LECTURE 1: DIFFERENTIAL FORMS

1. 1-forms on $\mathbb{R}^n$

In calculus, you may have seen the *differential* or *exterior derivative* df of a function $f(x, y, z)$ defined to be

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

The expression df is called a *1-form.* But what does this really mean?

Definition: A *smooth 1-form* ϕ on $\mathbb{R}^n$ is a real-valued function on the set of all tangent vectors to $\mathbb{R}^n$, i.e.,

$$\phi : T\mathbb{R}^n \to \mathbb{R}$$

with the properties that

1. ϕ is linear on the tangent space $T_x\mathbb{R}^n$ for each $x \in \mathbb{R}^n$.
2. For any smooth vector field $v = v(x)$, the function $\phi(v) : \mathbb{R}^n \to \mathbb{R}$ is smooth.

Given a 1-form ϕ, for each $x \in \mathbb{R}^n$ the map

$$\phi_x : T_x\mathbb{R}^n \to \mathbb{R}$$

is an element of the dual space $(T_x\mathbb{R}^n)^*$. When we extend this notion to all of $\mathbb{R}^n$, we see that the space of 1-forms on $\mathbb{R}^n$ is dual to the space of vector fields on $\mathbb{R}^n$.

In particular, the 1-forms $dx^1, \ldots, dx^n$ are defined by the property that for any vector $v = (v^1, \ldots, v^n) \in T_x\mathbb{R}^n$,

$$dx^i(v) = v^i.$$

The dx^i's form a basis for the 1-forms on $\mathbb{R}^n$, so any other 1-form ϕ may be expressed in the form

$$\phi = \sum_{i=1}^{n} f_i(x)\, dx^i.$$

If a vector field v on $\mathbb{R}^n$ has the form

$$v(x) = (v^1(x), \ldots, v^n(x)),$$

then at any point $x \in \mathbb{R}^n$,

$$\phi_x(v) = \sum_{i=1}^{n} f_i(x)\, v^i(x).$$

2. p-forms on $\mathbb{R}^n$

The 1-forms on $\mathbb{R}^n$ are part of an algebra, called the *algebra of differential forms* on $\mathbb{R}^n$. The multiplication in this algebra is called *wedge product*, and it is skew-symmetric:

$$dx^i \wedge dx^j = -dx^j \wedge dx^i.$$

One consequence of this is that $dx^i \wedge dx^i = 0$.

If each summand of a differential form ϕ contains p dx^i's, the form is called a *p-form*. Functions are considered to be 0-forms, and any form on $\mathbb{R}^n$ of degree $p > n$ must be zero due to the skew-symmetry.

A basis for the p-forms on $\mathbb{R}^n$ is given by the set

$$\{dx^{i_1} \wedge \cdots \wedge dx^{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}.$$

Any p-form ϕ may be expressed in the form

$$\phi = \sum_{|I|=p} f_I\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

where I ranges over all multi-indices $I = (i_1, \ldots, i_p)$ of length p.

Just as 1-forms act on vector fields to give real-valued functions, so p-forms act on p-tuples of vector fields to give real-valued functions. For instance, if ϕ, ψ are 1-forms and v, w are vector fields, then

$$(\phi \wedge \psi)(v, w) = \phi(v)\psi(w) - \phi(w)\psi(v).$$

In general, if $\phi_1, \ldots, \phi_p$ are 1-forms and $v_1 \ldots, v_p$ are vector fields, then

$$(\phi_1 \wedge \cdots \wedge \phi_p)(v_1, \ldots, v_p) = \sum_{\sigma \in S_p} \operatorname{sgn}(\sigma)\, \phi_1(v_{\sigma(1)})\, \phi_2(v_{\sigma(2)}) \cdots \phi_n(v_{\sigma(n)}).$$

3. The exterior derivative

The *exterior derivative* is an operation that takes p-forms to $(p+1)$-forms. We will first define it for functions and then extend this definition to higher degree forms.

Definition: If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, then the exterior derivative of f is the 1-form df with the property that for any $x \in \mathbb{R}^n,\ v \in T_x\mathbb{R}^n$,

$$df_x(v) = v(f),$$

i.e., $df_x(v)$ is the directional derivative of f at x in the direction of v.

It is not difficult to show that

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

The exterior derivative also obeys the Leibniz rule

$$d(fg) = g\, df + f\, dg$$

and the chain rule

$$d(h(f)) = h'(f)\, df.$$

We extend this definition to p-forms as follows:

Definition: Given a p-form $\phi = \sum_{|I|=p} f_I\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}$, the exterior derivative $d\phi$ is the $(p+1)$-form

$$d\phi = \sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

If ϕ is a p-form and ψ is a q-form, then the Leibniz rule takes the form

$$d(\phi \wedge \psi) = d\phi \wedge \psi + (-1)^p \phi \wedge d\psi.$$

Very Important Theorem: $d^2 = 0$. i.e., for any differential form ϕ,

$$d(d\phi) = 0.$$

Proof: First suppose that f is a function, i.e., a 0-form. Then

$$\begin{aligned} d(df) &= d(\sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i) \\ &= \sum_{i,j} \frac{\partial^2 f}{\partial x^i\, \partial x^j} dx^j \wedge dx^i \\ &= \sum_{i<j} (\frac{\partial^2 f}{\partial x^j\, \partial x^i} - \frac{\partial^2 f}{\partial x^i\, \partial x^j}) dx^i \wedge dx^j \\ &= 0 \end{aligned}$$

because mixed partials commute.

Next, note that dx^i really does mean $d(x^i)$, where x^i is the ith coordinate function. So by the argument above, $d(dx^i) = 0$. Now suppose that

$$\phi = \sum_{|I|=p} f_I\, dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Then by the Leibniz rule,

$$\begin{aligned} d(d\phi) &= d(\sum_{|I|=p} df_I \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p}) \\ &= \sum_{|I|=p} [d(df_I) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_p} - df_I \wedge d(dx^{i_1}) \wedge \cdots \wedge dx^{i_p} + \dots] \\ &= 0. \qquad \square \end{aligned}$$

Definition: A p-form ϕ is *closed* if $d\phi = 0$. ϕ is *exact* if there exists a $(p-1)$-form η such that $\phi = d\eta$.

By the Very Important Theorem, every exact form is closed. The converse is only partially true: every closed form is *locally* exact. This means that given a closed p-form ϕ on an open set $U \subset \mathbb{R}^n$, any point $x \in U$ has a neighborhood on which there exists a $(p-1)$-form η with $d\eta = \phi$.

4. Differential forms on manifolds

Given a smooth manifold M, a *smooth 1-form* ϕ on M is a real-valued function on the set of all tangent vectors to M such that

1. ϕ is linear on the tangent space T_xM for each $x \in M$.
2. For any smooth vector field v on M, the function $\phi(v) : M \to \mathbb{R}$ is smooth.

So for each $x \in M$, the map

$$\phi_x : T_xM \to \mathbb{R}$$

is an element of the dual space $(T_xM)^*$.

Wedge products and exterior derivatives are defined similarly as for $\mathbb{R}^n$. If $f : M \to \mathbb{R}$ is a differentiable function, then we define the exterior derivative of f to be the 1-form df with the property that for any $x \in M,\ v \in T_xM$,

$$df_x(v) = v(f).$$

A local basis for the space of 1-forms on M can be described as before in terms of any local coordinate chart $(x^1, \dots, x^n)$ on M, and it is possible to show that the coordinate-based notions of wedge product and exterior derivative are in fact independent of the choice of local coordinates and so are well-defined.

More generally, suppose that M_1, M_2 are smooth manifolds and that $F : M_1 \to M_2$ is a differentiable map. For any $x \in M_1$, the differential dF (also denoted F_*) : $T_xM_1 \to T_{F(x)}M_2$ may be thought of as a *vector-valued* 1-form, because it is a linear map from T_xM_1 to the vector space $T_{F(x)}M_2$. There is an analogous map in the opposite direction for differential forms, called the *pullback* and denoted F^*. It is defined as follows.

Definition: If $F : M_1 \to M_2$ is a differentiable map, then

1. If $f : M_2 \to \mathbb{R}$ is a differentiable function, then $F^*f : M_1 \to \mathbb{R}$ is the function
$$(F^*f)(x) = (f \circ F)(x).$$
2. If ϕ is a p-form on M_2, then $F^*\phi$ is the p-form on M_1 defined as follows: if $v_1, \dots , v_p \in T_xM_1$, then
$$(F^*\phi)(v_1, \dots , v_p) = \phi(F_*(v_1), \dots , F_*(v_p)).$$

In terms of local coordinates $(x^1, \dots , x^n)$ on M_1 and $(y^1, \dots , y^m)$ on M_2, suppose that the map F is described by
$$y^i = y^i(x^1, \dots , x^n), \quad 1 \le i \le m.$$
Then the differential dF at each point $x \in M_1$ may be represented in this coordinate system by the matrix
$$\left[\frac{\partial y^i}{\partial x^j}\right].$$
The dx^j's are forms on M_1, the dy^i's are forms on M_2, and the pullback map F^* acts on the dy^i's by
$$F^*(dy^i) = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j} dx^j.$$

The pullback map behaves as nicely as one could hope with respect to the various operations on differential forms, as described in the following theorem.

Theorem: Let $F : M_1 \to M_2$ be a differentiable map, and let ϕ, η be differential forms on M_2. Then

1. $F^*(\phi + \eta) = F^*\phi + F^*\eta$.
2. $F^*(\phi \wedge \eta) = F^*\phi \wedge F^*\eta$.
3. $F^*(d\phi) = d(F^*\phi)$.

5. The Lie derivative

The final operation that we will define on differential forms is the Lie derivative. This is a generalization of the notion of directional derivative of a function.

Suppose that $v(x)$ is a vector field on a manifold M, and let $\varphi : M\times(-\varepsilon,\varepsilon) \to M$ be the *flow* of v. This is the unique map that satisfies the conditions

$$\frac{\partial\varphi}{\partial t}(x,t) = v(\varphi(x,t))$$
$$\varphi(x,0) = x.$$

In other words, $\varphi_t(x) = \varphi(x,t)$ is the point reached at time t by flowing along the vector field $v(x)$ starting from the point x at time 0.

Recall that if $f : M \to \mathbb{R}$ is a smooth function, then the directional derivative of f at x in the direction of v is

$$\begin{aligned} v(f) &= \lim_{t\to 0} \frac{f(\varphi_t(x)) - f(x)}{t} \\ &= \lim_{t\to 0} \frac{(\varphi_t^*(f) - f)(x)}{t}. \end{aligned}$$

Similarly, given a differential form ϕ we define the *Lie derivative* of ϕ along the vector field $v(x)$ to be

$$\mathcal{L}_v\phi = \lim_{t\to 0} \frac{\varphi_t^*\phi - \phi}{t}.$$

Fortunately there is a practical way to compute the Lie derivative. First we need the notion of the left-hook of a differential form with a vector field. Given a p-form ϕ and a vector field v, the *left-hook* $v \lrcorner \phi$ of ϕ with v (also called the *interior product* of ϕ with v) is the $(p-1)$-form defined by the property that for any $w_1, \ldots, w_{p-1} \in T_x\mathbb{R}^n$,

$$(v \lrcorner \phi)(w_1, \ldots, w_{p-1}) = \phi(v, w_1, \ldots, w_{p-1}).$$

For instance,

$$\frac{\partial}{\partial x} \lrcorner (dx \wedge dy + dz \wedge dx) = dy - dz.$$

Now according to *Cartan's formula*, the Lie derivative of ϕ along the vector field v is

$$\mathcal{L}_v\phi = v \lrcorner d\phi + d(v \lrcorner \phi).$$

Exercises

1. Classical vector analysis avoids the use of differential forms on $\mathbb{R}^3$ by converting 1-forms and 2-forms into vector fields by means of the following one-to-one correspondences. ($\varepsilon_1, \varepsilon_2, \varepsilon_3$ will denote the standard basis $\varepsilon_1 = [1,0,0],\ \varepsilon_2 = [0,1,0],\ \varepsilon_3 = [0,0,1]$.)

$$f_1\,dx^1 + f_2\,dx^2 + f_3\,dx^3 \;\longleftrightarrow\; f_1\,\varepsilon_1 + f_2\,\varepsilon_2 + f_3\,\varepsilon_3$$
$$f_1\,dx^2\wedge dx^3 + f_2\,dx^3\wedge dx^1 + f_3\,dx^1\wedge dx^2 \longleftrightarrow\; f_1\,\varepsilon_1 + f_2\,\varepsilon_2 + f_3\,\varepsilon_3$$

Vector analysis uses three basic operations based on partial differentiation:

1. *Gradient* of a function f:
$$\mathrm{grad}(f) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}\,\varepsilon_i$$
2. *Curl* of a vector field $v = \sum_{i=1}^{3} v^i(x)\,\varepsilon_i$:
$$\mathrm{curl}(v) = \left(\frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3}\right)\varepsilon_1 + \left(\frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1}\right)\varepsilon_2 + \left(\frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2}\right)\varepsilon_3$$
3. *Divergence* of a vector field $v = \sum_{i=1}^{3} v^i(x)\,\varepsilon_i$:
$$\mathrm{div}(v) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i}$$

Prove that all three operations may be expressed in terms of exterior derivatives as follows:

1. $df \leftrightarrow \mathrm{grad}(f)$
2. If ϕ is a 1-form and $\phi \leftrightarrow v$, then $d\phi \leftrightarrow \mathrm{curl}(v)$.
3. If η is a 2-form and $\eta \leftrightarrow v$, then $d\eta \leftrightarrow \mathrm{div}(v)\,dx^1\wedge dx^2\wedge dx^3$.

Show that the identities
$$\mathrm{curl}(\mathrm{grad}(f)) = 0$$
$$\mathrm{div}(\mathrm{curl}(v)) = 0$$
follow from the fact that $d^2 = 0$.

2. Let f and g be real-valued functions on $\mathbb{R}^2$. Prove that
$$df\wedge dg = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix}\, dx\wedge dy.$$
(You may recognize this from the change-of-variables formula for double integrals.)

3. Suppose that ϕ, ψ are 1-forms on $\mathbb{R}^n$. Prove the Leibniz rule
$$d(\phi\wedge\psi) = d\phi\wedge\psi - \phi\wedge d\psi.$$

4. Prove the statement above that if $F : M_1 \to M_2$ is described in terms of local coordinates by

$$y^i = y^i(x^1, \dots, x^n), \quad 1 \leq i \leq m$$

then

$$F^*(dy^i) = \sum_{j=1}^{n} \frac{\partial y^i}{\partial x^j} dx^j.$$

5. Let (r, θ) be coordinates on $\mathbb{R}^2$ and (x, y, z) coordinates on $\mathbb{R}^3$. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$F(r, \theta) = (\cos\theta, \sin\theta, r).$$

Describe the differential dF in terms of these coordinates and compute the pullbacks $F^*(dx), F^*(dy), F^*(dz)$.