Ch 9. Action-angle variables

Recall: The Arnold-Liouville thm:

M^{2n} - sympl. mfd, $F_1, \ldots, F_n : M \to \mathbb{R}$ are in involution, ~~dt~~ $M_f = \{x \in M, F_i(x) = f_i\}$ (common level of $\{F_i\}$)

If 1) their diff's dF_i are linear indep $\forall x \in M_f$

2) the Hamilt fields $J\,dF_i$ $(i \le n)$ are complete on M_f

then a) $\forall$ conn. comp. of M_f is diffeo to $\mathbb{T}^k \times \mathbb{R}^{n-k}$

b) on $\mathbb{T}^k \times \mathbb{R}^{n-k}$ $\exists$ coord's $\varphi_1, \ldots \varphi_k$ mod 2π, $y_1, \ldots, y_{n-k}$

s.t. the i-th Ham. eq'n ~~$\dot{x} = J\,dF_i$~~ has the form $\dot{\varphi}_m = \omega_{mi}$, $\dot{y}_s = c_{si}$ ω, c - const.

The Hamilt. system with Ham. f'n F_i $\forall i = 1..n$ is called **completely integrable**.

The action-angle thm

Suppose the above assumptions hold, M_f is connected & cpt.

Then: 1) a small neighb. of $M_f \subset M$ is diffeo to $D \times \mathbb{T}^n$, where D - small domain in $\mathbb{R}^n$

2) in $D \times \mathbb{T}^n$ $\exists$ sympl. coord's I, φ mod 2π $(I \in D, \varphi \in \mathbb{T}^n)$ s.t. (in these variables) f's $F_1, \ldots, F_n$ depend only on I and s/s has the form $dI \wedge d\varphi$

Rm. In particular, Ham f'n $H = H(I)$,

$$\dot{I} = -H'_\varphi = 0, \quad \dot{\varphi} = H'_I = \omega(I)$$

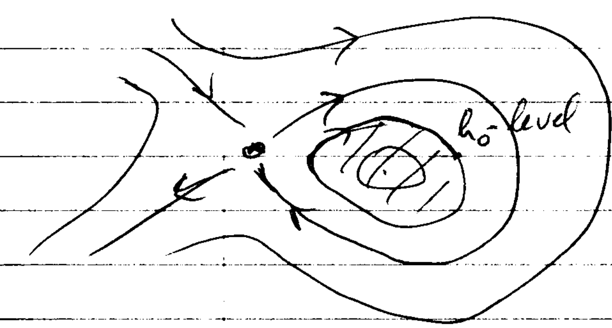
I - action variables (numbers invar tori) $I(t) = I_0$

φ - angle variables (uniformly change) $\omega(I) = \omega(I_0)$

Ex Ham. syst with 1 degree of freedom, $H : \mathbb{R}^2_{p,q} \to \mathbb{R}$

h_0 - regular value of H, $H = h_0$ - ~~bounded~~

$M_h = \{H = h\}$ for h close to h_0 diffeo to circles



$\forall h$ $\exists$ an angle coord, uniform
φ mod 2π chang. in time.

Then the conj variable is

$\frac{\Pi(h)}{2\pi}$, where $\Pi(h)$ = area inside M_h,

since $dp \wedge dq = \frac{1}{2\pi} d\Pi \wedge d\varphi$

Then $I(h) = \frac{1}{2\pi} \iint_{H \le h} dp \wedge dq = \frac{1}{2\pi} \oint_{H=h} p\,dq \Rightarrow I(h)$ is action f'l
(⇒ its name)

Pf First define $I_i = F_i$, φ_i mod 2π in a neigh. of $M_f \simeq \mathbb{T}^n$ (from Arn-Liouv. thm). Consider the matrix

$$\begin{pmatrix} \{I_i, I_j\}, & \{I_i, \varphi_j\} \\ \{\varphi_i, I_j\}, & \{\varphi_i, \varphi_j\} \end{pmatrix} = \begin{pmatrix} 0 & a_{ij} \\ -a_{ji} & b_{ij} \end{pmatrix}$$

By Arn-Liou $\{I_i, \varphi_j\} = L_{v_i} \varphi_j$ = const on M_f, i.e. $a_{ij} = a_{ij}(I)$

Prop ~~Note~~: b_{ij} also depends on I only. Indeed, by the Jacobi identity

$$\{F_m, \underbrace{\{\varphi_i, \varphi_j\}}_{\|}\} + \{\varphi_i, \underbrace{\{\varphi_j, F_m\}}_{\| \atop p(F)}\} + \{\varphi_j, \{F_m, \varphi_i\}\} = 0$$

Then $\{F_m, b_{ij}\} = \alpha^m_{ij}$ - indep of φ. On the other hand,

$$\alpha^m_{ij} = \sum_s \frac{\partial b_{ij}}{\partial \varphi_s} \{F_m, \varphi_s\} = \sum \frac{\partial b_{ij}}{\partial \varphi_s} a_{ms} \Rightarrow \text{find } \frac{\partial b_{ij}}{\partial \varphi_s} \text{ - f's of } \underline{I}$$

since $\det(a_{ms}) \neq 0$. Then $b_{ij} = \{\varphi_i, \varphi_j\} = \sum f^s_{ij}(I)\, \varphi_s + g_{ij}(I)$.

I.e. b_{ij} can be only linear f's of φ

Since b_{ij} - univalued f's $\Rightarrow f^s_{ij} \equiv 0$, i.e. $b_{ij} = b_{ij}(I)$ ☐

Now: change of variables: $I_s = I_s(J_1, \dots J_n)$ s.t. $\{J_i, \varphi_j\} = \delta_{ij}$
For this solve the system

$$a_{ij}(I) = \{I_i, \varphi_j\} = \sum \frac{\partial I_i}{\partial J_s} \delta_{sj} = \frac{\partial I_i}{\partial J_j}$$

The solubility cond'n

$$\frac{\partial a_{ij}}{\partial J_s} = \frac{\partial a_{is}}{\partial J_j} \Leftrightarrow \sum_k \frac{\partial a_{ij}}{\partial I_k} a_{ks} = \sum_k \frac{\partial a_{is}}{\partial I_k} a_{kj}$$

here we use that $b_{ij}(I)$

follows from the Jacobi identity applied to $I_i, \varphi_j, \varphi_k$
Finally, after construction of J_k, if φ_i do not commute, pass to ψ_i mod 2π s.t. $\varphi_i = \psi_i + f_i(J)$

$$b_{ij} = \{\varphi_i, \varphi_j\} = \{\psi_i + f_i(J), \psi_j + f_j(J)\}$$

$$= \{f_i(J), \psi_j\} + \{\psi_i, f_j(J)\} = \frac{\partial f_i}{\partial J_j} - \frac{\partial f_j}{\partial J_i}$$

The local solubility of this is closedness of the 2-form ($i,j = 1, \dots n$)

$$\overline{b} = \sum b_{ij}\, dI_i \wedge dI_j \Leftrightarrow \overline{b} = d\overline{f} \text{ for } \underset{\text{1-form}}{\overline{f}} = f_i\, dJ_i$$

The fact that $\overline{b}$ is a closed 2-form follows from $d\omega = 0$. Alternatively, use the Jacobi for $0 = \circlearrowleft \{\varphi_i \{\varphi_j, \varphi_k\}\} = \circlearrowleft \{\varphi_i, b_{jk}(J)\}$, which implies the relation on derivatives of b_{jk} ☒.

Rm. p, q - sympl. coord's in $\mathbb{R}^{2n}$,
$\gamma_1 \dots \gamma_n$ - basis cycles on $M_f \cong T^n$, contin. dep. on $f = f_1 \dots f_n$.

Then $\oint_{\gamma_s} p\,dq - \oint_{\gamma_s} I\,d\varphi = \oint_{\gamma_s} p\,dq - 2\pi I_s$ is const on $\mathbb{R}^{2n}$,

since $p\,dq - I\,d\varphi$ is closed. Hence $I_s = \frac{1}{2\pi}\oint_{\gamma_s} p\,dq$, $1 \le s \le n$

defines I_s (up to additive const).

Def. Ham-syst with $H(I)$ is nondegenerate in $D \times \mathbb{T}^n$ if

$$\frac{\partial \omega}{\partial I} = \det \left\| \frac{\partial^2 H}{\partial I^2} \right\| \neq 0 .$$

Then almost all invar tori are nonresonance, i.e.
$\sum k_i \omega_i = 0$ with $k_i \in \mathbb{Z}$ $\Rightarrow$ $\forall k_i = 0$. Traj's are dense on
Nonreson tori ~~have~~ Resonance tori are dense in $D \times \mathbb{T}^n$
(like rational among all #'s.)

Def Ham. system is properly completely degenerate if $\frac{\partial \omega}{\partial I} \equiv 0$.

Rm Non-commutative integrability

M^{2n} - sympl. mfd, $F_1, \dots, F_k : M \to \mathbb{R}$ - smooth, diff'ls indep. a.e. on M.
Assume $\{F_i, F_j\} = c_{ij}^k F_s$ - i.e. F's closed under $\{\ \}$, form a real Lie alg $\mathfrak{g}$, $\mathbb{R}k\,\mathfrak{g}$

Def A Hamilt. system is integrable in noncommutative sense if it admits an algebra $\mathcal{F}$ of first integrals satisfying the following cond'n:
at a generic pt $x \in M$ the subspace $K \subset T_x^* M$ generated by $dF(x)$, $F \in \mathcal{F}$ is coisotropic

Recall: Coisotropy condition $K^{\angle} \subset K$, where
$$K^{\angle} = \{\ell \in T_x^* M \mid \omega^{-1}(\ell, K) = 0\}$$

Non-commutative Liouville thm

(Nekhoroshev, Mischenko, Fomenko, Brailov)

Consider a common level of the integrals

$M_f = \{ F_i(x) = f_i, \ i = 1 \dots k \}$

If M_f is regular, cpt & connected, then it is a torus T^{2n-k} with quasiperiodic motion

Note. Comm integr $\in$ non-com. integr

Non-com. integr $\subset$ comm integr locally

Non commut. integrability:

M^{2n}, $F_1, \dots, F_k : M \to \mathbb{R}$, $\{F_i, F_j\} = \sum C_{ij}^k F_k$

$\mathfrak{g}$ - finite-dim Lie alg. formed by F_k

$\mathrm{rk}\, \mathfrak{g}$ = max rank of $a_{ij} = \{F_i, F_j\}$ over $\forall x \in M$

Thm Assume dF_i are linearly indep on

$M_f = \{ x \in M \mid F_i(x) = f_i \quad i = 1, \dots k \}$

and $\mathfrak{g}$ satisfies the condition $2 \dim \mathfrak{g} - \mathrm{rk}\, \mathfrak{g} = \dim M$

(Ex For Arnold-Liouv. $\dim \mathfrak{g} = n$, $\mathrm{rk}\, \mathfrak{g} = 0$)

If M_f - connected, cpt $\Rightarrow M_f \simeq \mathbb{T}^l$, $l = \frac{1}{2}(\dim M - \mathrm{rk}\, \mathfrak{g})$

if $F_1, \dots F_k$ - first int'ls of a system w/ Ham H, then $\exists$ angle coord's $\varphi_1 \dots \varphi_l$ mod 2π s.t. eq'n is $\dot{x} = J dH(x)$

$\Leftrightarrow \quad \dot{\varphi}_s = \omega_s = \text{const}.$