Geometric Fluid Dynamics

Henan University, Sept - Oct 2021

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Lecture 8

Arnold's setting for the Euler equation

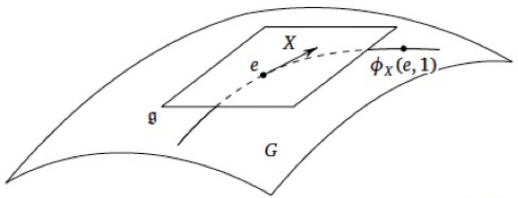
M — a Riemannian manifold with volume form μ
v — velocity field of an inviscid incompressible fluid filling M
The classical *Euler equation* (1757) on v:

$$\partial_t v + \nabla_v v = -\nabla p\,.$$

Here $\operatorname{div} v = 0$ and v is tangent to ∂M.
$\nabla_v v$ is the Riemannian covariant derivative.

Theorem (Arnold 1966)

The Euler equation is the geodesic flow on the group $G = \mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms w.r.t. the right-invariant L^2-metric $E(v) = \frac{1}{2}\int_M (v, v)\,\mu$ (fluid's kinetic energy).



Application: Other groups and energies

Group	Metric	Equation
$SO(3)$	$\langle \omega, A\omega \rangle$	Euler top
$E(3) = SO(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equation for a body in a fluid
$SO(n)$	Manakov's metrics	n-dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
$\mathrm{Diff}(S^1)$	$\dot{H}^{1/2}$	Constantin-Lax-Majda-type equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
$\mathrm{Diff}_\mu(M)$	L^2	Euler ideal fluid
$\mathrm{Diff}_\mu(M)$	H^1	averaged Euler flow
$\mathrm{Symp}_\omega(M)$	L^2	symplectic fluid
Diff(M)	L^2	EPDiff equation
$\mathrm{Diff}_\mu(M) \ltimes \mathrm{Vect}_\mu(M))$	$L^2 \oplus L^2$	magnetohydrodynamics
$C^\infty(S^1, SO(3))$	H^{-1}	Heisenberg magnetic chain

Remark These are Hamiltonian systems on $\mathfrak{g}^*$ with the quadratic Hamiltonian=kinetic energy for the Lie-Poisson bracket.

There are suitable functional-analytic settings of Sobolev (H^s for $s > 1 + n/2$) and tame Fréchet (C^∞) spaces.

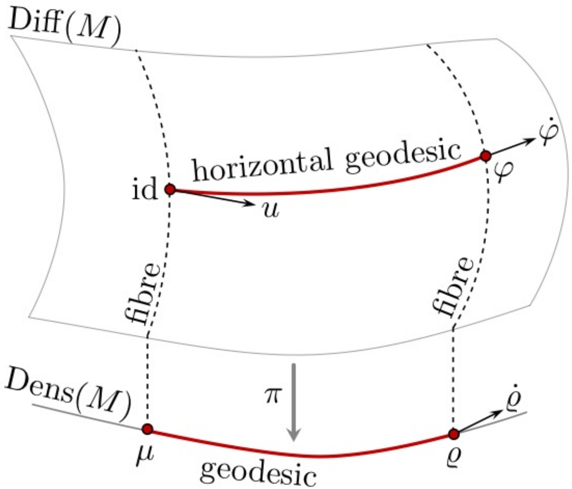
Exterior geometry of $\mathrm{Diff}_\mu(M) \subset \mathrm{Diff}(M)$

$\mathrm{Dens}(M)$ — the *space of smooth density functions* ("probability densities") on M:

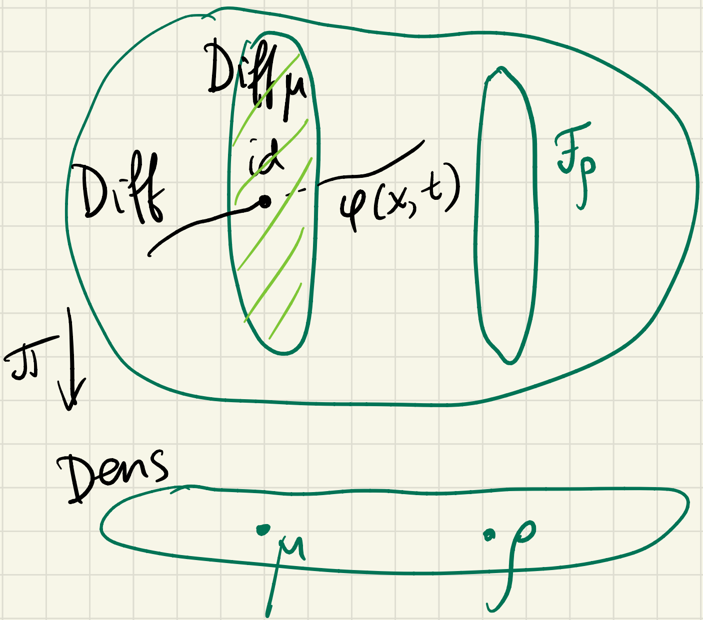
$$\mathrm{Dens}(M) = \{\rho \in C^\infty(M) \mid \rho > 0, \int_M \rho\mu = 1\}$$

Note:
$\mathrm{Dens}(M) = \mathrm{Diff}(M)/\mathrm{Diff}_\mu(M)$, the space of (left) cosets of $\mathrm{Diff}_\mu(M)$, with the projection $\pi\colon \mathrm{Diff}(M) \to \mathrm{Dens}(M)$.

Fibers are $\pi^{-1}(\varrho)$
$= \{\varphi \in \mathrm{Diff}(M) \mid \varphi_*\mu = \varrho\}$.



Rm We regard $\mathrm{Diff}_\mu(M) \subset \mathrm{Diff}(M)$ as a subgroup in the group of all diffeo's. Given a reference density μ the group Diff is fibered over the space of densities Dens.



Def For a curve $t \mapsto \varphi(\cdot, t) \in \mathrm{Diff}$ its length is $\ell(\varphi(\cdot,t)) = \int_0^1 \int_M (\partial_t \varphi, \partial_t \varphi)\mu \, dt$

For a flat M we think of $\varphi \in L^2(M,M)$, pre-Hilbert space with metric on $\partial_t \varphi(x,t) = v(\varphi(x,t),t)$

$$(v,v)_\varphi := (\partial_t \varphi, \partial_t \varphi) := \int_M (\partial_t \varphi, \partial_t \varphi)\mu = \|\partial_t \varphi\|^2_{L^2(M)}$$

Geometry of $\mathrm{Diff}(M)$

Remark Compare "the dimensions" of the fiber and the base:

$\dim(M) =$	1	2	3	...
$\mathrm{Diff}_\mu(M)$	$\approx \mathrm{Iso}(M)$	$\approx \mathrm{Ham}(M)$	$\approx \mathrm{Vect}_\mu(M)$	$\approx \mathrm{Vect}_\mu(M)$
	$\wedge$	$\wr$	$\vee$	$\vee$
$\mathrm{Dens}(M)$	$\approx C^\infty(M)$	$C^\infty(M)$	$C^\infty(M)$	$C^\infty(M)$

Define an *L^2-metric* on $\mathrm{Diff}(M)$ by

$$G_\varphi(\dot\varphi, \dot\varphi) = \int_M |\dot\varphi|_\varphi^2 \mu.$$

For a flat M this is a flat metric on $\mathrm{Diff}(M)$.

It is right-invariant for the $\mathrm{Diff}_\mu(M)$-action (but not $\mathrm{Diff}(M)$-action):
$G_\varphi(\dot\varphi, \dot\varphi) = G_{\varphi\circ\eta}(\dot\varphi \circ \eta, \dot\varphi \circ \eta)$ for $\eta \in \mathrm{Diff}_\mu(M)$.

The Euler geodesic property for a flat M

Let a flow $(t,x) \mapsto g(t,x)$ be defined by its velocity field $v(t,x)$:

$$\partial_t g(t,x) = v(t, g(t,x)),\ \ g(0,x) = x.$$

The chain rule immediately gives the acceleration

$$\partial^2_{tt} g(t,x) = (\partial_t v + \nabla_v v)(t, g(t,x)).$$

Geodesics on $\mathrm{Diff}(M)$ are straight lines, $\partial^2_{tt} g(t,x) = 0$, which is equivalent to the *Burgers equation*

$$\partial_t v + \nabla_v v = 0.$$

The Euler equation $\partial_t v + \nabla_v v = -\nabla p$ is equivalent to

$$\partial^2_{tt} g(t,x) = -(\nabla p)(t, g(t,x)),$$

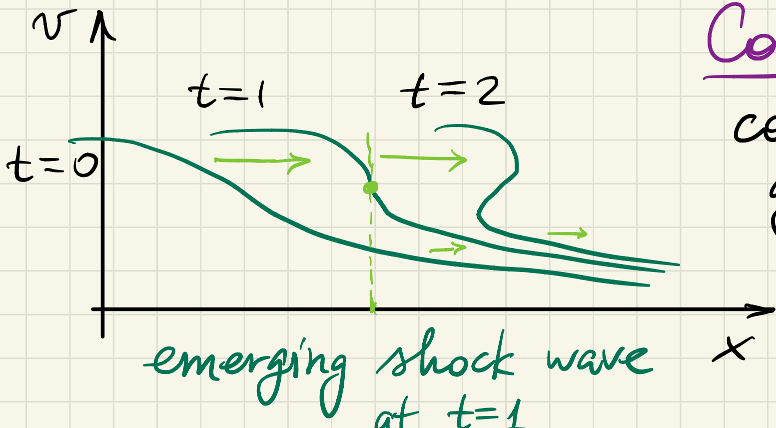
which means that the acceleration $\partial^2_{tt} g \perp_{L^2} \mathrm{Diff}_\mu(M)$.

Hence the flow $g(t,.)$ is a geodesic on the submanifold $\mathrm{Diff}_\mu(M) \subset \mathrm{Diff}(M)$ for the L^2-metric.

Example: The 1D Burgers eq'n is $\partial_t v + v v' = 0$,

$\Leftrightarrow \quad \partial^2_{tt} g = 0$, i.e (cf: $\partial_t v + v v' + v''' = 0$, KdV)

the acceleration of each particle is $= 0$, i.e. it flies with constant velocity.



Cor Geodesics on Dens(M) correspond to horizontal geodesics on Diff(M) $\leftrightarrows$ potential solutions of Burgers (recall: the Hodge decomp:

$$\mathrm{Vect}(M) = \mathrm{Vect}_\mu(M) \oplus_{L^2} \mathrm{Grad}(M))$$

Geometry of $\mathrm{Diff}(M)$ (cont'd)

Theorem (Otto 2000)

The left coset projection π is a Riemannian submersion with respect to the L^2-metric on $\mathrm{Diff}(M)$ and the Kantorovich-Wasserstein metric on $\mathrm{Dens}(M)$.

Definition of the Kantorovich-Wasserstein (L^2) metric

The *KW distance* between $\mu, \nu \in \mathrm{Dens}(M)$:

$$\mathrm{Dist}^2(\mu,\nu) := \inf\{\int_M \mathrm{dist}^2_M(x,\varphi(x))\,\mu \mid \varphi_*\mu = \nu\}\,.$$

The corresponding *Riemannian metric* on $\mathrm{Dens}(M)$:

$$\bar{G}_\rho(\dot\rho,\dot\rho) = \int_M |\nabla\theta|^2\rho\mu, \quad \text{for } \dot\rho + \mathrm{div}(\rho\nabla\theta) = 0,$$

where $\dot\rho \in C_0^\infty(M)$ is a tangent vector to $\mathrm{Dens}(M)$ at the point $\rho\mu$.

Rm For a map $x \mapsto g(x)$ taking density $\mu(x)$ to $\rho = g_* \mu$,

i.e $\rho(y) = h(y)\mu(y)$ for $y = g(x)$,

its jacobian satisfies $h(g(x)) \det \frac{\partial g}{\partial x} \equiv 1$.

One can show that in $\mathbb{R}^n$ an optimal map has the form $g = \nabla f$ for a convex function f

$\Rightarrow$ $\det(\text{Hess} f(x)) = \dfrac{1}{h(\nabla f(x))}$, the Monge-Ampere eq'n on optimal potential f.

Geodesics on Dens(M) are solutions of the optimal transport problem, projections of horiz. geodesics in Diff(M).

Hamiltonian view on a Riemannian submersion

Let $\pi : P \to B$ be a principal bundle with the structure group G.

A *Riemannian submersion* $\pi : P \to B$ preserves lengths of horizontal tangent vectors to P.
Geodesics on B can be lifted to horizontal geodesics in P, and the lift is unique for a given initial point in P.

For $P/G = B$ the symplectic reduction (over 0-momentum) is $T^*P//G = T^*B$.
If P is equipped with a G-invariant Riemannian metric $<,>_P$ it induces the metric $<,>_B$ on the base B.

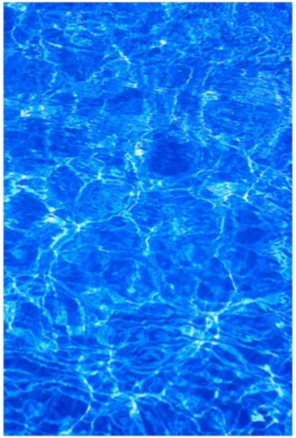
Proposition *The Riemannian submersion of P to the base B, equipped with the metrics $<,>_P$ and $<,>_B$ is the result of the symplectic reduction $T^*P//G = T^*B$ with metric identification of T and T^*.*

The Euler equation for barotropic fluids

v — velocity field of a compressible fluid filling M
ρ — density of the fluid
The equations of a compressible (barotropic) fluid (or gas dynamics) are

$$\begin{cases} \partial_t v + \nabla_v v + \frac{1}{\rho}\nabla P(\rho) = 0 \\ \partial_t \rho + \mathrm{div}(\rho v) = 0, \end{cases}$$

for the pressure function $P(\rho) = e'(\rho)\rho^2$.



Here $e(\rho)$ is the internal energy depending on fluid's properties.
For an ideal gas $P(\rho) = C \cdot \rho^a$ with $a = 5/3$ for monatomic gases (argon, krypton) and $a = 7/3$ for diatomic gases (such as nitrogen, oxygen, and hence approximately for air).

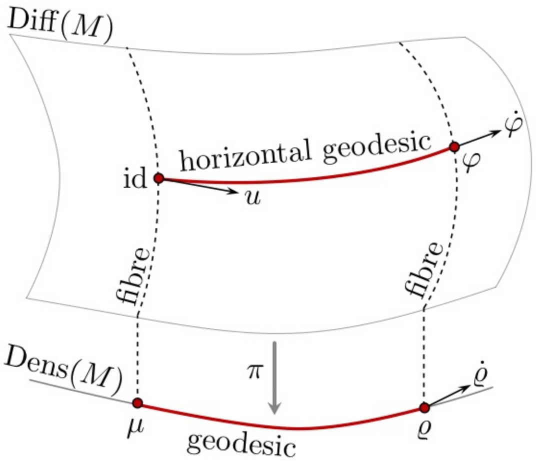
Barotropic fluid as a Newton's equation

Theorem (Smolentsev, K.-Misiolek-Modin)

The equations of a compressible barotropic fluid with internal energy $e(\rho)$ are equivalent to Newton's equations $\nabla_{\dot\varphi}\dot\varphi = -\nabla(\delta U/\delta\rho)\circ\varphi$ on $\varphi \in \mathrm{Diff}(M)$ for the potential $U(\rho) = \int_M e(\rho)\rho\,\mu$.

Equivalently, this is the Hamiltonian system on $T^*\mathrm{Diff}(M)$ with $H = K + \bar U$, where $\bar U(\varphi) = U(\rho)$ for $\rho = \det(D\varphi^{-1})$.

For $v = \nabla\theta$ the equation descends to $\mathrm{Dens}(M)$.



Other Newton's equations in the L^2-geometry

— *Classical mechanics*: $U(\rho) = \int_M V(x)\rho\mu$ for a smooth potential function V on $M \Longrightarrow$ Burgers equation with potential $\dot{v} + \nabla_v v + \nabla V = 0$.

— *Shallow water equations*: quadratic potential $U(\rho) = \frac{1}{2}\int_M \rho^2\mu \Longrightarrow$ $\dot{v} + \nabla_v v + \nabla\rho = 0$

— *Fully compressible fluids*: potential $U(\rho, \sigma)$, smaller symmetry group, larger quotient $\mathrm{Dens}(M) \times \Omega^n(M) \Longrightarrow \dot{v} + \nabla_v v + \rho^{-1}\nabla P(\rho, \sigma) = 0$ and the continuity equations for ρ and σ

— *Compressible MHD*: smaller symmetry group $\mathrm{Diff}_\mu(M) \cap \mathrm{Diff}_{\beta_0}(M)$; potential $U = \int_M e(\rho)\rho\mu + \frac{1}{2}\int_M \beta \wedge \star\beta$

— *Relativistic Burgers equation*: for $\varphi\colon [0,1] \times M \to M$ the action is

$$S(\varphi) = -\int_0^1 \int_M c^2 \sqrt{1 - \frac{1}{c^2}\,|\dot{\varphi}|^2}\,\mu dt$$

Alternative approach: semidirect products

Mantra: *see the continuity equation* $\Longrightarrow$ *look for a semidirect product group.*

Example

For the group $S = \mathrm{Diff}(M) \ltimes C^\infty(M)$ with product

$$(\varphi, f) \cdot (\psi, g) = (\varphi \circ \psi, \varphi_* g + f), \quad \varphi_* g = g \circ \varphi^{-1}$$

define the energy function on $\mathfrak{s}$

$$E(v, \varrho) = \int_M \left(\frac{1}{2}(v, v)\, \rho + \rho\, e(\rho) \right) \mu.$$

Then the Hamiltonian equation on $\mathfrak{s}^*$ gives the baropropic fluid with $P(\rho) = \rho^2 e'(\rho)$.

Similarly for MHD, a rigid body in a fluid, etc. See F.Dolzhansky, D.Holm, J.E.Marsden, R.Montgomery, T.Ratiu, A.Weinstein, ...

Hydrodynamics and Quantum Mechanics

Theorem (Madelung, von Renesse)

The (non)linear Schrödinger equation

$$i\partial_t\psi + \Delta\psi + V\psi + f(|\psi|^2)\psi = 0$$

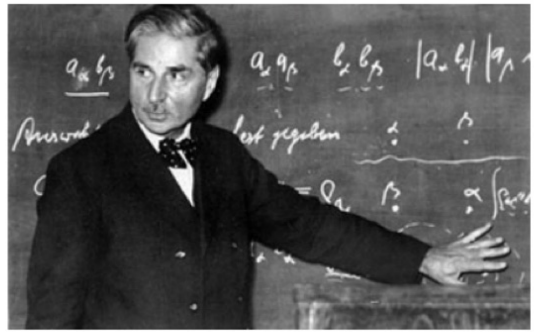
on the wave function $\psi : M \to \mathbb{C}$ *on an n-dim manifold* M*, where* $V : M \to \mathbb{R}$ *and* $f : \mathbb{R}_+ \to \mathbb{R}$*, is mapped by the transform* $\psi = \sqrt{\rho e^{i\theta}}$ *to the equations of a barotropic-type fluid*

$$\begin{cases} \partial_t v + \nabla_v v + 2\nabla\left(V + f(\rho) - \dfrac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \\ \partial_t\rho + \operatorname{div}(\rho v) = 0 \end{cases}$$

for $v = \nabla\theta$*.*

It is regarded as a hydrodynamical form of QM.

Madelung and his paper



Erwin Madelung

1881 - 1972

E. Schrödinger "An Undulatory Theory of the Mechanics of Atoms and Molecules" Physical Review, Dec. 1926.

E. Madelung "Quantentheorie in hydrodynamischer Form" Z. Phys. 1927.

Quantentheorie in hydrodynamischer Form.

Von **E. Madelung** in Frankfurt a. M.

(Eingegangen am 25. Oktober 1926.)

Geometry behind Madelung

The *Madelung transform* $\Phi : (\rho, \theta) \mapsto \psi = \sqrt{\rho e^{i\theta}}$. More precisely:

For (ρ, θ) we have $\int \rho = 1$, $\rho > 0$ and $[\theta] = \{\theta + C \mid \forall C \in \mathbb{R}\}$, i.e. we have $(\rho, [\theta]) \in T^*\mathrm{Dens}(M)$.

For ψ we have $\psi \neq 0$, $\|\psi\|_{L^2}^2 = 1$ and $[\psi] = \{\psi e^{i\alpha} \mid \forall \alpha \in \mathbb{R}\}$, i.e. we have $[\psi] \in \mathbb{P}C^\infty(M, \mathbb{C} \setminus 0)$.

Hence,

Definition

The Madelung transform is

$$\Phi : T^*\mathrm{Dens}(M) \to \mathbb{P}C^\infty(M, \mathbb{C} \setminus 0),$$

where

$$(\rho, [\theta]) \mapsto [\psi] \quad \text{for} \quad \psi = \sqrt{\rho e^{i\theta}}.$$

Madelung transform as a symplectomorphism

Consider the space of normalized densities $\mathrm{Dens}(M)$ and projectivize wave functions $\mathbb{P}C^\infty(M,\mathbb{C})$. Now regard $(\rho,[\theta]) \in T^*\mathrm{Dens}(M)$.

Theorem (K.-Misiolek-Modin)

The Madelung transform $\Phi : (\rho,[\theta]) \mapsto [\psi]$ for $\psi = \sqrt{\rho e^{i\theta}}$ induces a symplectomorphism

$$\Phi\colon\ T^*\mathrm{Dens}(M) \to \mathbb{P}C^\infty(M,\mathbb{C}\backslash\{0\})$$

for the canonical symplectic structure of $T^*\mathrm{Dens}(M)$ and the natural Fubini-Study symplectic structure of $\mathbb{P}C^\infty(M,\mathbb{C})$.

The Madelung transform is a symplectic submersion to the unit sphere in $L^2(M,\mathbb{C})$ (von Renesse).

Thus the Madelung transform maps Hamiltonian systems to Hamiltonian ones: the Hamiltonian

$$H(\psi) = \frac{1}{2}\int_M |\nabla\psi|^2\mu + \frac{1}{2}\int_M (V|\psi|^2 + F(|\psi|^2))\mu$$

of the Schrödinger equation on (the projectivization of) $C^\infty(M, \mathbb{C})$ for $F' = f$ is taken to the Hamiltonian

$$\tilde{H}(\rho, \theta) = \frac{1}{2}\int_M |\nabla\theta|^2\rho\mu + \frac{1}{2}\int_M \frac{|\nabla\rho|^2}{\rho}\mu + 2\int_M (V\rho + F(\rho))\mu\,.$$

on $T^*\mathrm{Dens}(M)$.

H^1-metrics on $\mathrm{Diff}(M)$ and information geometry

Example

For $M = S^1$ and right-invariant metrics on $\mathrm{Diff}(S^1)$:
the L^2-metric $E(v) = \frac{1}{2}\int v^2\, dx \Longrightarrow$ the Burgers equation

$$v_t + 3vv_x = 0;$$

the H^1-metric $\frac{1}{2}\int v^2 + (v')^2\, dx \Longrightarrow$ the Camassa–Holm equation

$$v_t + 3vv_x - v_{txx} - 2v_x v_{xx} - vv_{xxx} + cv_{xxx} = 0;$$

the $\dot{H}^1$-metric $\frac{1}{2}\int (v')^2\, dx \Longrightarrow$ the Hunter–Saxton equation

$$v_{xxt} + 2v_x v_{xx} + vv_{xxx} = 0$$

For any compact M the (degenerate) $\dot{H}^1$-metric on $\mathrm{Diff}(M)$ is given by $(v, v) = \frac{1}{4}\int_M (\mathrm{div}\ v)^2 \mu$ and it descends to $\mathrm{Dens}(M)$
The projection $\pi : \mathrm{Diff}(M) \to \mathrm{Dens}(M)$ is $\varphi \mapsto \rho = \sqrt{|\mathrm{Det}(D\varphi)|}$.

H^1-metrics (cont'd)

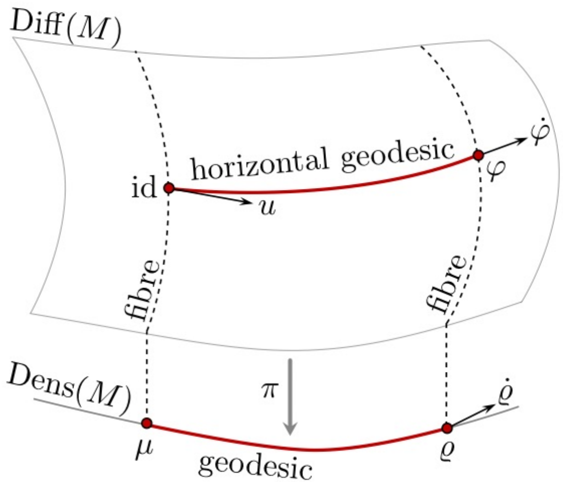
What is the induced metric on $\mathrm{Dens}(M)$?

Theorem (K., Lenells, Misiolek, Preston 2010)

There exists an isometry $\mathrm{Dens}(M) \approx U \subset S_r^\infty, \quad r = \sqrt{\mu(M)}$
(an open part of an inf-dim sphere).

Corollary

– *This is the* *Fisher-Rao metric* *on* $\mathrm{Dens}(M)$ *used in geometric statistics;*
– *It has constant curvature, explicit description of geodesics on* $\mathrm{Dens}(M)$, *their integrability.*



Summary of two metrics on $\mathrm{Dens}(M)$ so far

The **Kantorovich-Wasserstein metric**:

$$G_\rho^{KW}(\dot\rho,\dot\rho) = \int_M |\nabla\theta|^2\,\rho\mu \quad \text{for } \dot\rho + \mathrm{div}(\rho\nabla\theta) = 0$$

(depends on the Riemannian structure on M).

The **Fisher-Rao metric**:

$$G_\rho^{FR}(\dot\rho,\dot\rho) = \int_M \left(\frac{\dot\rho}{\rho}\right)^2 \rho\mu$$

(independent of the Riemannian structure on M).

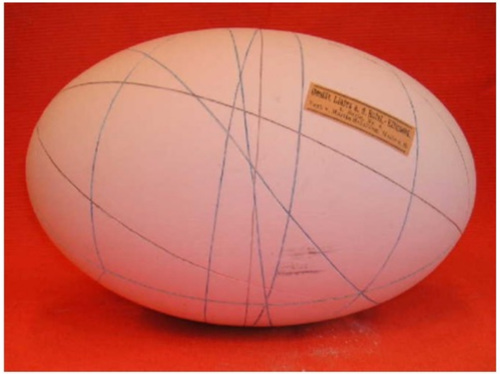
Newton's Equations for H^1-metrics

Step aside: the Neumann problem

The classical (finite-dimensional) Neumann problem is a system on the tangent bundle TS^n with the Lagrangian given by

$$L(q, \dot{q}) = \frac{(\dot{q}, \dot{q})}{2} - q \cdot Aq, \quad \text{where } q \in S^n \subset \mathbb{R}^{n+1}$$

and where A is a symmetric positive definite $(n+1) \times (n+1)$ matrix. This system is related to the geodesic flow on the ellipsoid $x \cdot Ax = 1$ and is integrable on T^*S^n.



Neumann problem (cont'd)

For the unit sphere $S^\infty(M) = \left\{ f \mid \int_M f^2 \mu = 1 \right\} \subset C^\infty(M) \cap L^2(M)$ take the quadratic potential $V(f) = \frac{1}{2}\langle \nabla f, \nabla f \rangle_{L^2} = \frac{1}{2}\int_M |\nabla f|^2 \mu$.

An infinite-dimensional Neumann problem: Find extremals $f : [0,1] \to S^\infty(M)$ minimizing the action functional

$$L(f, \dot f) = \frac{1}{2}\langle \dot f, \dot f \rangle_{L^2} - \frac{1}{2}\langle \nabla f, \nabla f \rangle_{L^2} = \frac{1}{2}\int_M (\dot f^2 + f \Delta f)\mu.$$

Consider the Fisher information functional on $\mathrm{Dens}(M)$:

$$I(\rho) = \frac{1}{2}\int_M \frac{|\nabla \rho|^2}{\rho}\, \mu,$$

Theorem (K.-Misiolek-Modin)

Newton's equations on $\mathrm{Dens}(M)$ with respect the Fisher-Rao metric and the Fisher information potential is equivalent to the infinite-dimensional Neumann problem, with the map $\rho \mapsto f = \sqrt{\rho}$ establishing the isomorphism.

Madelung as an isometry and a Kähler map

The Fisher-Rao metric on $\mathrm{Dens}(M)$ gives rise to the *Fisher-Rao-Sasaki metric* on $T^*\mathrm{Dens}(M)$:

$$G^{FRS}_{\rho,[\theta]}((\dot\rho,\dot\theta),(\dot\rho,\dot\theta)) := \int_M \frac{(\dot\rho)^2}{\rho}\mu \quad + \int_M (\dot\theta)^2 \rho\mu$$

Theorem (K.-Misiolek-Modin)

The Madelung transform Φ is an isometry (and hence a Kähler map) between the spaces $T^*\mathrm{Dens}(M)$ equipped with the Fisher-Rao-Sasaki metric and $\mathbb{P}C^\infty(M,\mathbb{C}\backslash\{0\})$ equipped with the Fubini-Study metric.

The (infinite-dimesional) *Fubini–Study metric* on $\mathbb{P}C^\infty(M,\mathbb{C})$ is

$$G^{FS}(\dot\psi,\dot\psi) := \frac{<\dot\psi,\dot\psi>}{\|\psi\|^2_{L^2}} - \frac{<\psi,\dot\psi><\dot\psi,\psi>}{\|\psi\|^4_{L^2}}$$

Madelung transform as a momentum map [D.Fusca]

Definition

The semidirect product group $S = \mathrm{Diff}(M) \ltimes C^\infty(M) \ni (\varphi, a)$ acts on the space $C^\infty(M, \mathbb{C}) \ni \psi$ of wave functions as follows:

$$(\varphi, a) \circ \psi = \sqrt{|\mathrm{Det}(D\varphi^{-1})|}\, e^{-ia/2}(\psi \circ \varphi^{-1}).$$

(ψ is pushed forward by a diffeomorphism φ as a complex-valued half-density, followed by a pointwise phase adjustment by $e^{-ia/2}$).

This action
– descends to the space of cosets $[\psi] \in \mathbb{P}C^\infty(M, \mathbb{C})$,
– is Hamiltonian.

Madelung transform as a momentum map (cont'd)

Theorem (D.Fusca 2017)

The momentum map

$$\mathbf{M}\colon C^\infty(M,\mathbb{C}) \to \mathfrak{s}^* = \Omega^1(M) \times \mathrm{Dens}(M)$$

for the group S-action on the space of wave functions $C^\infty(M,\mathbb{C})$ given by

$$\psi \mapsto (m,\rho) = \big(2\,\mathrm{Im}(\bar\psi\, d\psi), \bar\psi\psi\big)$$

is the inverse of the Madelung transform $(\rho,\theta) \mapsto \psi = \sqrt{\rho e^{i\theta}}$, where $\rho > 0$, in the following sense: if $\psi = \sqrt{\rho e^{i\theta}}$ then $\mathbf{M}(\psi) = (\rho\, d\theta, \rho)$.

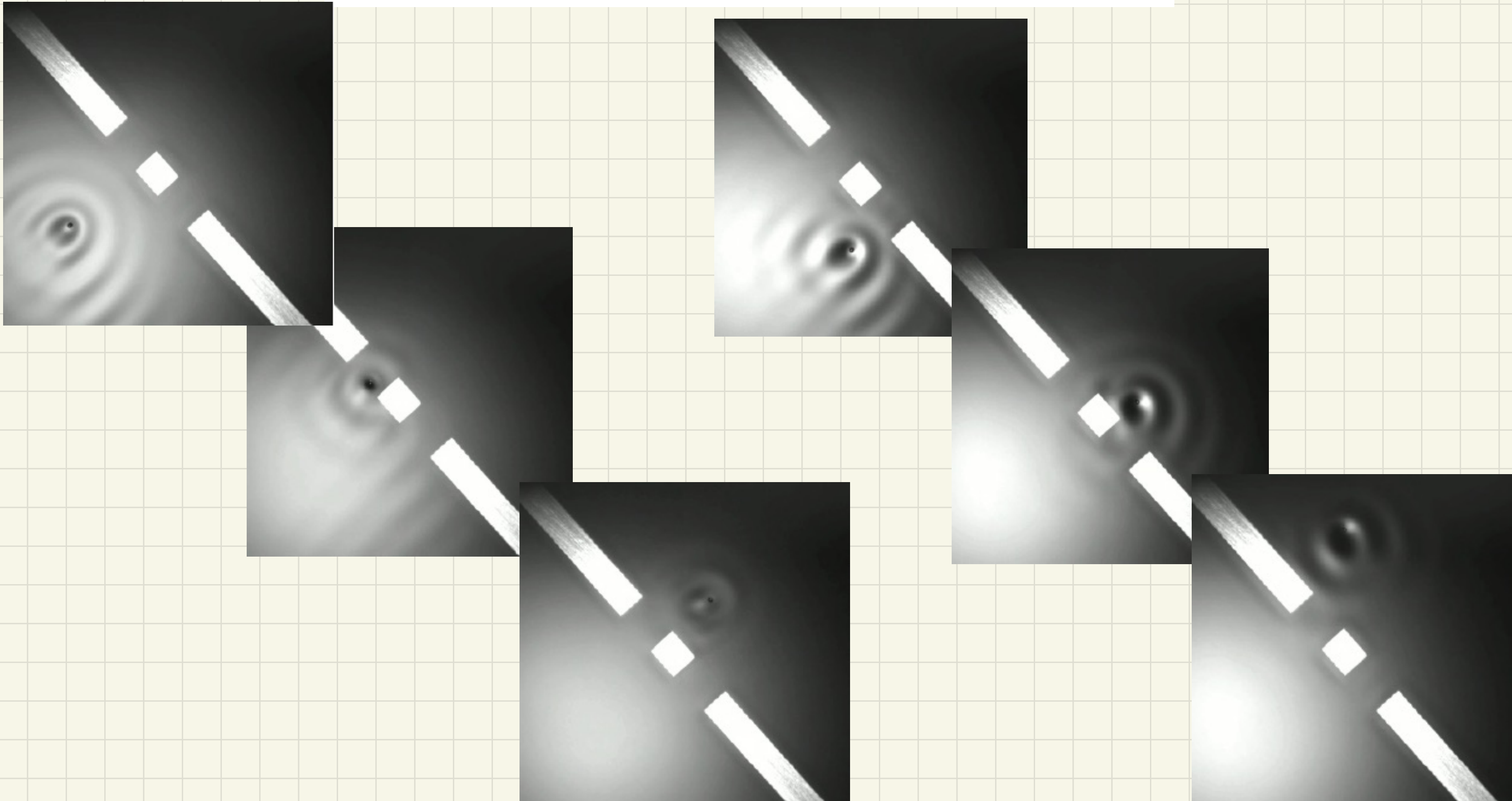
Remark This might resolve T.C.Wallstrom's critique (1994) of inequivalence between the Schrödinger equation and its hydrodynamic form, requiring a quantization condition around zeros of ψ: consider the map $\psi \mapsto (m,\rho)$ for $m = \rho\, d\theta$ rather than $\psi \mapsto (d\theta, \rho)$.

Madelung and bouncing droplets?

Corollary: The Madelung transform provides a Kähler map, a strong connection of QM and hydrodynamics.

Maybe this tighter Madelung connection could explain similarity of bouncing droplets and QM?

Beautiful pictures of pilot-wave hydrodynamics



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Thank you!

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