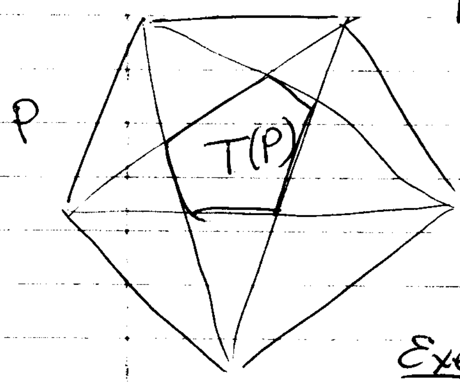
# Ch. 11. The pentagram map

Let P be a convex n-gon in $\mathbb{R}^2$.




$T(P)$ is convex hull of intersections of shortest diagonals of $P$

$C_n$ – space of convex n-gons modulo proj. transf's, i.e. action of $PGL(3,\mathbb{R})$

$T: C_n \to C_n$ – introduced by R. Schwartz, 1992

Exer. $T|_{C_5} = id$, $T|_{C_6}$ – involution.

Hint:

T exhibit quasiperiodic properties (experimental result)

Def. A twisted n-gon is a map $\varphi: \mathbb{Z} \to \mathbb{RP}^2$ s.t. $\varphi(k+n) = M \circ \varphi(k)$ for ~~some~~ fixed $M \in PGL(3,\mathbb{R})$ (project transf of $\mathbb{RP}^2$)

Ex $M = id \Rightarrow$ closed n-gon.

In general, it suffices to know $\varphi$ on $\{1, \dots, n\}$, then – reconstructable with $M$

$P_n$ – space of twisted polygons for all $M \in PGL(3,\mathbb{R})$ modulo proj. equivalence.

$T: P_n \to P_n$

Thm 1 (Ovsienko, Schwartz, Tabachnikov 2010)

$\exists$ a Poisson str. on $P_n$ of corank $= \begin{cases} 2 & \text{for odd } n \\ 4 & \text{for even } n \end{cases}$ invariant w.r.t. $T$

$\exists$ $2\underbrace{[\tfrac{n}{2}]}_{\text{the floor of } \frac{n}{2}} + 2$ alg. indep. invar. f's, so that their common level sets are Lagran (in sympl. leaves) and defines quasiper or period motion ($\Rightarrow$ integrability)

$P_n^{2n}$


n pts in $\mathbb{R}^2$    dim $PGL(3,\mathbb{R})$

Rm $\dim C_n = \underset{\text{n pts in } \mathbb{R}^2}{2n} - \underset{\dim PGL(3,\mathbb{R})}{8}$, $\dim P_n = \dim C_n + 8 = 2n$

For even $n$ we have 4 Casimirs + $(n-2)$ integrals of motion

Together this is $n+2 = \left(2\left[\frac{n}{2}\right]+2\right)$

Rm. $C_n \subset P_n$ is not a Poisson submfd

Thm (Fedorov, OST'11) Pentag map is integrable on $C_n$. ($\text{tori} \subset C_n$). The first int'ls have several relations & several linear relations on derivatives.

tori have dim $d = \begin{cases} n-4, & n \text{ is odd} \\ n-5, & n \text{ is even} \end{cases} = \begin{cases} \frac{1}{2}\dim C_n, & n \text{ odd} \\ \frac{1}{2}\dim C_n - 1, & n\text{-even} \end{cases}$

hyper-integrability

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Two sets of coordinates on $P_n$

1) The corner coordinates

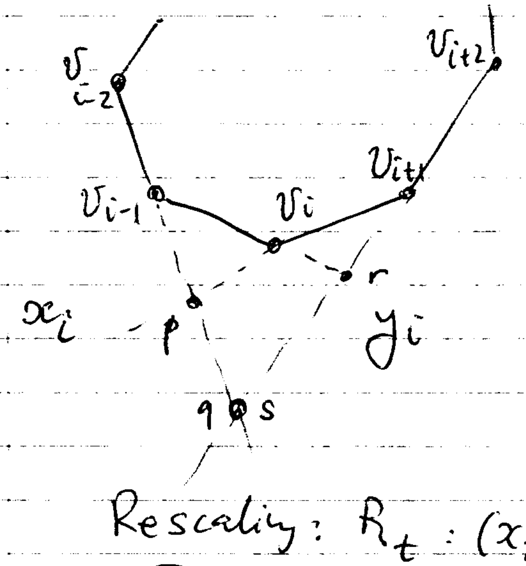
Def. The (inverse) cross ratio of 4 pts in $\mathbb{P}^1$ is

$t_1 \quad t_2 \quad t_3 \quad t_4$

$$[t_1, t_2, t_3, t_4] = \frac{(t_1-t_2)(t_3-t_4)}{(t_1-t_3)(t_2-t_4)}$$

– proj. invariant.

For a corner of a (twisted) polygon define




$x_{\cancel{2i}\,i} = [v_{i-2}, v_{i-1}, p, q]$

$y_i = [s, r, v_{i+1}, v_{i+2}]$

$x_{i+n} = x_i, \quad y_{i+n} = y_i$

$(x_1, y_1, \dots x_n, y_n)$ – coordinates on $P_n$,

$(n \not\vdots\, 3)$

Rescaling: $R_t : (x_1, y_1, \dots, x_n, y_n) \to (t x_1, t^{-1} y_1, \dots )$

T commutes with $R_t$

Various invariants: $O_n = \prod_{i=1}^{n} x_i$ , $E_n = \prod_{i=1}^{n} y_i$

$$O_{\frac{n}{2}} = \prod_{\text{even } i} x_i + \prod_{\text{odd } i} x_i \,, \quad E_{\frac{n}{2}} = \text{— " —}$$

Define $\mathcal{I}_1 = \frac{\text{trace}^3(M)}{\det M}$

$\mathcal{I}_2 = \frac{\text{trace}^3(M^{-1})}{\det (M^{-1})}$ } rational f's of $x_i, y_i$

consider different weights of $\mathcal{I}_{1,2}$ w.r.t the action of $R_t$ (weight $k \Rightarrow R_t^*(P) = t^k P$. Weighted parts are invariants of $T$ (Schwartz)

<u>Def</u> Poisson bracket: on $P_n$

$$\{x_i, x_{i\pm 1}\} = \mp x_i x_{i+1}$$
$$\{y_i, y_{i\pm 1}\} = \pm y_i y_{i+1}$$

Otherwise $= 0$.

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## 2) Global coordinates

Twisted polygons $\rightsquigarrow$ difference equations (similar to nondegenerate curves $\rightsquigarrow$ differential eq's)

Namely, lift $v_i \in \mathbb{R}^2$ (or $\mathbb{RP}^2$) $\to V_i \in \mathbb{R}^3$ so that

$$\det (V_i, V_{i+1}, V_{i+2}) = 1$$

Then $V_{i+3} = a_i V_{i+2} + b_i V_{i+1} + V_i$, where

$a_i, b_i$ – $n$-periodic (assume $n \not\equiv 0 \pmod 3$) $\iff \exists M \in SL(3,\mathbb{R})$ s.t. $V_{i+n} = M V_i$ (i.e. $\{V_i\}$ – <u>twisted</u>)

$(a_i, b_i)$ – global coord system on $P_n$.

Rm. Relation of coord systems:

$$x_i = \frac{a_{i-2}}{b_{i-2}\, b_{i-1}}, \quad y_i = -\frac{b_{i-1}}{a_{i-2}\, a_{i-1}}$$

Rm The Poisson bracket is

$$\{a_i, a_j\} = \sum_{k=1}^{m} \left(\delta_{i,j+3k} - \delta_{i,j-3k}\right) a_i a_j$$

$$\{a_i, b_j\} = 0$$

$$\{b_i, b_j\} = \sum_{k=1}^{m} \left(\delta_{i,j-3k} - \delta_{i,j+3k}\right) b_i b_j$$

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## The continuous limit

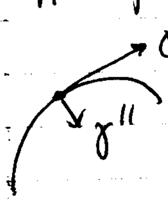
1) The space of curves

$n \to \infty$ for $n$-gon $\rightsquigarrow$ a smooth param. curve

$\gamma : \mathbb{R} \to \mathbb{RP}^2$ with monodromy $\gamma(x+1) = M(\gamma(x))$

$\forall x \in \mathbb{R}$, $M \in PGL(3, \mathbb{R})$ is fixed

Assumption: $\forall$ three conseq. pts are in general position




$\Rightarrow$ $\gamma'(x)$ & $\gamma''(x)$ are linearly indep $\forall x \in \mathbb{R}$

(i.e. no inflect pts.) – "nondegenerate curves"),

(up to projective equivalence.) $C$ – space of such curves,

Prop. $\exists$ a 1-1 corresp. between $C$ ( nondeg curves up to proj. equiv) and linear periodic DO

$$A = \frac{d^3}{dx^3} + \tilde{u}(x)\frac{d}{dx} + \tilde{v}(x)$$

with periodic coeff's $u(x)$, $w(x)$.

$$\left( = \frac{d^3}{dx^3} + \frac{1}{2}\left(u(x)\frac{d}{dx} + \frac{d}{dx}u(x)\right) + w(x) \right)$$

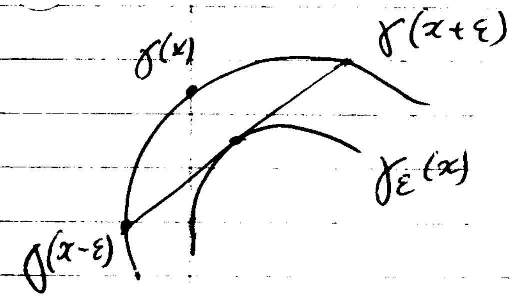
(here $w(x) = \tilde{v}(x) - \frac{\tilde{u}'(x)}{2}$)

"Pf" A nondeg. curve $\gamma(x) \subset \mathbb{RP}^2$ has a unique lift $\Gamma(x)$ to $\mathbb{R}^3$ s.t. $|\Gamma(x), \Gamma'(x), \Gamma''(x)| \equiv 1$ $\Rightarrow$ $\Gamma'''$ is a linear comb of $\Gamma, \Gamma', \Gamma''$, but

"$W(x)$"

$W \equiv \text{const} \Rightarrow$ does not depend on $\Gamma''$ $\boxtimes$

2) Continuous limit of the pent. map $T$:




Given $\gamma(x)$ draw the chord $(\gamma(x-\varepsilon), \gamma(x+\varepsilon))$, obtain $\gamma_\varepsilon(x)$ as the envelop of these chords.

Let $\gamma_\varepsilon(x) \rightsquigarrow u_\varepsilon, w_\varepsilon$ - periodic coef's for $\gamma_\varepsilon$

$$u_\varepsilon = u + \varepsilon^2 \tilde{u} + (\varepsilon^2), \quad w_\varepsilon = w + \varepsilon^2 \tilde{w} + (\varepsilon^2)$$

Define evolution: $\dot{u} = \tilde{u}, \ \dot{w} = \tilde{w}$

<u>Thm</u> (OST) The contin limit of the pentag. map $T$ is

$$\begin{cases} \dot{u} = w' \\ \dot{w} = -\dfrac{uu'}{3} - \dfrac{u'''}{12} \end{cases} \quad \text{or} \quad \ddot{u} + \frac{(u^2)''}{6} + \frac{u^{(4)}}{12} = 0$$

the clássical Boussinesq eq'n
(KdV-hierarch., eq'n (3, 2)-type on operators $A$ of order 3)

The Poisson bracket on $P_n \rightsquigarrow I^{st}$ GD bracket on $C$

<u>Problem</u> $\exists$? the second Poiss. br on $P_n$ corresp to $II^{nd}$ GD?

<u>Pf</u> The lift $\Gamma_\varepsilon$ of $\gamma_\varepsilon$ satisfies

$$\begin{cases} |\Gamma(x+\varepsilon), \Gamma(x-\varepsilon), \Gamma_\varepsilon(x)| = 0 \\ |\Gamma_\varepsilon(x), \Gamma(x+\varepsilon) - \Gamma(x-\varepsilon), \Gamma_\varepsilon'(x)| = 0 \end{cases}$$

Assume $\Gamma_\varepsilon(x) = \Gamma + \varepsilon A + \varepsilon^2 B + (\varepsilon^3)$, obtain

$$\Gamma_\varepsilon(x) = \left(1 + \frac{\varepsilon^2}{3} u\right)\Gamma + \frac{\varepsilon^2}{2}\Gamma'' + (\varepsilon^2),$$

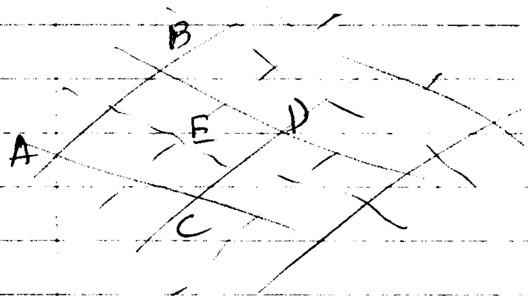
then find $u_\varepsilon, v_\varepsilon$ s.t. $\Gamma_\varepsilon''' + u_\varepsilon \Gamma_\varepsilon' + v_\varepsilon \Gamma = 0$. ∎

<u>Rm</u> Scaling symmetry: $u(x) \mapsto u(x)$
$w(x) \mapsto w(x) + t \qquad t$ - const.

Rm Pentagram map is closely related to cluster algebras (Glick 2011, Gekhtman, Shapiro, Tabachnikov, Vainshtein 2011)

In appropriate coordinates $T$ is given by rational transf's with positive coeff's, $\rightsquigarrow$ Poisson str's, combinatorics, simple Lie alg's, Teichmuller spaces, etc
Higher diagonals - corrugated curves (see GSTV)

Rm Frieze patterns




$\forall$ element $= \det$ of 4 neighbors

$$E = AD - BC$$

2 rows define everything (Conway - Coxeter) 1973

$\exists$ cluster structure

element = volumes, periodic polygons = finite width frieze

Rm $\exists$ Lax representation (F. Soloviev 2011) for the discrete system:

$$\begin{cases} L_{i,t}(z)\Psi_{i,t} = \Psi_{i+1,t}(z) \\ P_{i,t}(z)\Psi_{i,t} = \Psi_{i,t+1}(z) \end{cases}$$

$z$ spectral parameter

compatibility (zero-curvature) cond'n:

$$L_{i,t+1}(z) = P_{i+1,t}(z)\, L_{i,t}(z)\, P_{i,t}^{-1}(z)$$

$$\begin{tikzcd}
\Psi_{i,t+1} \arrow[r, "L_{i,t+1}"] & \Psi_{i+1,t+1} \arrow[r] & \cdots & {} \\
\Psi_{i,t} \arrow[u, "P_{i,t}"] \arrow[r, "L_{i,t}"] & \Psi_{i+1,t} \arrow[u, "P_{i+1,t}"'] \arrow[r] & \cdots & {} \arrow[u, "P_{i+n,t}"']
\end{tikzcd}$$

$\leftarrow$ initial polygon

$$L_{i,t}(z) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & b_i \\ 0 & z & a_i \end{pmatrix}^{-1} = \begin{pmatrix} -b_i & 1 & 0 \\ -a_i/z & 0 & 1/z \\ 1 & 0 & 0 \end{pmatrix}$$

Def Monodromy operators: $T_{0,t}, \dots T_{n-1,t}$ are defined by

$$T_{0,t} = L_{n-1,t} \dots L_{0t}$$
$$T_{1,t} = L_{0,t} L_{n-1,t} \dots L_{1t}$$
$$\vdots$$
$$T_{n-1,t} = L_{n-2,t} L_{n-3} \dots L_{n-1,t}$$

**Def.** A spectral curve of the monodromy $T_{i,t}(z)$ is
$$R(k,z) = \det\left(T_{i,t}(z) - kI\right) = 0$$

**Thm.** The spectral curve for the pent. map is
$$R(k,z) = k^3 - k^2 \operatorname{tr} T_{i,t} + k \operatorname{tr} T_{i,t}^{-1} z^{-n} - z^{-n} = 0$$
where
$$\operatorname{tr} T_{i,t}^{-1} = \sum_{j=0}^{q} I_j z^{q-j}, \quad \operatorname{tr} T_{i,t} = \sum_{j=0}^{q} J_j z^{j-q}$$
where $q = \left[\frac{n}{2}\right]$. $I_j, J_j$ - polynomials in $a_i, b_i$, $i = 0 \dots n-1$, invariants of the pent map $T$.