Ch 10. KAM Theory

§1. Nondegenerate Hamilt. systems

Recall: we start with an integr. Hamilt. system with $H_0(I)$

Its phase space is foliated by invar. tori $I = \text{const}$

The motion on $\forall$ torus is conditionally periodic with frequency vector $\omega(I) = \frac{\partial H_0}{\partial I}$

Def An invar. torus is nonresonance if from $\sum k_i \omega_i = 0$ with some $k_i \in \mathbb{Z}$ $\Rightarrow$ $\forall k_i = 0$, i.e. frequencies are rationally independent, any trajectory is dense on the torus.
Resonant tori are foliated into tori of lower dim.

Def. The Hamilton system with $H_0(I)$ is nondegenerate if the frequencies are functionally indep, i.e. $\det\left(\frac{\partial \omega}{\partial I}\right) := \det\left(\frac{\partial^2 H_0}{\partial I^2}\right) \neq 0$

Rm In nondegenerate systems:
- nonresonant tori are everywhere dense, the set of full measure
- resonant tori have measure zero, everywhere dense (like ration. # among all).
- sets of invar. tori with any # of rationally indep. frequencies from 1 to $n-1$ are everywhere dense. In particular, tori with closed orbits are dense

Def A system is isoenergetically nondegenerate on H_0=level if one of frequencies $\neq 0$ and the ratios of other $n-1$ freq's to it are function. indep on $\{H_0 = \text{const}\}$-level.

The condition is $\det\underbrace{\begin{pmatrix} \frac{\partial \omega}{\partial I} & \omega \\ \omega & 0 \end{pmatrix}}_{(n+1)\times(n+1) \text{ matrix}} := \det\begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2} & \frac{\partial H_0}{\partial I} \\ \frac{\partial H_0}{\partial I} & 0 \end{pmatrix} \neq 0$

It means nondegeneracy of mapping $I \mapsto (\omega_1(I) : \omega_2(I) : \ldots : \omega_n(I))$ as a map $(H_0 = \text{const}) \to \mathbb{R}P^{n-1}$

These two cond's are independent:

Ex1. Consider $H_0(I) = \sum a_i \ln I_i$ for $I \in \mathbb{R}^n_+$, $a_i \neq 0$, $\sum a_i = 0$

Then H_0 is nondegen: $\det \frac{\partial^2 H_0}{\partial I^2} = \mathrm{diag}\left(-\frac{a_i}{I_i^2}\right) \neq 0$

$$\omega_i(I) = \frac{a_i}{I_i}, \quad i=1,\dots,n$$

But H_0 is isoenerg. degenerate. Indeed,

$$\det\begin{pmatrix} \frac{\partial^2 H_0}{\partial I^2} & \frac{\partial H}{\partial I} \\ \frac{\partial H}{\partial I} & 0 \end{pmatrix} = \det\begin{pmatrix} -\frac{a_1}{I_1^2} & & & \frac{a_1}{I_1} \\ & \ddots & & \vdots \\ & & -\frac{a_n}{I_n^2} & \frac{a_n}{I_n} \\ \frac{a_1}{I_1} & \dots & \frac{a_n}{I_n} & 0 \end{pmatrix} = \frac{a_1 \dots a_n \overbrace{(a_1 + \dots + a_n)}^{0}}{I_1^2 \dots I_n^2} = 0$$

Note: $n=2$ $H_0(I) = \ln I_1 - \ln I_2 = \ln \frac{I_1}{I_2}$, i.e. on $H=\mathrm{const}$ the ratio of I_i's and the ratio ω_1/ω_2 is fixed

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Ex2 The Ham'n $H_0(I) = I_1 + \frac{1}{2}\sum_{i=2}^n I_i^2$

is isoenerg. nondeg. but degenerate everywhere

For $I = I^0$ the frequency vector is $\omega(I^0) = (1, I_2^0, \dots, I_n^0)$

i.e. $\det \frac{\partial \omega}{\partial I} = 0$, but ratios are independent!

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§ 2. The KAM theorem

Consider a perturbed system with Hamiltonian

$H(I, \varphi, \varepsilon) = H_0(I) + \varepsilon H_1(I, \varphi, \varepsilon)$. The corresp.

Hamilt. system is (for s/s $\omega = dI \wedge d\varphi$)

$$\dot I = -H'_\varphi = 0 \quad \boxed{-\varepsilon H'_{1\varphi}}$$
$$\dot \varphi = H'_I = \omega(I) \quad \boxed{+\varepsilon H_{1I}}$$

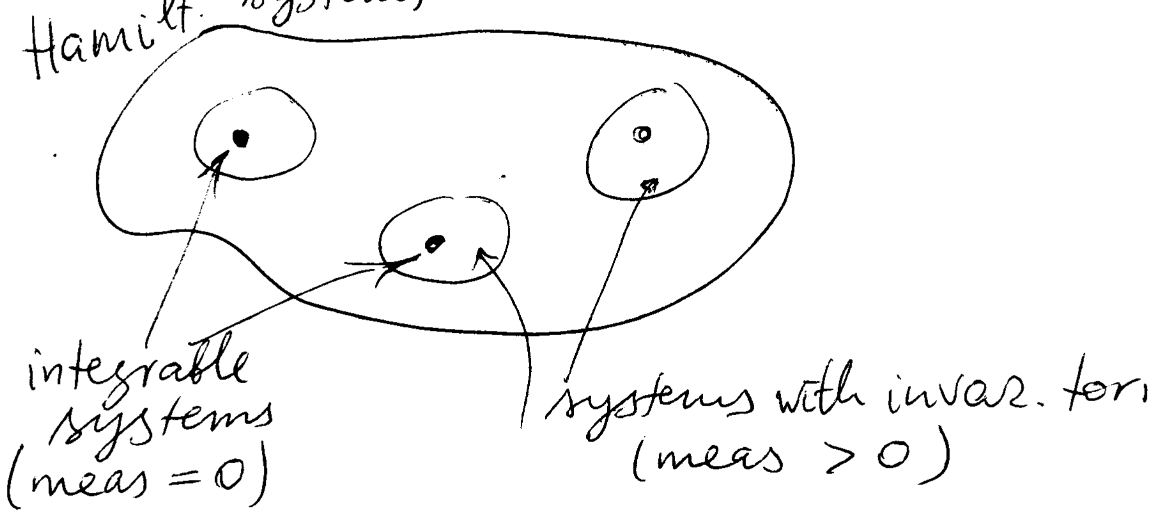
H_1 – 2π-periodic in φ

Kolmogorov's thm (extended by Arnold)

If the unperturbed Hamilt. system is nondegenerate (or isoenerget. nondegenerate), then under a suff. small Hamilt. perturbation most of the nonresonant invariant tori do not disappear, but are only slightly deformed, so that in the phase space of the perturbed system there exist invariant tori filled everywhere densely with phase curves, conditionally-periodically with # frequencies = # degrees of freedom. The invar. tori form a majority: measure of the complement is small with ε. (In the case of isoner. non-deg the invar. tori form a majority $\forall$ energy level.)

Rm. Invar. tori are often called Kolmogorov tori

In functional space of Hamilt. systems



Remarks to the KAM thm

1. Originally proved for analytic systems
   Moser - finite smoothness C^r ($r = 666$ as a joke)
   Now: holds for C^r with $r > 2n$
   (Pöchel, Salamon ~ 1980's)
2. The measure of the complement of the Kolmog. set is $\lesssim \sqrt{\varepsilon}$ (in any dim)
3. Let the unperturbed system be non-degen.
   Suppose we are given a large $\nu \in \mathbb{R}$ s.t. $n-1 < \nu < \frac{1}{2} r - 1$
   Under a suff small C^r perturbation, the frequency vectors of Kolmog. tori belong to the Cantor set

$$\Omega_{\varkappa} = \left\{ \xi \in \Omega \subset \mathbb{R}^n \;\middle|\; |\langle k, \xi \rangle| > \frac{\varkappa}{|k|^\nu} \quad \forall k \in \mathbb{Z}^n \setminus \{0\} \right\}$$

where unperturbed frequencies $\omega(I)$

frequencies must be "sufficiently far" from resonance (a diophantine or small divisor condition)

One can show that Lebesgue $\text{meas}(\Omega \setminus \Omega_{\varkappa}) \lesssim \varkappa$, i.e.

for "most" frequencies ξ $\exists \varkappa, \nu$ that found those frequencies ξ away from resonances.

(Similarly for the isoenerg. non-degen. case.)

Proof is a construction of normal form for a perturbed system by the fast-converg. Newton method. (Normal form for a perturb. torus $\Rightarrow$ the torus is preserved)

4. There are weaker forms of nondegeneracy, e.g. Rüssmann 1970 nondeg. (higher order derivatives $D^q \omega(I)$)
Also one uses perturbations removing the degeneracy (Hermann 1983, Fejoz), Pinzari-Chierchia-nondeg + reduction = KAM for the solar system

§3 Other versions of the KAM thm

A. Consider a *map* of $2n$-dim *annulus* close to an n-dim rotation:

$$I' = I + \varepsilon f(I, \varphi, \varepsilon) \qquad I \in B \subset \mathbb{R}^n$$
$$\varphi' = \varphi + h(I) + \varepsilon g(I, \varphi, \varepsilon) \qquad \varphi \bmod 2\pi \in \mathbb{T}^n$$

suppose the map is symplectic (i.e. preserves $\int I\, d\varphi$ closed countours) and unperturbed map ($\varepsilon = 0$) is nondegen., i.e. $\det \frac{\partial h}{\partial I} \neq 0$.

Thm (Arnold, Douady). Suppose the unperturb. map is analytic and nondeg. Then $\forall$ suff small perturb of class C^r, $r > 2n+1$ in the annulus $B \times \mathbb{T}^n$ there are invar. tori close to tori $I = \text{const}$, and the measure of the corresp complement is small with perturbation. The images of a pt $\in$ torus under iterations fills the torus densely.

Ex $n=1$, $\varepsilon = 0$. Nondeg $\Leftrightarrow$ rotation angle changes
α-rotation angle 2π-irrational $\Leftrightarrow$ circle is non-resonant

Nonresonant circles satisfying $\left|\alpha - \frac{2\pi p}{q}\right| > c\sqrt{\varepsilon}\, q^{-\nu}$ for some c, ν, $n+1 < \nu < \frac{1}{2}(r+1)$ $\forall p/q$
do not disappear but deform. Resonant circles are destroyed.

B. Small oscillations

Consider a Hamilt. system in $2n$ near an equilibrium
Suppose it is stable in linear approximation, so
$\exists\ n$ eigenfrequencies $\omega_1, \dots, \omega_n$

Assume no resonance up to order 4: i.e

$$k_1\omega_1 + \dots + k_n\omega_n \neq 0 \qquad \forall\, 0 < |k_1| + \dots + |k_n| \leq 4$$

Then there is the Birkhoff normal form

$$H = H_0(\tau) + \underset{\text{terms}}{\text{higher order }(>4)}, \text{ where } H_0(\tau) = \sum \omega_i \tau_i + {} + \frac{1}{2}\sum \omega_{ij}\tau_i\tau_j \text{ for } \tau_i = \frac{1}{2}(p_i^2 + q_i^2)$$

System is nondegenerate if $\det\left(\frac{\partial^2 H_0}{\partial \tau^2}\right) = \det \omega_{ij} \neq 0$

System is isoenerg. nondegen if $\det\begin{pmatrix} \frac{\partial^2 H_0}{\partial \tau^2} & \frac{\partial H}{\partial \tau} \\ \frac{\partial H_0}{\partial \tau} & 0 \end{pmatrix} = \det\begin{pmatrix} \omega_{ij} & \omega_i \\ \omega_j & 0 \end{pmatrix} \neq 0$

Either case $\Rightarrow$ the Hamiltonian is of general elliptic type

H_0 - integrable Hamilt, dynamics on tori $\tau = \text{const}$
H - near-integrable

Thm (Arnold, Moser) A Hamilt of general elliptic type has invariant tori close to tori of the linearized system. Those tori have relative measure, which $\to 1$ as $\varepsilon \to 0$.

For $n=2$ isoener. nondeg. $\Rightarrow$ Lyapuna stability
($\Rightarrow$ quadr. part of H_0 is not divisible by H_0) (see Arnold's diffusion)

§4. Toy model of small divisors:

The Poincare thm on normal form of a vect. field near zero

Consider $v(x) = Ax + \dots$, $A \in Mat_{\mathbb{C}}(n)$, $x \in \mathbb{C}^n$

Assume $\lambda = (\lambda_1, \dots, \lambda_n)$ - distinct eigenvalues of A,
$e_1, \dots, e_n$ - eigenvectors

Def. $\lambda = (\lambda_1, \dots, \lambda_n)$ is resonant if $\exists$ relation $\lambda_s = (m, \lambda)$ ← resonance
where $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$,
$m_k \geq 0$, $\sum m_k \geq 2$. $\|m\|$ is the resonance order

Ex. $\lambda_1 = 2\lambda_2$ - resonance of order 2
$2\lambda_1 = 3\lambda_2$ - no resonance
$\lambda_1 + \lambda_2 = 0$ - resonance of order 3, since
$\lambda_1 = 2\lambda_1 + \lambda_3$, etc.

Thm (Poincaré)
If A is nonresonant, then $\dot{x} = Ax + \dots$ by a formal change of variable $x = y + \dots$ can by transformed to $\dot{y} = Ay$

Pf idea: Let $h(y)$ be vector polynomial, $h(0) = h'(0) = 0$. The variable change

Lemma 1 $x = y + h(y)$ takes $\dot{y} = Ay \longmapsto \dot{x} = Ax + v(x) + \dots$
where $v(x) = \frac{\partial h}{\partial x} Ax - Ah(x)$ $(=: L_A h)$

Pf of Lem 1
$$\dot{x} = \frac{\partial x}{\partial y}\dot{y} = \left(E + \frac{\partial h}{\partial y}\right)Ay = \left(E + \frac{\partial h}{\partial y}\right)A(x - h(x) + \dots)$$
$$= Ax + \left[\frac{\partial h}{\partial x}Ax - Ah(x)\right] + \dots \quad \blacksquare$$

Rm $[\cdot\ldots]$ is the Lie bracket $[Ax, h(x)]$ of vect. fields

Notation: $[AX, \cdot] =: L_A$.

<u>Def</u> The equation $L_A h = v$ on h is called the <u>homological equation</u>.

<u>Lemma 2</u> Let $x^m = x_1^{m_1} \ldots x_n^{m_n}$, x_i -- coord's in basis $(e_1, .., e_n)$. If A is diagonal, L_A is diagonal in the basis $X^m e_s$ and $L_A X^m e_s = [(m, \lambda) - \lambda_s] X^m e_s$

<u>Pf of Lem 2</u> - evident for $h = X^m e_s$:

$$\frac{\partial h}{\partial x} A x = \frac{\partial X^m}{\partial X} A X \overset{e_s}{=} \sum_i \frac{m_i}{X_i} X^m \lambda_i X_i \overset{e_s}{=} (m, \lambda) X^m e_s$$

while $Ah(X) = \lambda_s h(X) = \lambda_s X^m e_s$ ∎

<u>Corol.</u> Given v, the eq'n $L_A h = v$ is solvable for h, provided A is nonresonant.

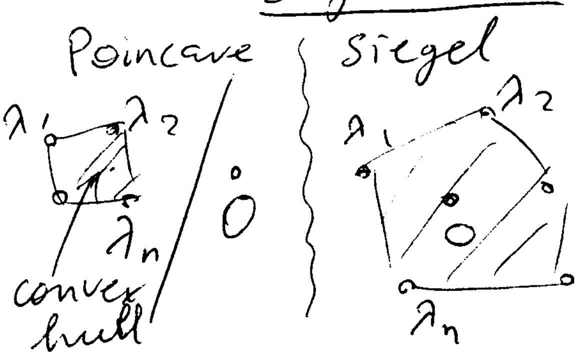
Now apply consecutively to kill higher order terms in v ∎

<u>Rm 1</u> One can kill all non-resonant terms.

<u>Rm 2</u> Similar consideration for nondiagonal A: since L_A - invertible.

Rm 3 <u>Poincare domain</u> of A: convex hull of A does not contain 0

<u>Siegel domain</u>: convex hull contains 0.



In Poincare case $\exists$ holom. normal form

In Siegel need to have $|\lambda_s - (m, \lambda)| \geq \frac{C}{|m|^\nu}$ for holom normal form

$\lambda = (\lambda_1, .., \lambda_n)$ - of (C, ν)-type