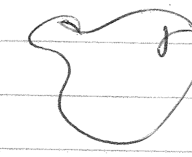
Filament (binormal, or LIA) eq'n

In $\mathbb{R}^3$ the Euler vorticity eq'n $\dot{\xi} = -L_v \xi$ for $\xi = \operatorname{curl} v$

Consider vorticity ξ supported on a (closed) curve γ



Note: Euler dynamics of γ is nonlocal: need to find $v = \operatorname{curl}^{-1} \xi$ - integral oper'n

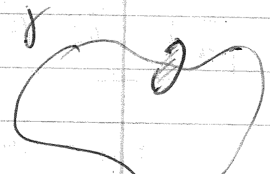
<u>Def</u> Let $\gamma \subset \mathbb{R}^3$ be arc-length-parametrized. The eq'n $\partial_t \gamma = \gamma' \times \gamma''$ is called the (vortex) filament eq'n (or binormal, LIA, Da Rios (1906) eq'n)

<u>Rm</u> What is localised induction approximation eq'n?

Assume $\xi = C\delta_\gamma$ - 2-form supported on $\gamma \subset \mathbb{R}^3$ of length L, θ-arc-length parameter, $\Rightarrow \xi(x,t) = C\int_0^L \delta(x - \gamma(\theta,t))\frac{\partial \gamma}{\partial \theta}d\theta$

δ is δ-f'n on $\mathbb{R}^3$, C = flux ξ across a small contour around γ

γ



The Biot-Savart law gives for $v(x,t) = \operatorname{curl}^{-1}\xi(x,t)$:

$$v(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-\bar{x}) \times \xi(\bar{x})}{|x-\bar{x}|^3} d^3\bar{x} = -\frac{C}{4\pi}\int_\gamma \frac{x - \gamma(\bar{\theta},t)}{|x-\gamma(\bar{\theta},t)|^3} \times \frac{\partial \gamma}{\partial \bar{\theta}} d\bar{\theta}$$

Since the Euler eq'n ξ evolution is given by the velocity v, $\frac{\partial \gamma}{\partial t}(\theta,t) = v(\gamma(\theta,t),t)$ we have

$$\frac{\partial \gamma}{\partial t}(\theta,t) = -\frac{C}{4\pi}\int_\gamma \frac{\gamma(\theta,t) - \gamma(\bar{\theta},t)}{|\gamma(\theta,t) - \gamma(\bar{\theta},t)|^3} \times \frac{\partial \gamma}{\partial \bar{\theta}} d\bar{\theta}$$

The integral diverges: goes to ∞ for small $|\theta - \bar{\theta}|$

The Taylor exp'n: $\gamma(\theta) = \gamma(\bar\theta) + \frac{\partial \gamma}{\partial \bar\theta}(\theta - \bar\theta) + \frac{1}{2}\frac{\partial^2 \gamma}{\partial \bar\theta^2}(\theta-\bar\theta)^2 + \dots$

Then
$$\frac{\partial \gamma}{\partial t} = -\frac{c}{4\pi}\int_\gamma \frac{\frac{\partial \gamma}{\partial \bar\theta}(\theta-\bar\theta) + \frac{1}{2}\frac{\partial^2\gamma}{\partial\bar\theta^2}(\theta-\bar\theta)^2}{|\theta-\bar\theta|^3} \times \frac{\partial\gamma}{\partial\bar\theta}\, d\bar\theta$$

$$= \frac{c}{8\pi}\left[\frac{\partial\gamma}{\partial\theta}\times\frac{\partial^2\gamma}{\partial\theta^2}\right]\left[\int_0^L \frac{d\bar\theta}{|\theta-\bar\theta|} + O(1)\right] \text{ as } \theta \to \bar\theta$$

cut-off beyond $|\theta-\bar\theta| > \varepsilon$ $\rightarrow$ $\int_{\gamma \setminus [-\varepsilon,\varepsilon]} \frac{d\bar\theta}{|\theta - \bar\theta|} + O(1)$ as $\varepsilon \to 0$

$\simeq \ln \varepsilon$

rescale time $t \to t \cdot \ln \varepsilon$. Obtain the (local) filament eq'n: $\partial_t \gamma = \frac{\partial\gamma}{\partial\theta}\times\frac{\partial^2\gamma}{\partial\theta^2}$, i.e. $\boxed{\partial_t\gamma = \gamma' \times \gamma''}$

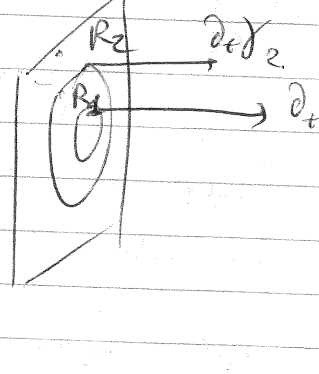
Rm. For arc-length param $\vec{t} = \gamma'$ (tangent unit)

$\gamma'' = k \cdot \vec{n}$ ← unit normal, k ← curvature

For $\vec{b} = \vec{t}\times\vec{n}$ – unit binormal, the eq'n is $\boxed{\partial_t\gamma = k\cdot\vec{b}}$

binormal eq'n, valid in any parametrization.

Ex



If $R_1 < R_2$ ← radius of γ_i

$k_1 > k_2$ ← curvature

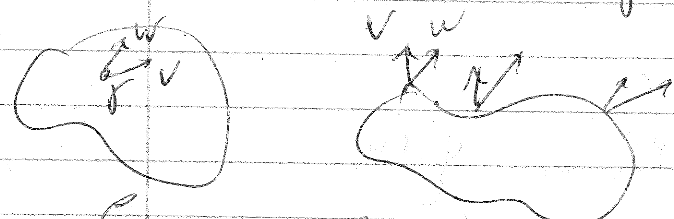
$\partial_t\gamma_1 > \partial_t\gamma_2$ ← velocity

The Marsden–Weinstein symplectic structure on the space of knots in $\mathbb{R}^3$

Consider a curve $\gamma \subset \mathbb{R}^3$, space of knots diffeom'c to γ: $\mathcal{C}$
μ – volume-form

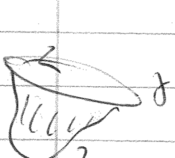
Def. The Marsden–Weinstein sympl. str-ke on $\mathcal{C}$ is

$$\omega_\gamma^{MW}(V, W) = \int_\gamma i_V i_W \mu = \text{volume of collar}(v, w, \gamma')$$



Similarly, $\exists$ s/s on codim 2 submf's in (M^4, μ)

Def. γ can be regarded as a linear f'l on div-free vect fields in $\mathbb{R}^3$: $l_\gamma(v) = \text{Flux } v|_\sigma$, where



σ – surface bounded by γ, $\partial\sigma = \gamma$.

Prop. $l_\gamma(v)$ does not depend on the choice of σ, provided $\partial\sigma = \gamma$.

Recall, that for $\mathfrak{g} = \text{SVect}(\mathbb{R}^3)$, $\mathfrak{g}^* = \Omega^1 / d\Omega^0 \simeq d\Omega^1 \simeq Z^2(\mathbb{R}^3)$
closed 2-forms

Consider δ-type 2-form ω_γ supported on γ

Then $d^{-1}\omega_\gamma = u_\sigma$ – 1-form supported on σ

(ambiguity in $\sigma \subsetneq$ ambiguity in $d^{-1}\omega_\gamma = u_\sigma$)

Prop The pairing $[u_\sigma]$ with $v \in \text{SVect}$ coincides with pairing γ with v.

Indeed, $\langle [u_\sigma], v \rangle = \int_{\mathbb{R}^3} i_v u_\sigma \wedge \mu = \int_{\mathbb{R}^3} u_\sigma \wedge i_v \mu = \int_\sigma i_v \mu = \text{Flux } v|_\sigma$ $\blacksquare$

Consider $\gamma \in \Omega^1/d\Omega^0 = \mathfrak{g}^*$ and look at its orbit $\mathcal{O}_\gamma \subset \mathfrak{g}^*$.

<u>Prop</u> The Kirillov-Kostant s/str (Lie-Poisson str) on $\mathcal{O}_\gamma$ coincides with the Marsden-Weinstein s/s on $\{\gamma\}$

<u>Pf</u>

$$\omega_{KK}(\xi_\gamma)(V,W) = \langle d^{-1}\xi, [V,W]\rangle = \langle [u_\gamma], [V,W]\rangle$$
$$= \langle u_\gamma, i_{[V,W]}\mu\rangle$$

Note: $i_{[V,W]}\mu = d i_V i_W \mu$ for div-free v.f's V, W

Then $\omega_{KK}(\xi_\gamma)(V,W) = \int_{\mathbb{R}^3} u_\gamma \wedge d i_V i_W \mu = \int_{\mathbb{R}^3} \underbrace{du_\gamma}_{\omega_\gamma = \delta_\gamma} \wedge i_V i_W \mu = \int_\gamma i_V i_W \mu$
$= \omega_\gamma^{MW}(V,W)$. $\boxtimes$

Now to define dynamics on γ.

fix the Euclidean ~~m~~metric in $\mathbb{R}^3$. Define Hamiltonian f'n $H(\gamma) = \text{length}(\gamma) = \int_{S^1} \sqrt{(\gamma'(\theta), \gamma'(\theta))}\, d\theta$

Let θ be arc-length parameter

<u>Lemma</u> The filament eq'n $\partial_t \gamma = \gamma' \times \gamma''$ is Hamiltonian on $\mathcal{O}_\gamma$ with Ham f'n $H(\gamma)$ and the Marsden-Weins s/s ω^{MW}

<u>Pf</u> Find the variational derivative $\frac{\delta H}{\delta \gamma}$ ("gradient") of $H(\gamma)$ on $\mathcal{O}$:

By def, it is defined by

$$H(\gamma + \varepsilon v) = H(\gamma) + \varepsilon \langle \frac{\delta H}{\delta \gamma}, v\rangle + O(\varepsilon^2)$$

Here $H(\gamma+\varepsilon v) = \int\limits_{S^1} \sqrt{(\gamma'+\varepsilon v', \gamma'+\varepsilon v')}\, d\theta = \int\limits_0^L \sqrt{\underset{\|}{(\gamma',\gamma')} + 2\varepsilon(\gamma',v') + O(\varepsilon^2)}\, d\theta$

Use: $(1+x)^\alpha = 1+\alpha x + \dots \longrightarrow$ $\qquad 1$

$$\int_0^L \left(1 + \varepsilon(\gamma', v') + O(\varepsilon^2)\right) d\theta$$

$\Rightarrow \dfrac{\delta H}{\delta \gamma} = -\gamma''$ for arc-length param θ.

Then $\partial_t \gamma = \text{sgrad}\, H = -J_\gamma\left(\dfrac{\delta H}{\delta \gamma}\right) = \gamma' \times \gamma''$

where J – the operator of rotation by $\pi/2$ in the normal plane to γ'.

<u>Cor</u> In particular, lenth(γ) is preserved. □

<u>Rm</u>. For $L = \gamma'$ (Euler map) obtain: $\dot{\gamma}' = \gamma' \times \gamma''' \Leftrightarrow \dot{L} = L \times L''$

It is Landau-Lifschitz eq'n, Euler-Arnold on $\widetilde{SO}(3)$ w/ H^{-1}-metric

<u>Rm</u>. For γ consider k, τ – curvature & torsion of γ

For $v = \tau$, $\rho = k^2$ obtain eq's $\begin{cases} \partial_t v + v v' = -\nabla \varphi(\rho) \\ \partial_t \rho + (\rho v)' = 0 \end{cases}$ – eq's of 1D compressible fluid

continuity eq'n

<u>Rm</u> Hasimoto (1972) showed that the f'n

$$\psi(x) = k(x)\, e^{i \int_{x_0}^{x} \tau(\tilde{x})\, d\tilde{x}}$$

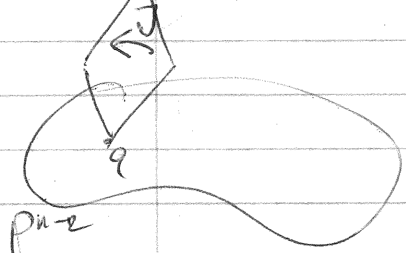
satisfies the

NLS (nonlinear Schroedinger eq'n): $i\partial_t \psi + \psi'' + |\psi|^2 \psi = 0$

Thus the binomal eq'n is completely integrable,

One can pull back first integrals:

$\int k, \int k^2, \int \tau k^2, \dots$



Higher-dimensional version, Skew-mean-curvature flow

$P^{n-2} \subset \mathbb{R}^n$ μ-volume form. – closed, codim 2 submfd

Def Marsden-Weinstein s/s : $\omega_P^{MW}(V, W) = \int_{P^{n-2}} \underbrace{i_V i_W \mu}_{(n-2)\text{-form}}$

Hamiltonian f'n $H(P) = \text{volume}_{n-2}(P)$

Thm Hamilt. eq'n for $H(P)$ and s/s ω_P^{MW} is the skew-mean-curvature flow: $\dot{q} = J(\vec{MC}_P(q))$ for $q \in P^{n-2} \subset \mathbb{R}^n$. Here J - (almost complex str're) - operator of $\frac{\pi}{2}$-rotation in $N_q P$

Def: For $q \in P^l \subset \mathbb{R}^n$, the mean curvature vector $\vec{MC}(q) \in N_q P$ is 1) the $\underset{l}{\text{trace}}$ of the second fundamental form $\mathrm{II}_q : T_q P \to N_q P$ (i.e. sum over basis $e_i \in T_q P$)

2) Equivalently, $\vec{MC}(q)$ = average geodesic curvature of P in directions $\vec{u} \in S^{l-1} \subset T_q P$
unit tangent sphere

Ex. $n=3$ $\vec{MC} = k \cdot \vec{n}$, $J(\vec{MC}) = k \cdot \underset{\vec{t} \times \vec{n}}{\vec{b}}$

Pf idea grad $H = \frac{\delta H}{\delta P} = \vec{MC}$ (mean curvat. direction $\Leftrightarrow$ fastest volume change) $\Rightarrow$
sgrad $H = J(\vec{MC})$, since ω^{MW} is the averaging s/s in each $N_q P$.

Q What is the higher-dim Hasimoto transform?

Note: for $\rho = |MC|^2$ is it true $W = \int \rho = \int |MC|^2$ is

Q: preserved during the evolution?

W is the willmore energy, in 1D it is preserved:

$$\partial_t W^{\pi} = \int_{S^1} \partial_t (\gamma'', \gamma'') = -2 \int_{S^1} (\partial_t \gamma', \gamma''') = \int_{S^1} (\gamma' \times \gamma''', \gamma''') = 0$$
