§ 1. Nondegenerate Hamilton systems

Recall: we start with an integrable Hamiltonian system with $H_0(I)$. Its phase space is foliated by invariant tori $I = \text{const}$. The motion on a torus is conditionally periodic with frequency vector $\omega(I) = \frac{\partial H_0}{\partial I}$.

**Def.** An invariant torus is nonresonant if from $\sum k_i \omega_i = 0$ with some $k_i \in \mathbb{Z}$ it follows $\forall k_i = 0$, i.e. frequencies are rationally independent, any trajectory is dense on the torus. Resonant tori are foliated into tori of lower dimension.

**Def.** The Hamilton system with $H_0(I)$ is nondegenerate if the frequencies are functionally independent, i.e. $\det \left( \frac{\partial \omega}{\partial I} \right) = \det \left( \frac{\partial^2 H_0}{\partial I^2} \right) \neq 0$.

**Rem.** In nondegenerate systems:
- nonresonant tori are everywhere dense, the set of full measure.
- resonant tori have measure zero, everywhere dense (like rationals among all).
- sets of invariant tori with any number of rationally independent frequencies from 1 to $n - 1$ are everywhere dense. In particular, tori with closed orbits are dense.

**Def.** A system is isoenergetically nondegenerate on $H_0$-level if one of frequencies is zero and the ratios of other $n - 1$ freqs to it are function independent on $H_0 = \text{const}$ $\ell$-level. The condition is $\det \left( \begin{pmatrix} \frac{\partial \omega}{\partial I} & \omega \\ \omega & 0 \end{pmatrix} \right) = \det \left( \begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2} & \frac{\partial H_0}{\partial I} \\ \frac{\partial H_0}{\partial I} & 0 \end{pmatrix} \right) \neq 0$.

It means nondegeneracy of mapping $I \mapsto (\omega_1(I), \omega_2(I), \ldots, \omega_n(I))$ as a map $(H_0 = \text{const}) \to \mathbb{R}^{n - 1}$. 
These two cond's are independent:

**Ex1** Consider \( H_0(I) = \sum a_i \ln I_i \) for \( I \in \mathbb{R}^n_+ \), \( \sum a_i = 0 \).

Then \( H_0 \) is nondegen: \( \det \frac{\partial^2 H_0}{\partial I_i \partial I_j} = \text{diag} \left( -\frac{a_i}{I_i^2} \right) \neq 0 \)

\[ \omega_i(I) = \frac{a_i}{I_i}, \ i = 1, \ldots, n \]

But \( H_0 \) is isoenergy, degenerate. Indeed,

\[ \det \left( \begin{array}{cc} \frac{\partial^2 H_0}{\partial I_i \partial I_j} & \frac{\partial H_0}{\partial I_j} \\ \frac{\partial H_0}{\partial I_i} & 0 \end{array} \right) = \det \left( \begin{array}{ccc} -\frac{a_i}{I_i^2} & \frac{a_n}{I_n^2} & \frac{a_n}{I_n} \\ \frac{a_i}{I_i} & \frac{a_n}{I_n} & 0 \\ \frac{a_i}{I_i} & \frac{a_n}{I_n} & 0 \end{array} \right) = \frac{a_1 - a_n (q_1 + \ldots + q_n)}{I_1^2 \ldots I_n^2} = 0 \]

Note: \( n = 2 \) \( H_0(I) = \ln I_1 - \ln I_2 = \ln \frac{I_1}{I_2} \), i.e. on \( H = \text{const} \), the ratio of \( I_i \)'s and the ratio \( \omega_1/\omega_2 \) is fixed.

**Ex2** The Ham'\( \hbox{'n} \) \( H_0(I) = I_1 + \frac{1}{2} \sum_{i=2}^n I_i^2 \)

is isoenergy nondeger. but degenerate everywhere.

For \( I = I^0 \) the frequency vector is \( \omega(I^0) = (1, I_2^0, \ldots, I_n^0) \)

i.e. \( \det \frac{\partial \omega}{\partial I} = 0 \), but ratios are independent!

§ 2 The KAM theorem

Consider a perturbed system with Hamiltonian

\[ H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon). \]

The corresponding Hamiltonian system is (for \( dI \) in \( d\varphi \))

\[ \dot{\varphi} = H_1' = \omega(I) + \varepsilon H_1' \]

\[ \dot{I} = -H_0' = 0 \quad H_1' - 2\pi\text{-periodic in } \varphi \]
Kolmogorov's thm (extended by Arnold)

If the unperturbed Hamilton system is nondegenerate (or isoenergetic nondegenerate), then under a sufficiently small Hamilton perturbation most of the nonresonant invariant tori do not disappear but are only slightly deformed, so that in the phase space of the perturbed system there exist invariant tori filled everywhere densely with phase curves, conditionally-periodically with \# frequencies = \# degrees of freedom. The invariant tori form a majority: measure of the complement is small with \( \varepsilon \). (In the case of isosce resonant the invariant tori form a majority \( \forall \) energy level.)

Rmk. Invariant tori are often called Kolmogorov tori.

In functional space of Hamilton systems

integrable systems (meas = 0)

systems with invariant tori (meas > 0)
Remarks to the KAM Thm

1. Originally proved for analytic systems
   Moser - finite smoothness $C^r$ ($r=666$ as a joke)
   Now: holds for $C^r$ with $r>2n$
   (Pöchel, Salamon ~1980's)

2. The measure of the complement of the Kolmogorov set is $\leq \sqrt{\varepsilon}$ (in any dim)

3. Let the unperturbed system be non-degen.
   Suppose we are given $\nu \in \mathbb{R}$ s.t. $n-1 < \nu \leq \frac{1}{2} n-1$
   Under a small $C^r$ perturbation, the frequency vectors of Kolmogorov tori belong to the Cantor set
   $$\mathcal{L}_\varepsilon = \left\{ \frac{\mathbf{k}}{\nu} \in \mathbb{R}^n \mid \left| \mathbf{k} \cdot \mathbf{z} \right| > \frac{x}{|k|} \quad \forall k \neq 0 \right\}$$
   where unperturbed frequencies $\omega(I)$ must be "sufficiently far" from resonance
   (a Diophantine or small divisor condition)

One can show that Lebesgue measure
$$\text{meas}(\mathcal{L} \setminus \mathcal{L}_\varepsilon) \leq C \varepsilon$$
for "most" frequencies $\mathbf{k}$, $\mathbf{r}$, $\mathbf{p}$ that bound those frequencies away from resonances.

(Similarly for the isoenergetic, non-degen. case.)

Proof is a construction of normal form for a perturbed system by the fast-convergent Newton method. (Normal form $\Rightarrow$ the torus is preserved for a perturbed torus)
4. There are weaker forms of nondegeneracy, e.g. Rüssmann nondej. (higher order derivatives $J^W(I)$) Also one uses perturbations removing the degeneracy (Herman, Fejzor, Pianzoli-Chierchia 1983). Nondej. + reduction = KAM for the solar system.

§3 Other versions of the KAM theorem

A. Consider a map of 2n-dim annulus close to an n-dim rotation:

$I' = I + \varepsilon f(I, \Psi, \varepsilon) \quad I \in B \subset \mathbb{R}^n$

$\Psi' = \Psi + h(I) + \varepsilon g(I, \Psi, \varepsilon) \mod 2\pi \subset T^n$

Suppose the map is symplectic (i.e. preserves $\int \text{Id}\Psi$) and unperturbed map ($\varepsilon = 0$) is nondegen., i.e. $\det \frac{\partial h}{\partial I} \neq 0$.

Then (Arnold, Moser). Suppose the unperturbed map is analytic and nondegen. Then $A$ suff small perturb of class $C^r$, $r > 2n + 1$ in the annulus $B \times T^n$, there are invariant tori close to tori $I = \text{const}$, and the measure of the corresponding complement is small with perturbation. The images of a pt. torus under iterations fills the torus densely.

Ex. $n = 1, \varepsilon = 0$. Nondeg. $\iff$ rotation angle changes $2\pi$-rational $\iff$ circle is nonresonant.

Nonresonant circles satisfying $|x - \frac{2\pi \Psi}{q}| > c \sqrt{\varepsilon} q^{-1}$ for some $c, q$, $n + 1 < q < \frac{1}{2} (n + 1)$ do not disappear but deform $\Psi/\Psi$. Resonant circles are destroyed.
B. Small oscillations

Consider a Hamilton system in 2n near equilibrium. Suppose it is stable in linear approximation, so

\[ \exists n \text{ eigen frequencies } w_1, \ldots, w_n \]

Assume no resonance up to order 4; i.e.

\[ k_1 w_1 + \ldots + k_n w_n \neq 0 \quad \forall 0 < |k_1| + \ldots + |k_n| \leq 4 \]

Then there is the Birkhoff normal form

\[ H = H_0 (\tau) + \text{higher order (> 4), where } H_0 (\tau) = \sum w_i \tau_i + \text{terms} \]

System is nondegenerate if

\[ \det \left( \frac{\partial^2 H_0}{\partial \tau^2} \right) = \det w_{ij} \neq 0 \]

for

\[ \tau_i = \frac{1}{2} (p_i^2 + q_i^2) \]

System is isoenergy-nondegenerate if

\[ \det \left( \frac{\partial^2 H_0}{\partial \tau \partial \tau^*} \right) = \det \left( \begin{pmatrix} w_i & 0 \\ 0 & w_i \end{pmatrix} \right) \neq 0 \]

Either case \( \Rightarrow \) the Hamiltonian is of general elliptic type

\( H_0 \) - integrable Hamilton, dynamics on tori \( \tau = \text{const} \)

\( H \) - near-integrable

Thus (Arnold, Moser) A Hamilton of general elliptic type has invariant tori close to tori of the linearized system. Those tori have relative measure, which \( \to 1 \) as \( \varepsilon \to 0 \).

For \( n = 2 \) isoenerg. nondeg. \( \Rightarrow \) Lyapunov stability

(\( \Rightarrow \) quadratic part of \( H_0 \)

is not divisible by \( H_0 \)) (see Arnold's diffusion)
§ 4. Toy model of small divisors:

The Poincaré thin uniform form of a vector field near zero.

Consider \( v(x) = Ax + \ldots \), \( A \in \text{Mat}_c(n) \), \( x \in \mathbb{C}^n \)

Assume \( \lambda = (\lambda_1, \ldots, \lambda_n) \) - distinct eigenvalues of \( A \),

\( e_1, \ldots, e_n \) - eigenvectors.

Def. \( \lambda = (\lambda_1, \ldots, \lambda_n) \) is resonance if \( J \) relation \( J_s = (m, \lambda) \)

where \( m = (m_1, \ldots, m_n) \in \mathbb{Z}^n \)

\( m_k \geq 0, \Sigma m_k \geq 2 \). \( \| m \| \) is the resonance order.

Ex. \( \lambda_1 = 2\lambda_2 \) - resonance of order 2.

\( 2\lambda_1 = 3\lambda_2 \) - no resonance.

\( \lambda_1 + \lambda_2 = 0 \) - resonance of order 3, since \( \lambda_1 = 2\lambda_2 + \lambda_3 \), etc.

Thin (Poincaré)

If \( A \) is nonresonant, then \( \dot{x} = Ax + \ldots \) by a formal change of variable \( x = y + \ldots \) can be transformed.

Pf idea: Let \( h(y) \) be vector polynomial.

Lemma 1. \( h(0) = h'(0) = 0 \). The variable change \( x = y + h(y) \) takes \( \dot{y} = Ay \rightarrow \dot{x} = Ax + v(x) + \ldots \)

where \( v(x) = \frac{\partial h}{\partial x} Ax - Ah(x) \) (\( =: L_A h \))

Pf of Lemma 1.

\[ \dot{x} = \frac{\partial x}{\partial y} \dot{y} = (E + \frac{\partial h}{\partial y}) Ay = (E + \frac{\partial h}{\partial y}) A(x - h(x) + \ldots) \]

\[ = Ax + \left[ \frac{\partial h}{\partial x} Ax - Ah(x) \right] + \ldots \]
\( R_m [\ldots] \) is the Lie bracket \([Ax, h(x)]\) of vector fields.

Notation: \([Ax, \cdot] =: LA\).

**Def.** The equation \( LA h = \nu \) on \( h \) is called the homological equation.

**Lemma 2.** Let \( x^m = x^m_1 \ldots x^m_n \), \( x^a \) - coord's in basis (\( e_1, \ldots, e_n \)). If \( A \) is diagonal, \( LA \) is diagonal in the basis \( X^m e_s \) and \( LX^m e_s = [(m, \lambda) - \lambda s] X^m e_s \).

**Pf of Lem 2.** - evident for \( h = X^m e_s \):

\[
\frac{\partial h}{\partial x} Ax = \frac{\partial X^m}{\partial x} Ax^s \equiv \sum_i m_i x^m_{x_i} X_i x^e = (m, \lambda) X^m e_s
\]

while \( Ah(x) = \lambda s h(x) = \lambda h e_{s} e_s \). 

**Corol.** Given \( \nu \), the eq \( LA h = \nu \) is solvable for \( h \), provided \( A \) is non-resonant.

**Now apply consecutively to kill higher order terms in \( h \).**

**Rm 1.** One can kill all non-resonant terms.

**Rm 2.** Similar consideration for non-diagonal \( A \) since \( LA \) - invertible.

**Rm 3.** Poincare domain \( J A \): convex hull of \( A \) does not contain 0.

Siegell domain: convex hull contains 0.

**Poincare**

**Siegell**

In Poincare case \( J \) holom. normal form

In Siegel need to have

\[
|\lambda s - (m, \lambda)| \geq \frac{C}{|m|^p}
\]

for holom. normal form

\( \lambda = (\lambda_1, \ldots, \lambda_n) \) of \((C, \rho)\)-type.