Geometric Fluid Dynamics

Henan University, Sept - Oct 2021

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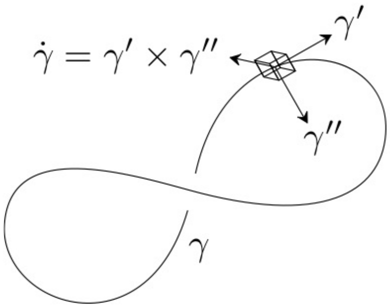
Lecture 7

The binormal equation

Let $\gamma \subset \mathbb{R}^3$ be a closed arc-length parametrized curve, $\gamma = \gamma(s,t)$.
The *vortex filament* equation is

$$\partial_t \gamma = \gamma' \times \gamma'',$$

where $\gamma' := \partial\gamma/\partial s$.



Other names: Localized Induction Approximation (LIA) equation,
Da Rios equations (1906)

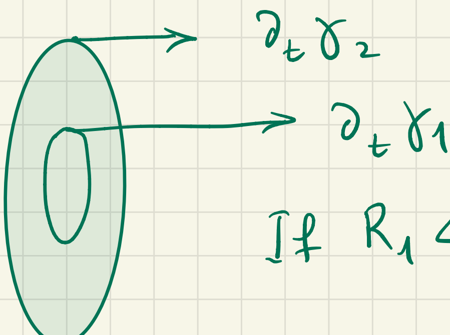
Vortex rings in nature

In any parametrization it is the *binormal equation*

$$\partial_t \gamma = \kappa \, \mathbf{b} \, ,$$

where κ is the curvature of γ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ is the binormal unit vector at any point of γ.

Rings of smaller radius move faster!



If $R_1 < R_2 \Rightarrow k_1 > k_2 \Rightarrow \partial_t \gamma_1 > \partial_t \gamma_2$

$k_1 = 1/R_1$, $k_2 = 1/R_2$

Properties of the binormal equation

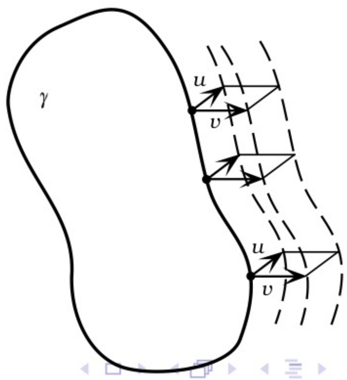
– **it is Hamiltonian**:

The Hamiltonian function is the *length* $H(\gamma) = \int_\gamma |\gamma'(s)|\, ds$ of γ.

The symplectic structure is the *Marsden-Weinstein symplectic structure* ω^{MW} on the space of knots:

$$\omega^{MW}(\gamma)(u, v) = \int_\gamma i_u i_v \mu = \int_\gamma \mu(u, v, \gamma')\, ds,$$

where u and v are two vector fields attached to γ, and μ is the volume form in $\mathbb{R}^3$.



Properties of the binormal equation

– **it is integrable**:

To a curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ with curvature κ and torsion τ, the *Hasimoto transformation* assigns the following wave function $\psi : \mathbb{R} \to \mathbb{C}$

$$(k(s), \tau(s)) \mapsto \psi(s) = \kappa(s) e^{i \int_{s_0}^{s} \tau(x)\, dx},$$

where s_0 is some fixed point on the curve.
(The ambiguity in the choice of s_0 defines the wave function ψ up to a phase.)

This Hasimoto map takes the binormal equation to the 1D nonlinear Schrödinger (NLS) equation on for $\psi(\cdot, t) : \mathbb{R} \to \mathbb{C}$:

$$i\partial_t \psi + \psi'' + \frac{1}{2}|\psi|^2\psi = 0\,.$$

Properties of the binormal equation

– it is equivalent to a barotropic-type fluid

Introduce the density $\rho = \kappa^2$ and the velocity $v = 2\tau$ for a curve γ governed by the binormal flow. Then ρ and v satisfy the system of compressible 1D fluid equations:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho v) = 0, \\ \partial_t v + v\, v' + \left(-\rho - 2\frac{\sqrt{\rho}''}{\sqrt{\rho}}\right)' = 0. \end{cases}$$

Thus there is an equivalence of three evolution equations:

– the binormal equation

– the 1D nonlinear Schrödinger equation

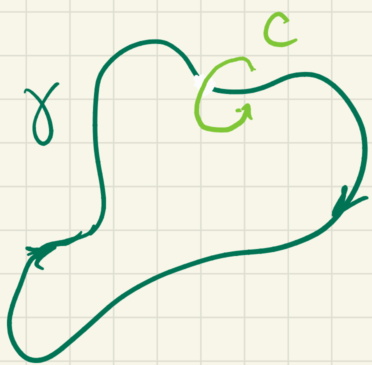
– the 1D barotropic-type fluid equation.

What of this remains in higher dimensions?

What is Localized Induction Approximation (LIA)?

Recall that in $\mathbb{R}^3$ the Euler eq'n has the vorticity form: $\partial_t \xi = -L_v \xi$ for the field $\xi = \operatorname{curl} v$.

Let ξ be a singular vorticity, supported on a closed curve $\gamma \subset \mathbb{R}^3$.



Note: the Euler dynamics of γ is nonlocal: one needs to find $v = \operatorname{curl}^{-1} \xi$, which is an integral operator.

Set $\xi = c\,\delta_\gamma$ for the 2-form δ_γ supported on $\gamma \subset \mathbb{R}^3$,
c is the flux of ξ across a small contour around γ.

Symbolically, $\xi(x,t) = c\int_0^L \delta\left(x - \gamma(\theta,t)\right)\frac{\partial\gamma}{\partial\theta}\,d\theta$

where δ is the δ-function in $\mathbb{R}^3$, θ is the arc-length parameter on γ (of length L).

The Biot-Savart law gives for $v(x,t) = \mathrm{curl}^{-1}\xi(x,t)$:

$$v(x,t) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{(x-\bar{x})\times\xi(\bar{x})}{\|x-\bar{x}\|^3}\,d^3\bar{x} = -\frac{c}{4\pi}\int_\gamma \frac{x-\gamma(\bar{\theta},t)}{\|x-\gamma(\bar{\theta},t)\|^3}\times\frac{\partial\gamma}{\partial\bar{\theta}}\,d\bar{\theta}$$

Since the Euler equation is the evolution given by the velocity v, $\partial_t \gamma(\theta,t) = v(\gamma(\theta,t),t)$, we have

$$\partial_t \gamma(\theta,t) = -\frac{c}{4\pi}\int_\gamma \frac{\gamma(\theta,t)-\gamma(\bar\theta,t)}{\|\gamma(\theta,t)-\gamma(\bar\theta,t)\|^3} \times \frac{\partial\gamma}{\partial\theta}\, d\bar\theta$$

This integral diverges: it goes to ∞ for small $\theta-\bar\theta$!

Indeed, consider the Taylor expansion:

$$\gamma(\theta) = \gamma(\bar\theta) + \frac{\partial\gamma}{\partial\theta}(\theta-\bar\theta) + \frac{1}{2}\frac{\partial^2\gamma}{\partial\theta^2}(\theta-\bar\theta)^2 + \cdots$$

Then $$\partial_t\gamma = -\frac{c}{4\pi}\int_\gamma \frac{\frac{\partial\gamma}{\partial\theta}(\theta-\bar\theta) + \frac{1}{2}\frac{\partial^2\gamma}{\partial\theta^2}(\theta-\theta)^2 + \cdots}{|\theta-\bar\theta|^3} \times \frac{\partial\gamma}{\partial\theta}\, d\bar\theta,$$ i.e.

$$\partial_t\gamma = \frac{c}{8\pi}\left(\frac{\partial\gamma}{\partial\theta}\times\frac{\partial^2\gamma}{\partial\theta^2}\right)\left[\int_0^L \frac{d\bar\theta}{|\theta-\bar\theta|} + O(1)\right] \text{ as } \theta\to\bar\theta$$

This integral is ∞, but we apply a cut-off beyond

$|\theta-\bar\theta| > \varepsilon$, i.e. $\displaystyle\int_0^L \frac{d\bar\theta}{|\theta-\bar\theta|} \approx \int_{[-\varepsilon,\varepsilon]} \frac{d\bar\theta}{|\theta-\bar\theta|} + O(1) \simeq \ln\varepsilon + O(1)$

as $\varepsilon \to 0$

Now rescale time $t \rightsquigarrow t\cdot\ln\varepsilon$ and obtain the (local)

filament equation $\partial_t\gamma = \dfrac{\partial\gamma}{\partial\theta} \times \dfrac{\partial^2\gamma}{\partial\theta^2}$, i.e. $\boxed{\partial_t\gamma = \gamma' \times \gamma''}$

Rm For arc-length parameter

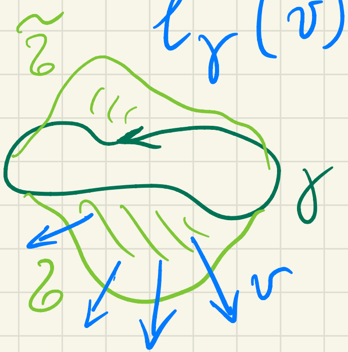
$\gamma' = \vec{t}$ (unit tangent), $\gamma'' = k\cdot\vec{n}$ (unit normal), $\vec{b} = \vec{t}\times\vec{n}$ (unit binormal)

(k ↑ curvature of γ)

the filament equation is $\boxed{\partial_t\gamma = k\cdot\vec{b}}$, binormal equation, valid $\forall$ parameter.

The Marsden-Weinstein symplectic structure on knots

Def. An oriented curve $\gamma \subset \mathbb{R}^3$ is a linear functional l_γ on div-free vector fields in $\mathbb{R}^3$:



$$l_\gamma(v) = \text{Flux}\, v\big|_S = \int_S i_v \mu,$$ where μ - volume form in $\mathbb{R}^3$,

S - oriented surface bounded by γ, $\partial S = \gamma$.

Prop $l_\gamma(v)$ does not depend on the choice of S, provided that $\partial S = \gamma$.

Pf $$\int_S i_v \mu - \int_{\tilde{S}} i_v \mu = \int_{S \cup \tilde{S}} i_v \mu = 0,$$ as a closed 2-form $i_v\mu$ over a closed surface $S \cup \tilde{S}$.

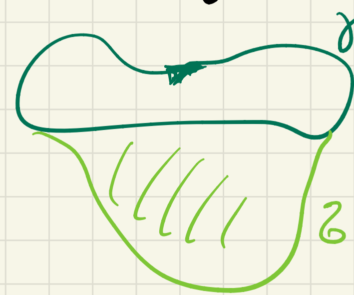
(Recall: $d i_v \mu = L_v \mu = 0$)

QED.

Rm Recall that for the Lie algebra $\mathfrak{g} = \mathrm{Vect}_\mu(\mathbb{R}^3)$ the dual space is $\mathfrak{g}^* = \Omega^1 / d\Omega^0 (\mathbb{R}^3) \simeq d\Omega^1(\mathbb{R}^3) \simeq Z^2(\mathbb{R}^3)$, the space of closed 2-forms in $\mathbb{R}^3$.

Let ω_γ be the δ-type 2-form supported on γ.

Then $d^{-1}\omega_\gamma = u_\sigma$, i.e. δ-type 1-form supported on σ, where $\partial\sigma = \gamma$



Note:

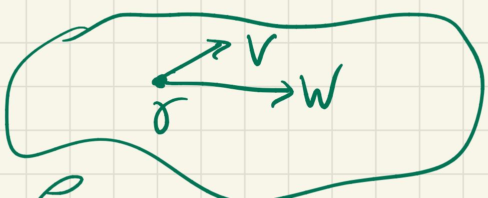
different choices of σ, s.t. $\partial\sigma = \gamma$ $\leftrightarrows$ different choices of $u_\sigma = d^{-1}\omega_\gamma$

Prop The pairing of $[u_\mathcal{C}] \in \mathfrak{g}^*$ with $v \in \mathfrak{g} = \mathrm{Vect}_\mu(\mathbb{R}^3)$ coincides with the pairing of ℓ_γ and v.

Pf $\langle [u_\mathcal{C}], v \rangle = \int_{\mathbb{R}^3} i_v u_\mathcal{C} \wedge \mu = \int_{\mathbb{R}^3} u_\mathcal{C} \wedge i_v \mu = \int_\mathcal{C} i_v \mu = \mathrm{Flux}\, v \big|_\mathcal{C}$.

QED

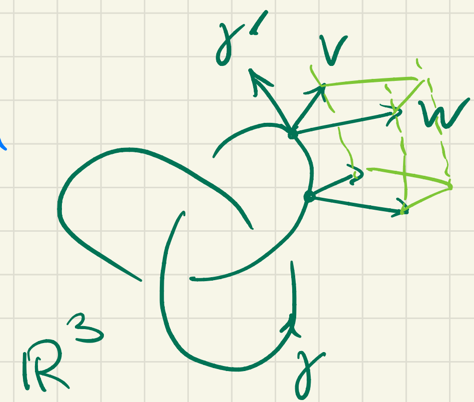
Exer The Kirillov-Kostant sympl. structure (or Lie-Poiss. str) on the orbit $\mathcal{O}_\gamma$ coincides with the Marsden-Weinstein symplectic structure on the space of knots $\mathcal{C}$.



$\mathcal{C}$ space of knots

$$\omega_\gamma^{MW}(v,w) = \int_\gamma i_v i_w \mu$$

$= \text{volume of collar}\,(v, w, \gamma')$



Pf sketch: $\omega_{KK}(\xi_\gamma)(V,W) := \langle d^{-1}\xi, [V,W] \rangle$

$$= \langle [u_\gamma], [V,W] \rangle = \langle u_\gamma, i_{[V,W]}\mu \rangle$$

Note: for div-free vect. fields V, W one has the identity: $\quad i_{[V,W]}\mu = d\, i_V i_W \mu$

Then $\omega_{KK}(\xi_\gamma)(V,W) = \int_{\mathbb{R}^3} u_\gamma \wedge d\, i_V i_W \mu$

$$= \int_{\mathbb{R}^3} \underbrace{du_\gamma}_{\omega_\gamma = \delta_\gamma} \wedge\, i_V i_W \mu = \int_{\mathbb{R}^3} \delta_\gamma \wedge i_V i_W \mu = \int_\gamma i_V i_W \mu .$$

QED

To define dynamics on knots we fix the Euclidean metric in $\mathbb{R}^3$. Let $H(\gamma) = \text{length}(\gamma) = \int_{S^1} \sqrt{(\gamma'(\theta), \gamma'(\theta))}\, d\theta$ be the Hamilton. f'n (θ - arc-length)

Prop The binormal eq'n $\partial_t \gamma = \gamma' \times \gamma''$ is Hamiltonian on $\mathcal{C}$ with Hamilton. f'n $H(\gamma)$ and the Marsden-Weinstein symplectic str're ω^{MW}.

Pf sketch. $H(\gamma + \varepsilon v) = H(\gamma) + \varepsilon \langle \frac{\delta H}{\delta \gamma}, v \rangle + O(\varepsilon^2)$, $\varepsilon \to 0$

Then the variational derivative $\frac{\delta H}{\delta \gamma} = -\gamma''$ for arc-length θ.

Hence $\partial_t \gamma = \text{sgrad}\, H = -J_\gamma \left(\frac{\delta H}{\delta \gamma} \right) = \gamma' \times \gamma''$, where J is $\frac{\pi}{2}$-rotation in the normal plane to γ', the almost complex str're. QED

$$J * := \gamma' \times *$$

Rm To see that $\frac{\delta H}{\delta \gamma} = -\gamma''$ expand

$$H(\gamma + \varepsilon v) = \int_{S^1} \sqrt{(\gamma' + \varepsilon v', \gamma' + \varepsilon v')}\, d\theta =$$

$$= \int_{S^1} \sqrt{(\gamma', \gamma') + 2\varepsilon(\gamma', v') + O(\varepsilon^2)}\, d\theta$$

"1" for arc-length θ

$$= \int_{S^1} \left(1 + \tfrac{1}{2} \cdot 2\varepsilon(\gamma', v') + O(\varepsilon^2)\right) d\theta = H(\gamma) - \varepsilon \int_{S^1} (\gamma'', v)\, d\theta + O(\varepsilon^2)$$

Hence $\frac{\delta H}{\delta \gamma} = -\gamma''$

Vortex membranes

Definition

Let $\Sigma^n \subset \mathbb{R}^{n+2}$ be a codimension 2 membrane (i.e., a compact oriented submanifold of codimension 2 in $\mathbb{R}^{n+2}$).
The *skew-mean-curvature (or, binormal) flow* of Σ is

$$\partial_t p = -J(\mathbf{MC}(p)),$$

where $p \in \Sigma$, $\mathbf{MC}(p)$ is the mean curvature vector to Σ at p, the operator J is the positive $\pi/2$ rotation in the 2-dim normal plane $N_p\Sigma$ at p.

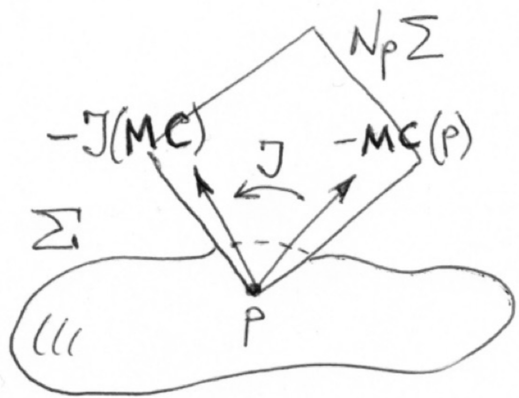
Note: It is a generalization of the binormal equation: in 1D $\Sigma = \gamma$ is a curve, $\mathbf{MC} = \kappa\,\mathbf{n}$, where κ is the curvature of γ,
$-J(\mathbf{MC}) = -J(\kappa\,\mathbf{n}) = \kappa\,\mathbf{b}$.

Theorem (Haller-Vizman, Shashikanth, K.)

The skew-mean-curvature flow $\partial_t p = -J(\mathbf{MC}(p))$ *is the Hamiltonian flow on the membrane space equipped with the Marsden-Weinstein structure and with the Hamiltonian given by the volume functional* vol.

The mean curvature vector $\mathbf{MC}(p)$ at $p \in \Sigma \subset \mathbb{R}^{n+2}$ is the average geodesic curvature of Σ over all directions in $T_p\Sigma$.

Corollary: The skew-mean-curvature flow preserves $\mathrm{vol}(\Sigma)$.



Properties of the flow

– **it is Hamiltonian:**

The Hamiltonian function is the n-dim *volume* $\mathrm{vol}(\Sigma)$ of $\Sigma \subset \mathbb{R}^{n+2}$.

The *Marsden-Weinstein symplectic structure* ω^{MW} on the space of codimension 2 membranes is

$$\omega^{MW}(\Sigma)(u, v) = \int_\Sigma i_u i_v \mu,$$

where u and v are two vector fields attached to the membrane Σ, and μ is the volume form in $\mathbb{R}^{n+2}$.

Idea of proof:

The Marsden-Weinstein symplectic structure is the averaging of the symplectic structures in all 2-dim normal planes $N_p\Sigma$ to Σ. Hence the skew-gradient is obtained from the gradient field attached at $\Sigma \subset \mathbb{R}^{n+2}$ by applying the fiberwise $\pi/2$-rotation operator J in $N_p\Sigma$.

On the other hand, the gradient for the volume functional $\mathrm{vol}(\Sigma)$ is $-\mathbf{MC}(p)$ at $p \in \Sigma$. Hence the Hamiltonian field on membranes is given by $-J(\mathbf{MC}(p))$ at any point $p \in \Sigma$. QED

Question: Is there an analogue of Hasimoto?

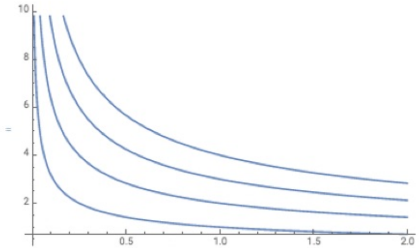
The binormal flow for products of spheres

Let $F : \Sigma = \mathbb{S}^m(a) \times \mathbb{S}^\ell(b) \hookrightarrow \mathbb{R}^{m+1} \times \mathbb{R}^{\ell+1} = \mathbb{R}^{m+\ell+2}$ be the product of two spheres of radiuses a and b.

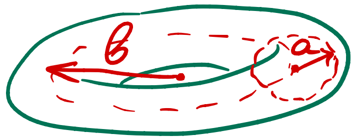
Theorem (Yang-K.)

The evolution F_t of this surface Σ in the binormal flow is the product of spheres $F_t(\Sigma) = \mathbb{S}^m(a(t)) \times \mathbb{S}^\ell(b(t))$ at any t with radiuses changing monotonically according to the ODE system:

$$\begin{cases} \dot{a} &= -\ell/b, \\ \dot{b} &= +m/a. \end{cases}$$

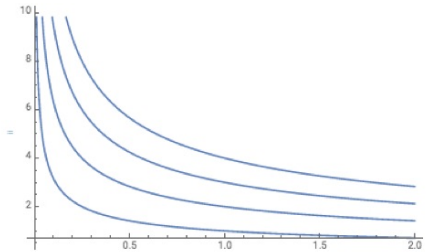


Clifford tori as vortex membranes



$a \to 0$
$b \to \infty$

$$\begin{cases} \dot{a} &= -\ell/b, \\ \dot{b} &= +m/a. \end{cases}$$



Corollary

For $m = \ell$ one has $a(t) = ae^{-lt/(ab)}$ and $b(t) = be^{mt/(ab)}$, the solutions exist for all $t \in \mathbb{R}$.

Example: Clifford torus $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$.

Example of collapse for products of spheres

Corollary

For $0 < m < \ell$ the corresponding solution F_t is

$$a(t) = a^{m/(m-\ell)} \left(a - (\ell - m) b^{-1} t\right)^{\ell/(\ell-m)} \text{ and}$$

$$b(t) = b^{\ell/(\ell-m)} \left(b + (m - \ell) a^{-1} t\right)^{m/(m-\ell)}.$$

It exists only for finite time and collapses at $t = a(0)b(0)/(\ell - m)$.

Example: The simplest case of $0 < m < \ell$ is $m = 1, \ell = 2$ for $\mathbb{S}^1(a) \times \mathbb{S}^2(b) \subset \mathbb{R}^5$.

Remark: Since the skew-mean-curvature flow is the LIA of the Euler equation, this collapse in 5D might be indicative for the Euler singularity problem in higher dimensions.

The Euler equation of an ideal fluid

For an inviscid incompressible fluid filling a Riemannian manifold M the fluid motion is described by the classical *Euler equation* on its velocity v:

$$\partial_t v + \nabla_v v = -\nabla p \,.$$

Here $\mathrm{div}\, v = 0$ and v is tangent to ∂M. $\nabla_v v$ is the Riemannian covariant derivative.

In any dimension, the *vorticity is the 2-form* $\xi := dv^\flat$, where $v^\flat$ is the 1-form metric-related to the vector field v. In 3D $\xi = \mathrm{curl}\; v$.
The *vorticity form of the Euler equation* is

$$\partial_t \xi + L_v \xi = 0 \,,$$

where L_v is the Lie derivative. It means that the vorticity is transported by (or "frozen into") the fluid flow.

Generalized Biot-Savart formula

Consider the vorticity 2-form $\xi_\Sigma = \delta_\Sigma$ supported on a membrane $\Sigma^n \subset \mathbb{R}^{n+2}$.

We need to find the divergence-free field v with prescribed vorticity 2-form ξ, i.e. $\xi_\Sigma = dv^\flat \in \Omega^2(\mathbb{R}^{n+2})$. In 3D $v = \mathrm{curl}^{-1}\xi$ is the field-potential given by the Biot-Savart formula.

Theorem (Shashikanth for 4D, K. for any D)

In any dim, vector field v in $\mathbb{R}^{n+2}$ satisfying $\mathrm{curl}\, v = \xi_\Sigma$ and $\mathrm{div} v = 0$ is given by the generalized Biot-Savart formula: $\forall q \notin \Sigma$

$$v(q) := C_n \cdot \int_\Sigma J\left(\mathrm{Proj}_N \nabla_p G(q,p)\right) \mu_\Sigma(p)\,,$$

where $\mathrm{Proj}_N \nabla_p G(\cdot, p)$ is the projection of $\nabla_p G(\cdot, p)$ of the Green function $G(\cdot, p)$ to the normal plane $N_p\Sigma$ at $p \in \Sigma$, and μ_Σ is the induced Riemannian volume on $\Sigma \subset \mathbb{R}^{n+2}$.

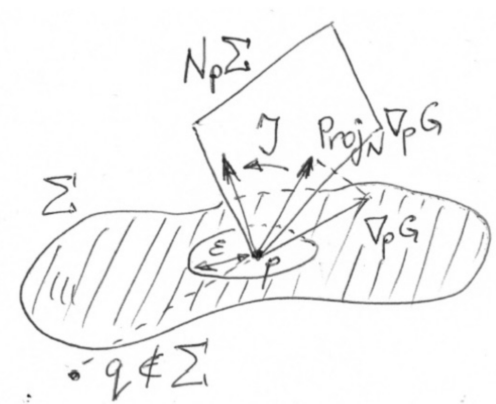
Regularization of velocity

As $q \to \Sigma$ the vector field $v(q) \to \infty$. Given $\epsilon > 0$, consider the truncation: for $q \in \Sigma$ take the integral not over Σ but over all points $p \in \Sigma$ at the distance at least ϵ from q:

$$v_\epsilon(q) := C_n \cdot \int_{\{p \in \Sigma\,, \|q-p\| \geq \epsilon\}} J\left(\mathrm{Proj}_N \nabla_p G(q,p)\right) \mu_\Sigma(p)\,.$$

It is a *localized induction approximation* of v. Similarly *regularize the energy*:

$$E_\epsilon(v) := \frac{1}{2} \int_{\mathbb{R}^{n+2}} (v, v_\epsilon)\, \mu\,,$$



Localized Induction Approximation theorem

Theorem (Shashikanth for 4D, K. for any D)

For any dim and a membrane $\Sigma \subset \mathbb{R}^{n+2}$
i) the velocity v *satisfying* $\xi_\Sigma = dv^\flat$ *has the LIA truncation* v_ϵ*: for* $q \in \Sigma \subset \mathbb{R}^{n+2}$ *one has*

$$\lim_{\epsilon \to 0} \frac{v_\epsilon(q)}{\ln \epsilon} = C_n \cdot J(\mathbf{MC}(q)) \ ;$$

ii) the regularized energy $E_\epsilon(v)$ *for the velocity of* Σ *has the asymptotics:*

$$\lim_{\epsilon \to 0} \frac{E_\epsilon(v)}{\ln \epsilon} = C_n \cdot \int_\Sigma \mu_P = C_n \cdot \mathrm{volume}\,(\Sigma)\,.$$

Question: Relation to 5D Euler?