(cf. Palais, BAMS Symmetry of solitons 1997)

Ch. 8. The inverse scattering method/transform

1. Toy model: the Fourier transform on ~~periodic~~ Schwartz f's $\{P\}$ on $\mathbb{R}$ (fast decay)

$v \mapsto w = \mathcal{F}(v)$,

$$w(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(x) e^{-i\alpha x}\, dx \quad \text{– Fourier transform}$$

$w \mapsto v = \mathcal{IF}(w)$,

$$v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} w(\alpha) e^{i\alpha x}\, d\alpha$$

Property: A diff. operator $F(\partial) := \sum \overset{\text{const}}{a_k} \partial^k \xrightarrow{\mathcal{F}}$

An operator of multiplication by $F(i\alpha)$

Strategy: transform a diff. eq'n to alg. eq'n, solve, transform back.

2. Lax eq'n: KdV as an isospectral flow

Return to the Lax eq'n: H – Hilbert space,

$U(t)$ – family of unitary transf's of H, $U(0) = I$.

$B(t) = U_t(t) U^{-1}(t) \Rightarrow B^* = -B$, i.e. $B(t)$ is skew-adjoint

(note: $U_t = BU$, $U_t^* = -U^*B$) (diff'te $U U^* = I$).

Suppose $L(0)$ – self-adjoint oper. on H, $L(t) = U(t) L(0) U(t)^{-1}$

Then $L(t)$ satisfies the Lax eq'n $L_t = [B, L]$.

Indeed, $L(0) = U^* L(t) U$, differ.'te it $\Rightarrow$

$0 = U_t^* L U + U^* L_t U + U^* L U_t = U^*(-BL + L_t + LB) U \Rightarrow$ Lax.

Rm ~~Given $\forall \psi(0) \in H$ define $\psi(t) = U(t)\psi(0)$~~

~~Since $U(t)$~~ Let $\psi(0) \in H$ be an eigenvector of $L(0)$ with eigenvalue λ.

Define $\Psi(t) := U(t)\,\Psi(0)$ $\qquad (*)$

Then $\Psi(t)$ is an eigenvector of $L(t)$ with the same eigenvector λ.

Indeed, since ~~$U(t)L(0) = L(t)U(t)$~~ $L(0) \xrightarrow{U(t)} L(t)$,

then $\Psi(0) \xrightarrow{U(t)} \Psi(t)$.

Note: $\Psi_t = B\,\Psi(t)$ (diff'te $(*)$) – a LODE defining Ψ with initial cond $\Psi(0)$

Obtain:

Isospectral Principle: Let $L(t)$, $B(t)$ be smooth families of self-adj & skew-adj. operators on H, satisfying $L_t = [B, L]$.
Let $\Psi(t) \in H$ – a sol'n of LDE $\Psi_t = B\Psi$.
Then if $\Psi(0)$ is an eigenvector of $L(0)$ with e.v. $\lambda \Rightarrow$
$\Psi(t)$ is –"– –"– $L(t)$ with the same λ.

Apply to KdV: $H = L^2(\mathbb{R})$ $\left(\text{Schwartz } f\text{'s in } L^2 \ S(\mathbb{R})\right)$

$L(t) = -\partial^2 + u(x,t)$ – self-adj.

$B(t) = -4\partial^3 + 3(u\partial + \partial u)$ – skew-adj.

Check the Lax form:

$L_t = [B, L] \iff u_t - 6uu_x + u_{xxx} = 0$

Indeed, $L_t = u_t$,

$[B, L] = 4[\partial^3, \partial^2] - 4[\partial^3, u] - 3[u\partial, \partial^2] + 3[u\partial, u]$

$\qquad - 3[u\partial, \partial^2] + 3[\partial u, u]$

Compute: $[\partial^3, \partial^2] = 0$, $[\partial^3, u] = u_{xxx} + 3u_{xx}\partial + 3u_x\partial^2$

$[u\partial, \partial^2] = -u_{xx}\partial - 2u_x\partial^2$, $[u\partial, u] = uu_x$

$[\partial u, \partial^2] = -3u_{xx}\partial - 2u_x\partial^2 - u_{xxx}$, $[\partial u, u] = -uu_x$

$\Rightarrow [B, L] = 6uu_x - u_{xxx}$

Thus we have:

<u>KdV Isospectrality thm</u>: Suppose $u(x,t)$ is a sol'n of the KdV:

$$u_t - 6uu_x + u_{xxx} = 0 \quad \text{with initial values } u(x,0) \in S(\mathbb{R})$$

Let $\psi(x)$ be an eigenf'n of the Schrödinger eq'n with potential $u(x,0)$ with eigenvalue λ:

$$-\partial_x^2 \psi(x) + u(x,0)\psi(x) = \lambda \psi(x). \quad \Leftrightarrow \quad L\psi = \lambda\psi$$

Assume $\psi(x,t)$ be the sol'n of the evolution eq'n $\psi_t = B\psi$, i.e.

$\psi_t = B\psi$

$$\Leftrightarrow \psi_t = -4\psi_{xxx} + 3\left(u(x,t)\psi_x + \left(u(x,t)\psi(x,t)\right)_x\right) \qquad (1)$$

with the initial value $\psi(x,0) = \psi(x)$.

Then $\psi(x,t)$ is an eigenf'n for Schr. eq'n with potential $u(x,t)$, and same e.v. λ:

$L\psi(t) = \lambda\psi(t)$

$$-\psi_{xx}(x,t) + u(x,t)\psi(x,t) = \lambda\psi(x,t).$$

Moreover, the L^2-norm of $\psi(\cdot,t)$ is independent of t.

Finally, $\psi(x,t)$ satisfies $\quad \psi_t + (4\lambda + 2u)\psi_x + u_x\psi = 0 \qquad (**)$

<u>Pf</u>. This is appl'n of Isospect. prin. except for $(**)$. The latter: diff'te (in x) the eigenvalue eq'n for ψ: $\psi_{xxx} = u_x\psi + (u-\lambda)\psi_x$

Plug this in the eq'n (1) for ψ_t. ∎

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3. <u>The scattering data and its evolution.</u>

Now Fix a potential u in $S(\mathbb{R})$

Consider $\quad L^u(\psi) = -\psi_{xx} + u\psi - \lambda\psi \quad$ - ~~Schr. operator~~ on $C^\infty(\mathbb{R})$

$$E_\lambda(u) \text{ - space of sol's } L^u(\psi) = 0 \qquad (2 \text{ dim } \forall\lambda)$$

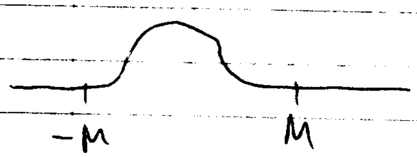
" "λ eigenf's" of Schr. operator $-\partial^2 + u$

Ignore $\lambda = 0$, consider $\lambda < 0$, $\lambda > 0$ separately

I. $\lambda = -k^2$, $k>0$. If u vanishes on some interval,

$$\psi(x) = a e^{kx} + b e^{-kx}$$

If u has cpt support, for $x < -M$, $\psi^{\pm}_{\lambda,-\infty}(x) = e^{\pm kx}$



for $x > M$ $\quad \psi^{\pm}_{\lambda,\infty}(x) = e^{\pm kx}$

For $u \in S(\mathbb{R})$ (fast decaying), we say

$$\psi^{\pm}_{\lambda,-\infty} \sim e^{\pm kx}, \quad \psi^{\pm}_{\lambda,\infty} \sim e^{\pm kx} \quad \text{as } x \to \infty$$

(i.e. $\lim_{x\to-\infty} \psi^{\pm}_{\lambda,-\infty} \cdot e^{\mp kx} = 1$, etc.)

Define f's $f(\lambda)$ and $c(\lambda)$ by $\psi^{+}_{\lambda,-\infty} = f(\lambda)\psi^{+}_{\lambda,\infty} + c(\lambda)\psi^{-}_{\lambda,\infty}$

Let us detect those λ (a "discrete eigenvalue" of L^u)

when $E_\lambda(u)$ has a non zero element $\psi \in L^2(\mathbb{R})$ (of norm $=1$)

ψ must be like $\psi^{+}_{\lambda,-\infty}$ and $\psi^{-}_{\lambda,+\infty}$ – decays exp'ly at $\pm\infty$.

$$\Rightarrow \psi^{+}_{\lambda,-\infty} = c(\lambda)\psi^{-}_{\lambda,+\infty}, \quad f(\lambda) = 0$$

Thus discrete eigenvalues of L^u are roots of $f(\lambda)$.

Sturm – Liouville theory $\Rightarrow$ $\exists$ finitely many $\lambda_1, \dots, \lambda_N$

corresp. to (normalised) eigenf's $\psi_1, \dots, \psi_N$,

determine "normalization constants" $c_1, \dots, c_N$ by

$\psi_n = c_n \psi^{-}_{\lambda_n,\infty}$. I.e. for $\lambda_n = -k_n^2$, c_n

are defined by $\quad \psi_n \sim c_n e^{-k_n x}$ as $x \to \infty$

(c_n is determ. upto $\pm$, but we use c_n^2 only later).

II. $\lambda = k^2$, $k > 0$. Similarly, ~~find~~ define the bases

$$\psi^{\pm}_{\lambda,-\infty}(x) \sim e^{\pm ikx} \quad \text{as } x \to -\infty$$

$$\psi^{\pm}_{\lambda,\infty}(x) \sim e^{\pm ikx} \quad \text{as } x \to \infty$$

Then $\psi^{-}_{\lambda,-\infty} = \alpha\, \psi^{-}_{\lambda,\infty} + \beta\, \psi^{+}_{\lambda,\infty}$, where α can be shown $\neq 0$.

Divide by α, obtain the Jost sol'n with the assymptotics

$$\psi_k(x) \sim a(k)\, e^{-ikx} \quad \text{as } x \to -\infty$$

$$\psi_k(x) \sim e^{-ikx} + b(k)\, e^{ikx} \quad \text{as } x \to \infty$$

Def $a(k)$ is the transmission coef't
$b(k)$ is the reflection coef't.

Scattering data : reflection coef't $b(k)$
$S(u)$ + normalizing const's $c_1, \dots, c_n$

Thm (Evolution of the Scattering data). Let $u(t) = u(x,t)$ be a smooth curve in $S(\mathbb{R})$ (Schwartz space), satisfying KdV $u_t - 6uu_x + u_{xxx} = 0$. Assume that the Schrödinger oper'n with potential u has discrete eigenvalues $-k_1^2, \dots, -k_n^2$ whose normalized eigenf's have normal. const's $c_1(t), \dots, c_n(t)$. Let $a(k,t)$, $b(k,t)$ be resp. transmission & refl. coef's.

Then: 0) transm. coef's are first int's: $a(k,t) = a(k,0)$

1) refl. coef's satisfy $b(k,t) = b(k,0)\, e^{8ik^3 t}$

2) normalizing const's satisfy $c_n(t) = c_n(0)\, e^{4k_n^3 t}$

Rm. The Inverse scattering method

To solve the KdV IVP $u_t - 6uu_x + u_{xxx} = 0$ with given initial potential $u(x,0)$ in $S(\mathbb{R})$

1) Apply the "direct scattering transform"

$u(x,0) \rightsquigarrow$ discr. eigenvalues $-k_1^2, \dots, -k_N^2$ $\rightsquigarrow$ compute scat. data $S(u)$: normalizing const's $c_n(0)$, refl. coef's $b(k,0)$

2) Define $c_n(t) = c_n(0) e^{4k_n^3 t}$, $b(k,t) = b(k,0) e^{8ik^3 t}$

3) Use the "inverse scat. transform" to compute $u(t)$ from $c_n(t)$, $b(k,t)$.

Idea of Pf (of evolution thm, $c(t)$ only).

One can show that if ψ is an eigenf'n of L^u for $\lambda = -k^2$ i.e. $-\psi_{xx} + u\psi = +\underset{-k^2}{\lambda}\psi$, then $\psi(x,t) \sim c(t) e^{+kx}$ as $x \to -\infty$

Also, $\psi_t(x,t) \sim c'(t) e^{kx}$ ($c(t)$ is diff'le in t - some integral f'las for c)

Moreover, $\psi_x(x,t) \sim -c(t) k e^{kx}$

(Here one uses $\psi = e^{-kx}\varphi$, $\varphi^{(x,t)} \to c(t)$ as $x \to -\infty$

$\Rightarrow \varphi = \psi e^{+kx}$

$\psi_x e^{kx} = \underset{\searrow 0 \text{ as } x\to-\infty}{\varphi_x} - k\psi e^{kx}$

$\Rightarrow \psi_x \sim -k\psi \sim -k c(t) e^{kx}$ as $x \to -\infty$)

From the KdV Isospectrality thm:

If u satisfies KdV $\Rightarrow$ $\psi(x,t)$ satisfies in $S(\mathbb{R})$

$$\Psi_t - (-4k^2 + 2u)\Psi_x + u_x \Psi = 0 \qquad \Big| \cdot e^{-kx}$$

as $x \to -\infty$: $\quad c'(t) + 4k^2(-k c(t)) = 0 \Rightarrow c'(t) - 4k^3 c(t) = 0$

$\Rightarrow c(t) = c(0)e^{4k^3 t}$ – evolution of normalizing const's $\boxed{}$

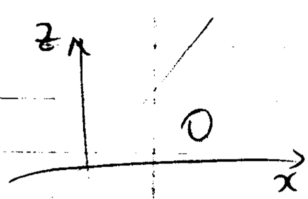
Similarly for $a(k,t)$, $b(k,t)$.

3. The Inverse scattering transform

Forget about t: Given discrete eigenvalues $-k_1^2, \dots, -k_N^2$ (for the Sturm–Liouville operator $-\frac{d^2}{dx^2} + u(x)$) with normal const' $c_1, \dots, c_N$ and reflect. coeff's $b(k)$ how to find $u(x)$? (Gelfand-Levitan & Marchenko) ~mid 50's.

Define $\displaystyle B(\xi) = \sum_{n=1}^{N} c_n^2 e^{-k_n \xi} + \frac{1}{2\pi}\int_{-\infty}^{\infty} b(k) e^{ik\xi}\, dk$

Inverse scattering thm: The potential u can be recovered via $u(x) = -2\frac{d}{dx}K(x,x)$, where $K(x,z)$ is the unique f'n on $\mathbb{R}\times\mathbb{R}$ $= 0$ for $z < x$ and satisfies the Gelfand–Levitan–Marchenko integral eq'n:



$$K(x,z) + B(x+z) + \int_{(x)\,-\infty}^{\infty} K(x,y)B(y+z)\,dy = 0$$

{and $K(x,z) \to 0$ as $z \to \infty$)

Rmk: Define the map $\mathcal{F}^B : C(\mathbb{R}\times\mathbb{R}) \to C(\mathbb{R}\times\mathbb{R})$ $\mathbb{R}^2/_2$ – bounded, contin. real-valued, sup norm on

$$\mathcal{F}^B(K)(x,z) = -B(x+z) - \int_{-\infty\,(\text{or } x)}^{\infty} K(x,y)B(y+z)\,dy$$

K satisfies the GLM eq'n $\Leftrightarrow$ it is a fixed pt of $\mathcal{F}^B$

$\mathcal{F}^B$ is Lipschitz with const $\|B\|_{L^1}$, so if $\|B\|_{L^1} < 1$

$\Rightarrow$ apply the Banach contraction principle $\Rightarrow \exists!$ sol'n, it is the lim of K_n defined by $K_1(x,z) = -B(x+z)$, $K_{n+1} = \mathcal{F}^B(K_n)$

Rm 2 Assume that B is "separable":

$$B(x+z) = \sum_{n=1}^{N} X_n(x)\, Z_n(z), \text{ then the GLM eq'n is}$$

$$(1) \qquad K(x,z) + \sum_{n=1}^{N} X_n(x) Z_n(z) + \sum_{n=1}^{N} Z_n(z) \int_x^\infty K(x,y) X_n(y)\, dy = 0$$

i.e. $K(x,z) = \sum_{n=1}^{N} L_n(x) Z_n(z)$ for some f's $L_n(x)$

By substit. K into (1) obtain a system of N linear eq's on L_n

$$X_n(x) + \sum_{m=1}^{N} A_{nm}(x) L_m(x) = 0 \text{ where } A_{n,m} = \delta_{nm} + a_{nm}(x),$$

$$\text{and } a_{nm}(x) = \int_x^\infty Z_m(y) X_n(y)\, dy$$

while $K(x,x) = -\sum_{n=1}^{N} Z_n(x) \sum_{m=1}^{N} A^{-1}_{nm}(x) X_m(x)$

Example A potential u is "reflectionless" if all the refl. coef's $=0$. Since $b(k,t) = b(k,0)\, e^{8ik^3 t} \Rightarrow$ if u is reflless at $t=0 \Rightarrow$ it is reflless $\forall t$

For reflless u with discrete eigenvalues $-\kappa_1^2, \dots, -\kappa_N^2$ & normal const's $c_1, \dots, c_N$

$$B(\xi) = \sum_{n=1}^{N} c_n^2 e^{-\kappa_n \xi} \Rightarrow$$

$$X_n(x) = c_n^2 e^{-\kappa_n x}$$

$$B(x+z) = \sum_{n=1}^{N} X_n(x) Z_n(z), \text{ where } \quad Z_n(x) = e^{-\kappa_n z}$$

separable!

$$\text{Then } a_{nm}(x) = \int_x^\infty Z_m(y) X_n(y)\,dy = c_n^2 \int_x^\infty e^{-(\kappa_n+\kappa_m)y}\,dy$$

$$= \frac{c_n^2 e^{-(\kappa_n+\kappa_m)x}}{\kappa_n+\kappa_m}$$

<u>Prop</u>. In this case $K(x,x) = \frac{d}{dx} \log \det A(x)$ (for $A = (a_{nm})$)

and $u(x) = -2\frac{d}{dx}K(x,x) = -2\frac{d^2}{dx^2} \log \det A(x)$

<u>Pf</u>. $A_{nm}(x) = \delta_{nm} + a_{nm}(x) = \delta_{nm} + \dfrac{c_n^2 e^{-(\kappa_n+\kappa_m)x}}{\kappa_n+\kappa_m}$

$$\frac{d}{dx} A_{nm}(x) = -c_n^2 e^{-(\kappa_n+\kappa_m)x},$$

$$K(x,x) = -\sum_{n=1}^{N} Z_n(x) \sum_{m=1}^{N} A^{-1}_{nm}(x) X_m(x)$$

$$= \sum_{n=1}^{N} e^{-\kappa_n x} \sum_{m=1}^{N} A^{-1}_{nm}(x)\left(-c_m^2 e^{-\kappa_m x}\right) = \sum_{n=1}^{N}\sum_{m=1}^{N} A^{-1}_{nm} \frac{d}{dx} A_{mn}(x)$$

$$= \operatorname{tr}\left(A^{-1}(x) \frac{d}{dx} A(x)\right) = \frac{1}{\det A(x)} \frac{d}{dx} \det A(x) = \frac{d}{dx} \log \det A(x)$$

<u>Rm</u>. $A(x)$ satisfies the DE-system

$Y' = (A'A^{-1})Y \Rightarrow$ by Abel's the Wronskian $W = \det A$ satisfies

$$(\det A)' = \operatorname{tr}(A'A^{-1})(\det A)$$

$$\operatorname{tr}(A'A^{-1}) = \operatorname{tr}(A^{-1}A')$$

∎

4. Solitons

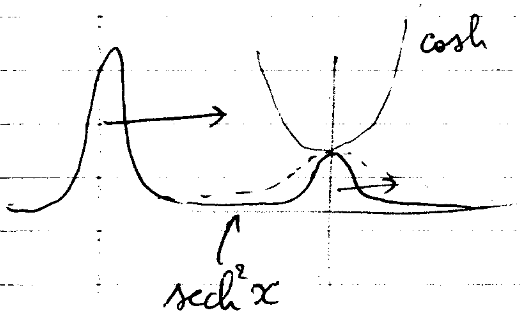
For the KdV eq'n $u_t + 6uu_x + u_{xxx} = 0$
look for travelling waves sol's $u(x,t) = f(x-ct)$
$\Rightarrow \quad -cf' + 6ff' + f''' = 0$.

Boundary cond's $f \to 0$ at $x \to \pm\infty$ $\Rightarrow$ 1-param family

$u(x,t) = 2a^2 \underset{\text{"}\cosh^{-1}\text{"}}{\operatorname{sech}^2}(a(x-4a^2t))$ – <u>one soliton sol'n of KdV</u>

<u>Rm</u>. Amplitude $2a^2 \sim \frac{1}{2}$ speed $4a^2$ $\Rightarrow$ taller waves move faster.

Solitons $\quad \operatorname{sech}^2 x = (\cosh^{-1} x)^2 = \left(\frac{2}{e^x + e^{-x}}\right)^2$



(Zabusky, Kruskal 1965)

After interaction solitons are virtually unaffected in size or shape, but the phase changes.

<u>Rm</u> For $N=1$, $\quad K = K_1$, $\quad$ and using $\quad c(t) = c(0)e^{4K^3t}$

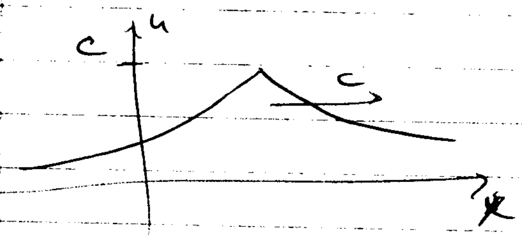
$u(x) = -2\frac{d^2}{dx^2}\log\det A(x)$ gives

$u(x,t) = -\frac{K^2}{2}\operatorname{sech}^2(K(x-K^2t))$.

In general, N-soliton sol's.

Rm. For the CH eq'n $u_t + u_{xxt} + 3uu_x = 2u_x u_{xx} + u u_{xxx}$

solitary waves are peakons: $u(x,t) = c e^{-|x-ct|}$



Multipeakons: