda^3\mathrm{R}^{(0)}_{\lambda,z}(\phi) + \lambda^2\mathrm{R}^{(2)}_{\lambda,z}(\phi) \tag{A.1}$$

where R is the Riemann curvature tensor and the remainders $\mathrm{R}^{(\mathrm{j})}_{\lambda,\mathrm{z}}(\phi), \mathrm{j}=0,2$, are local terms depending on ϕ and $\partial\phi$ and satisfying the estimates

$$\|\partial^i_\lambda\partial^\alpha_z\mathrm{R}^{(\mathrm{r})}_{\lambda,\mathrm{z}}(\phi,\partial\phi)\|_{\mathrm{H}^{\mathrm{k}-1}} \lesssim \sum_{\mathrm{j}\le\mathrm{i},\beta\le\alpha}\|\partial^{\mathrm{j}}_\lambda\partial^\beta_{\mathrm{z}}\phi\|^{\mathrm{r}}_{\mathrm{H}^{\mathrm{k}}} \tag{A.2}$$

provided $\|\phi\|_{H^k} \lesssim 1$ and $i+|\alpha|\le 2$, and

$$\lambda H(\theta_{\rho,z}) = n - \frac{1}{3}Ric_z\lambda^2 - (\Delta_{\mathbb{S}^n}+n)\phi + \lambda^2 M'_{\lambda,z}\phi + \lambda^2 N'_{\lambda,z}(\phi) + \lambda^3 H_{\lambda,z}, \tag{A.3}$$

where $Ric_z : \omega \to Ric_z(\omega,\omega)$ ($Ric(\cdot,\cdot)$ is the Ricci curvature tensor of M), $M'_{\lambda,z}\phi$, $N'_{\lambda,z}(\phi)$ and $H_{\lambda,z}$ are linear non-linear terms in ϕ and in its derivatives up to *order two, respectively, and independent of ϕ term, satisfying the estimates:*

$$\|\partial_\lambda^i \partial_z^\alpha M'_{\lambda,z}(\phi)\|_{H^{k-2}} \lesssim \sum_{j\le i,\beta\le\alpha} \|\partial_\lambda^j \partial_z^\beta \phi\|_{H^k}, \tag{A.4}$$

$$\|\partial_\lambda^i \partial_z^\alpha N'_{\lambda,z}(\phi)\|_{H^{k-2}} \lesssim \sum_{j\le i,\beta\le\alpha} \|\partial_\lambda^j \partial_z^\beta \phi\|_{H^k}^2, \tag{A.5}$$

$$\|\partial_\lambda^i \partial_z^\alpha H_{\lambda,z}\|_{H^{k-2}} \lesssim 1, \tag{A.6}$$

and similarly for $N'_{\lambda,z}(\phi') - N'_{\lambda,z}(\phi)$, for $i + |\alpha| \le 2$, $k > \frac{n}{2} + 2$. (Above, if ϕ is independent of λ and z, then only the term with $j = |\beta| = 0$ survives on the r.h.s..)

Proof. Both expressions (A.1) and (A.3) can be read from [24]. Bounds (A.4) are immediate by definition. In order to derive estimate (A.6) for the non-linearity $N_{\lambda,z}(\phi)$ we have to first examine its structure. Recall that the mean curvature of $\theta_{\rho,z}$ is given by[4]

$$H(\theta_{\rho,z}) = \sum_{i,j} g^{ij} g(\nabla_{\zeta_i}\nu, \zeta_j) = \mathrm{div}_{\theta_{\rho,z}}\nu \tag{A.8}$$

where g^{ij} is the inverse matrix to the metric $g_{ij} := g(\zeta_i, \zeta_j)$ and ν the outward unit normal vector field on $\theta_{\rho,z}$:

$$\nu = \frac{-\omega + \lambda \sum_{i,j} g^{ij}\partial_i\phi\zeta_j}{\sqrt{1-\lambda^2 g(\nabla_{\mathbb{S}^n}\phi, \nabla_{\mathbb{S}^n}\phi)}} \tag{A.9}$$

Note that ν is well defined for λ and $\|\phi\|_{H^k}$ appropriately small. Hence, we observe that the non-linear terms in ϕ arising in the expansion of $H(\theta_{\rho,z})$ are of the form $b(\lambda\phi(\omega), \lambda\partial\phi(\omega))\lambda\partial^2\phi(\omega)$, where $b(s,t)$ is a simple function, uniformly bounded together with its derivatives, provided $|s| \ll 1$ and $|t| \ll 1$. Using these estimates it is not hard (but somewhat tedious) to show that $\|b(\lambda\phi, \lambda\partial\phi)\lambda\partial^2\phi\|_{H^{k-2}} \lesssim \lambda^r\|\phi\|_{H^k}^r$ for some $r \ge 2$, provided $\|\phi\|_{H^k} \ll 1$ (here we use the condition $k > \frac{n}{2} + 2$). This completes the proof of the lemma. $\square$

Eq (A.1) implies

$$\lambda^{-n}(1+\phi)^{-n}\sqrt{\mathrm{det} g_{\theta_{\rho,z}}} = 1 + \frac{1}{6}\mathrm{Ric}(\omega,\omega)\lambda^2(1+\phi)^2 + \frac{1}{2}\frac{|\nabla_{\mathbb{S}^n}\phi|^2}{(1+\phi)^2} + \lambda^3 \mathrm{S}^{(0)}_{\lambda,z}(\phi) + \lambda^2 \mathrm{S}^{(2)}_{\lambda,z}(\phi) \tag{A.10}$$

---

[4]Let g be the ambient metric with its associated Christoffel symbols, Γ^j_{kl}, and $\bar{g}$ be the pull back metric of g onto $\psi(\mathbb{S}^n)$. In any local coordinate, if ψ is any immersion from $\mathbb{S}^n$ to M^{n+1},

$$H = \bar{g}^{\alpha\beta} g_{ij}\nu^i(\partial_\alpha\partial_\beta\psi^j + \Gamma^j_{kl}\partial_\alpha\psi^i\partial_\beta\psi^j) \tag{A.7}$$

where the Latin indices $i, j = 1, ..., n+1$ are for coordinates in the ambient manifold and the Greek ones $1, ..., n$ are on $\mathbb{S}^n$ and ν^j is the unit normal vector on $\psi(\mathbb{S}^n)$. At $\psi(\omega)$, we see that all geometric quantities are smooth functions of the ambient metric and at most 2 derivatives of the immersion ψ, all evaluated at ω. We remark that, however, dependence on top (2nd) order derivative is linear and comes from the $\nabla_\alpha\nabla_\beta\psi$ term above.

where Ric is the Ricci curvature tensor of M and $\mathrm{S}^{(0)}_{\lambda,\mathrm{z}}, \mathrm{r}=0,2$, are local terms depending on ϕ and $\partial\phi$ satisfying the estimates

$$\|\partial_\lambda^i \partial_z^\alpha \mathrm{S}^{(\mathrm{r})}_{\lambda,\mathrm{z}}(\phi,\partial\phi)\|_{\mathrm{H}^{\mathrm{k}-1}} \lesssim \|\phi\|^{\mathrm{r}}_{\mathrm{H}^{\mathrm{k}}} \tag{A.11}$$

provided $\|\phi\|_{H^k} \lesssim 1$ and $i+|\alpha| \le 2$. Furthermore, multiplying (A.10) by $\lambda^n(1+\phi)^n$, integrating over $\mathbb{S}^n$ and taking into account that $\int_{\mathbb{S}^n}\phi = 0$, and using the co-area formula and polar coordinates, we obtain:

Corollary A.2. *We have the following expansions*

$$A(\Phi_\lambda(z)) = \lambda^n\Big(a_n\big[1-\frac{1}{6(n+1)}R(z)\lambda^2\big] + \lambda^3 \mathrm{Q}^{(0)}_{\lambda,\mathrm{z}}(\phi) + \lambda^2 Q^{(2)}_{\lambda,z}(\phi) \tag{A.12}$$

where a_n denotes the area of the Euclidean unit sphere $\mathbb{S}^n$ and R is the scalar curvature of M, and

$$V_{enc}(\Phi_\lambda(z)) = \lambda^{n+1}\Big(\frac{a_n}{n+1}\big[1-\frac{1}{6(n+3)}R(z)\lambda^2\big] + \lambda^3 \mathrm{T}^{(0)}_{\lambda,\mathrm{z}}(\phi) + \lambda^2 T^{(2)}_{\lambda,z}(\phi) \tag{A.13}$$

The remainders $\mathrm{Q}^{(\mathrm{r})}_{\lambda,\mathrm{z}}$ and $\mathrm{T}^{(\mathrm{r})}_{\lambda,\mathrm{z}}$ above are local expression in $\phi, \partial\phi$ satisfying the estimates of the type of (4.2).

Appendix B. Existence of barycenter

In this section, we show that barycenter exists. That is, we show that equation (3.4) has a solution. To begin, we state some useful estimates.

Lemma B.1. *For the terms defined in equations* (3.2) - (3.3)*, we have the estimate*

$$\|\sigma - 1\|_{H^{k-1}} \lesssim \lambda\|\phi_{\lambda,z}\|_{H^k} \tag{B.1}$$

$$\|\sigma(\xi) - \sigma\|_{H^{k-1}} \lesssim \lambda\|\xi\|_{H^k} \tag{B.2}$$

$$\|f_i - \omega_i\|_{H^{k-1}} \lesssim \lambda(1 + \|\phi_{\lambda,z}\|_{H^k} + \|\partial_{z^i}\phi_{\lambda,z}\|_{H^k}) \text{ for } i = 1,...,n+1 \tag{B.3}$$

$$\|f_i(\xi) - f_i\|_{H^{k-1}} \lesssim \lambda\|\xi\|_{H^k} \text{ for } i = 1,...,n+1 \tag{B.4}$$

$$\|f_0 - 1\|_{H^{k-1}} \lesssim \lambda\|\partial_\lambda\phi_{\lambda,z}\|_{H^k} + \|\phi_{\lambda,z}\|_{H^k} \tag{B.5}$$

$$\|f_i(\xi) - f_i\|_{H^{k-1}} \lesssim \lambda\|\xi\|_{H^k} \tag{B.6}$$

Proof. Recall that $\psi(\omega) = \exp_z(\rho(\xi)\omega)$, where $\rho(\xi) := \lambda(1+\phi_{\lambda,z}+\xi)$. Using the definitions (3.2), we compute the difference, $\sigma(\xi)-\sigma$. To this end, it suffices to compute the differences of each factor in (3.2):

$$\begin{aligned}\sigma(\xi)-\sigma =& g_\psi(\nu(\theta_{\rho(\xi),z}(\omega)), \partial_s\theta_{s,z}(\omega)\mid_{s=\rho(\xi)}) - g_{\theta_{\lambda,z})}(\nu(\theta_{\rho(0),z}(\omega)), \partial_s\theta_{s,z}(\omega)\mid_{s=\rho(0)})\\ =& g_\psi(\nu(\theta_{\rho(\xi),z}(\omega)), \partial_s\theta_{s,z}(\omega)\mid_{s=\rho(\xi)}) - g_\psi(T\nu(\theta_{\rho(0),z}(\omega)), T\partial_s\theta_{s,z}(\omega)\mid_{s=\rho(0)})\end{aligned}$$

where T is the parallel transport from $\theta_{\lambda,z}(\omega)$ to $\psi(\omega)$. Continuing the estimate, we have

$$
\begin{aligned}
=&g_\psi(\nu(\theta_{\rho(\xi)}) - T\nu(\theta_{\rho(0),z}(\omega)), \partial_s\theta_{s,z}(\omega)\mid_{s=\rho(\xi)})\\
&+ g_\psi(T\nu(\theta_{\rho(0),z}(\omega)), \partial_s\theta_{s,z}(\omega)\mid_{s=\rho(\xi)} - T\partial_s\theta_{s,z}(\omega)\mid_{s=\rho(0)}).
\end{aligned}
$$

Recalling that $\theta_{\lambda,z} = \exp_z(\rho(0)\omega)$ and letting $\omega^0 := 1$, we find furthermore

$$
\begin{aligned}
&g(\nu(\theta_{\rho(0),z}(\omega)), \partial_s\theta_{s,z}(\omega)\mid_{s=\rho(0)})\mid_{\lambda=0} = g(\nu(\theta_{\lambda,z}(\omega))\mid_{\lambda=0}, \partial_s\theta_{s,z}(\omega)\mid_{s=0}) = 1\\
&g(\nu(\theta_{\rho(0),z}(\omega)), \partial_{z^i}\theta_{\rho(0),z}(\omega))\mid_{\lambda=0} = \omega^i,\ \ i = 0, 1, ..., n+1.
\end{aligned}
$$

Similarly, we write $f_i(\xi) - f_i$. Hence, using this, we only need to estimate the following items to complete the proof:

$$
\begin{aligned}
&g_p - g_{p'}\\
&T(s) - T(s')\\
&\partial_s \exp_z(s\omega)\mid_{s=s_1} - T(s_1 - s_2, \exp_z(s_2\omega))\partial_s \exp_z(s\omega)\mid_{s=s_2}\\
&\partial_z \exp_z(s_1\omega) - T(s_1 - s_2, \exp_z(s_2\omega))\partial_z \exp_z(s_2\omega)\\
&\rho(\xi) - \rho(0)\\
&\partial_\lambda\rho - \partial_\lambda\rho\mid_{\phi_{\lambda,z}=0}\\
&\rho(\xi)\\
&\nu(\psi) - T(\lambda, \theta_{\lambda,z})\nu(\theta_{\lambda,z})
\end{aligned}
$$

where g_p is the value of the metric at p, $T(s,z)$ is parallel transport from z along ω for time s. Then the difference of quantities in the statement of the question are composition and smooth functions of the above with at most two derivatives of $\phi_{\lambda,z}$ and ξ. Note that the first four quantities only depends on the ambient geometry of M^{n+1}. Since we are working lcoally, we may assume that we are working on a compact subset of M. Thus, the first four are smooth functions, the first four quantity exhibits Lipschitz estimates in the difference in their argument. For example,

$$
|\partial_z \exp_z(s_1\omega) - T(s_1 - s_2, \exp_z(s_2\omega))\partial_z \exp_z(s_2\omega)| \lesssim |s_1 - s_2|
$$

uniformly on M^{n+1}. The next 3 expressions have the obvious estimate

$$
\begin{aligned}
&\|\rho(\xi) - \rho(0)\|_{H^k} = \lambda\|\xi\|_{H^k}\\
&\|\partial_\lambda\rho - \partial_\lambda\rho\mid_{\phi_{\lambda,z}=0}\|_{H^k} \lesssim \lambda\|\partial_\lambda\phi_{\lambda,z}\|_{H^k} + \|\phi_{\lambda,z}\|_{H^k}\\
&\|\partial_z\rho - \partial_z\rho\mid_{\phi_{\lambda,z}=0}\|_{H^k} \lesssim \lambda(\|\partial_z\phi_{\lambda,z}\|_{H^k} + \|\phi_{\lambda,z}\|_{H^k})\\
&\|\rho(\xi)\|_{H^k} \lesssim \lambda(\|\xi\|_{H^k} + \|\phi_{\lambda,z}\|_{H^k}).
\end{aligned}
$$

Finally, the last one follows from the fact $\nu(\psi)$ is a none singular rational function of ψ. $\square$

Now, let

$$\begin{aligned} P(\lambda, z\psi) :&\mathbb{R} \times \mathbb{R}^{n+1} \times H^k \to \mathbb{R} \times \mathbb{R}^{n+1} \\ =&(\langle f_0\sigma, \xi\rangle, \cdots, \langle f_{n+1}\sigma, \xi\rangle) \end{aligned}$$

where ξ is defined by equation (3.1).

Lemma B.2. *Any ψ sufficiently close to a CMC $\theta_{\lambda_0,z_0} = \exp_{z_0}(\lambda_0(1 + \phi_{\lambda_0,z_0}))$ can be written in the form (2.2), with (3.1) and λ and z satisfying the equation*

$$P(\lambda, z, \psi) = 0$$

Proof. We note that $P(\lambda_0, z_0, \theta_{\lambda_0,z_0}) = 0$. By the implicit function theorem, it suffices to show that

(*i*) P is C^1, and
(*i*) $(\partial_{\lambda,z}P)(\lambda_0, z_0, \theta_{\lambda_0,z_0})$ is an invertible matrix.

(*i*) We check, by definition of f_i and ξ, that they are real functions of λ, z and $\phi_{\lambda,z}$ up to 1 derivative. Since $\phi_{\lambda,z}$ is C^1 in λ and z (c.f. Proposition 2.4), we see that f_i and σ are C^1 in λ and z. Since P is linear in ξ, using the estimate of Proposition 2.4 again, we see that P is C^1 in ψ as well.

(*ii*) We compute

$$(\partial_{\lambda,z}P)(\lambda_0, z_0, \theta_{\lambda_0,z_0}) = (\langle f_i\sigma, \partial_{\lambda,z}\xi\rangle) \tag{B.7}$$

since $\xi = 0$ for $\psi = \theta_{\lambda_0,z_0}$. To compute $\partial_\lambda\xi$, we use the fact that, by definition of ξ,

$$\exp_{z_0}(\lambda(1 + \phi_{\lambda,z_0} + \xi)\omega) = \exp_{z_0}(\lambda_0(1 + \phi_{\lambda_0,z_0})\omega) \tag{B.8}$$

for λ sufficiently close to λ_0. (Note that we suppressed the identification I_z between T_zM and $\mathbb{R}^{n+1}$ here as z is not varied.) Taking ∂_λ and evaluate at $\lambda = 0$, we get the result of

$$\partial_s \exp_{z_0}(s\omega)\mid_{s=\lambda_0(1+\phi_{\lambda_0,z_0})} (1 + \phi_{\lambda_0,z_0} + \lambda_0\partial_\lambda\phi_{\lambda,z_0}\mid_{\lambda=\lambda_0} + \lambda_0\partial_\lambda\xi\mid_{\lambda=\lambda_0}) = 0 \tag{B.9}$$

Contracting with $\nu(\theta_{\lambda_0,\theta_0})$, we get

$$0 = f_0 + \lambda_0\sigma\partial_\lambda\xi, \tag{B.10}$$

where the last line follow from Proposition 2.4. To compute $\partial_z\xi\mid_{\lambda_0,z_0}$, we consider curves $z(t) : (-\epsilon, \epsilon) \to M$ with $z(0) = z_0$ and $\dot{z}(0) = v$ for any $v \in T_{z_0}M$ fixed. Then any variation of

$$\exp_{z(t)}(\lambda_0(1 + \phi_{\lambda_0,z(t)} + \xi)I_{z(t)}(\omega)) \tag{B.11}$$

is tangential. Taking derivative with respect to t and setting $t = 0$, we see that the expression (B.11) becomes

$$\begin{aligned} &[(\partial_x \exp_x(\lambda_0(1 + \phi_{\lambda_0,z_0})I_x(\omega)))\mid_{x=z_0} \\ &+ \partial_s \exp_{z_0}(s\omega)\mid_{s=\lambda_0(1+\phi_{\lambda_0,z_0})} (\lambda_0\partial_z\phi_{\lambda_0,z}\mid_{z=z_0} + \lambda_0\partial_z\xi\mid_{z=z_0})]\dot{z}(0) \end{aligned}$$

Using the fact that varying z is only tangential, it follows that

$$0 = g(\nu(\theta_{\lambda_0,z_0}), [(\partial_x \exp_x(\lambda_0(1+\phi_{\lambda_0,z_0})I_x(\omega))) \mid_{x=z_0} \tag{B.12}$$

$$+ \partial_s \exp_{z_0}(s\omega) \mid_{s=\lambda_0(1+\phi_{\lambda_0,z_0})} (\lambda_0 \partial_z \phi_{\lambda_0,z} \mid_{z=z_0})]v) \tag{B.13}$$

$$+ g(\nu(\theta_{\lambda_0,z_0}), \lambda_0 \partial_z \xi \mid_{z=z_0})v) \tag{B.14}$$

for any $v \in T_z M$. By definition of f_i,

$$f_i + \lambda_0 \sigma \partial_{z^i} \xi = 0 \tag{B.15}$$

It follows that

$$(\partial_{\lambda,z} P)(\lambda_0, z_0, \theta_{\lambda_0,z_0}) = -\lambda_0^{-1} \langle f_i, f_j \rangle \tag{B.16}$$

is invertible by Lemma B.1. □

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