# Geometric Fluid Dynamics

Henan University, Sept - Oct 2021

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Lecture 6

# Geometry of 2D fluid

Consider an ideal fluid in $\mathbb{R}^2$, $\mu = dx\wedge dy$

The fluid velocity field $v$ is div-free $\Rightarrow$ Hamiltonian, on $\mathbb{R}^2$

Let $\psi$ be the stream (= Hamiltonian) f'n for $v$,

i.e. $v = \operatorname{sgrad}\psi = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right)$.

The vorticity function is $\omega = \operatorname{curl} v = \Delta\psi$.

Equivalent forms of the 2D Euler equation:

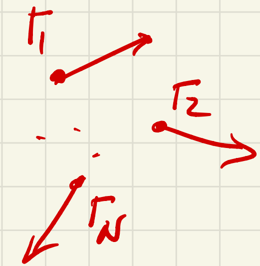
$\partial_t\omega = -L_v\omega$ or $\partial_t\omega + \{\psi, \omega\} = 0$

for $\omega = \operatorname{curl} v$      for $\omega = \Delta\psi$

($\Leftrightarrow$ frozenness of the vorticity).

# Kirchhoff equations for point vortices

Assume that the singular vorticity $\omega = \sum_{j=1}^{N} \Gamma_j \delta(z - z_j)$ is supported on $N$ point vortices $z_j = (x_j, y_j) \in \mathbb{R}^2 \simeq \mathbb{C}$ of strengths $\Gamma_j$. Their Euler evolution is given by the Kirchhoff equations (1876):




$$\Gamma_j \dot{x}_j = \frac{\partial \mathcal{H}}{\partial y_j}, \quad \Gamma_j \dot{y}_j = -\frac{\partial \mathcal{H}}{\partial x_j}, \quad j = 1, \dots, N$$

for $\mathcal{H}(z_1, \dots, z_N) = -\frac{1}{4\pi} \sum_{j<k}^{N} \Gamma_j \Gamma_k \ln |z_j - z_k|$

<u>Exer</u> These equations are Hamiltonian in $\mathbb{R}^{2N}$ with the Hamilt. f'n $\mathcal{H}$ and the sympl. structure $\sum_{j=1}^{N} \Gamma_j dx_j \wedge dy_j$ (or the Poiss. bracket $\{f, g\} = \sum_{j=1}^{N} \frac{1}{\Gamma_j} \left( \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial x_j} \right)$)

**Hint:** Here $\Psi = \Delta^{-1}\omega = \frac{1}{4\pi}\sum_{j=1}^{N} \Gamma_j \ln|z - z_j|$ (Green's f'n in $\mathbb{R}^2$)

$$v(z_j) = \operatorname{sgrad} \Psi\Big|_{z=z_j} = J \operatorname{grad}\Big|_{z=z_j}\left(\frac{1}{4\pi}\sum_{\substack{k=1\\k\neq j}} \Gamma_k \ln|z - z_k|\right) = \frac{1}{\Gamma_j} J \frac{\partial \mathcal{H}}{\partial z_j}$$

**Rm.** For rigorous derivation of the vortex model (i.e. if $\omega_0 \to \sum_j \Gamma_j \delta(z - z_j)$ then $\omega(t) \to \sum_j \Gamma_j \delta(z - z_j(t))$) see Marchioro-Pulverenti 1994.

**Rm** The system is invariant w.r.t. the group of Euclidean motions $E(2) = SO(2) \ltimes \mathbb{R}^2$ (rotations & translations)

The corresponding Noether integrals are

$$Q = \sum \Gamma_j x_j, \quad P = \sum \Gamma_j y_j \quad \text{(translations)}$$

$$F = \sum \Gamma_j (x_j^2 + y_j^2) \quad \text{(rotations)}$$

**Note:** e.g. $\{Q, P\} = \sum \Gamma_j$ (not in involution for $\sum \Gamma_j \neq 0$)

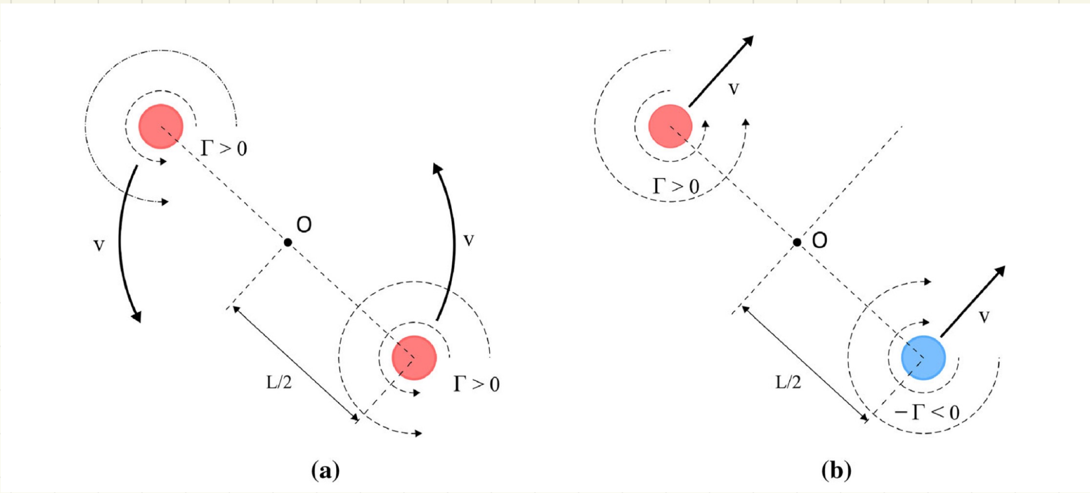
There are 3 integrals in involution: $\mathcal{H}, F, Q^2+P^2$ on $\mathbb{R}^{2N}$

**Cor** The system of $N$ point vortices on $\mathbb{R}^2$ is integrable for $N = 1, 2, 3$ (and for $N=4$ if $\sum \Gamma_j = 0$ and $\sum \Gamma_j z_j = 0$)

**Thm (Ziglin)** For $N \geq 4$ and generic $\Gamma_j$ it is nonintegrable.

<u>Ex</u> $N=1$ A single vortex stays at rest in $\mathbb{R}^2$

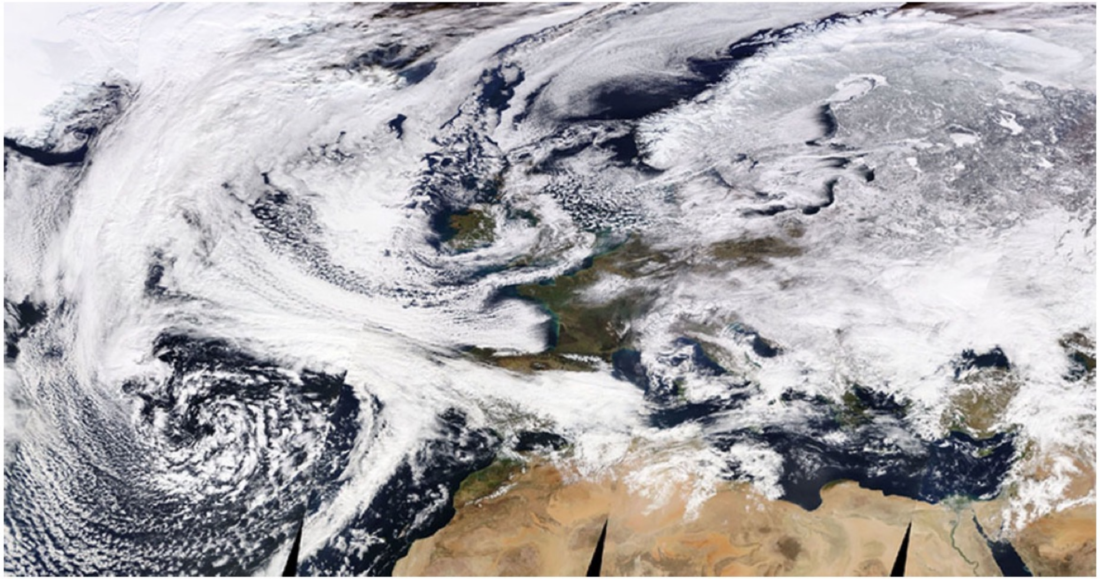
$N=2$ A pair of vortices rotates about their vorticity center $z_c = \dfrac{\Gamma_1 z_1 + \Gamma_2 z_2}{\Gamma_1 + \Gamma_2}$




Note: A collapse of 2 vortices is impossible!

vortex pair $\Gamma_1 = \Gamma_2$

dipole $\Gamma_1 = -\Gamma_2$

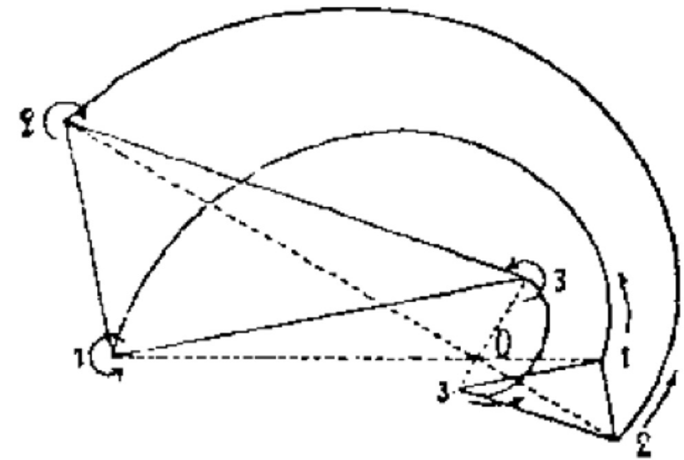


The interaction of Cyclone Emma (approaching from the southwest) and Anticyclone Hartmut (covering Europe from the northeast) on February 27, 2018. (Courtesy of NASA, Wiki-Commons.)

$N = 3$ The motion of 3 point vortices is integrable and there are self-similar collapsing solutions

(Gröbli, Aref, P. Newton, ..)

There are similar results for $S^2$ instead of $\mathbb{R}^2$.

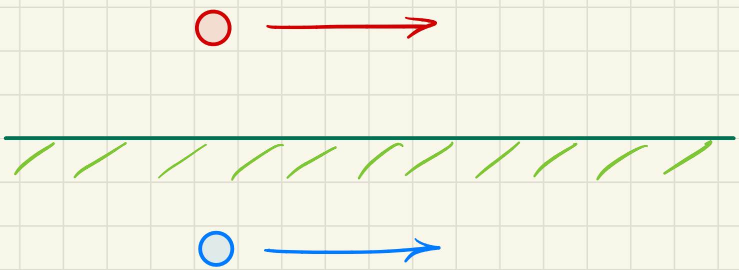



Self-similar motion in which the vortex triangle changes its size but not its shape (a drawing from W. Gröbli's 1877 dissertation). This self-similar expansion corresponds to vortices of strengths $\Gamma_1 = 3, \Gamma_2 = -2$, and $\Gamma_3 = 6$;

# Point vortices in the half-plane

Vortices "interact" with the boundary

$N=1$ A single vortex moves along the boundary (it is equivalent to a dipole, the mirror image)



Indeed, the Green function for half-plane has two mirror terms to provide zero boundary condition (impermeable boundary).

$N=2$ Two point vortices can have a variety of motions, including leapfrogging, depending on the interaction parameter $W$

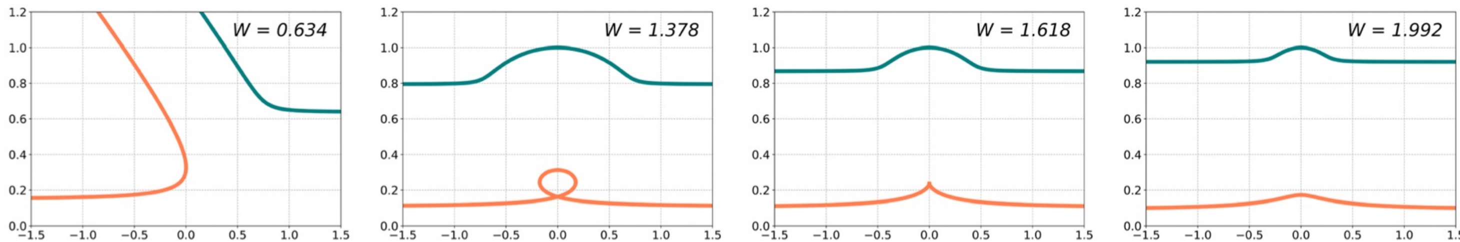



**Figure 6.** As the interaction weakens, either the vortices in the dipole go to infinity, or one of them makes a kink, or they pass around each other. (The orange and aquamarine colors correspond to the two vortices.)

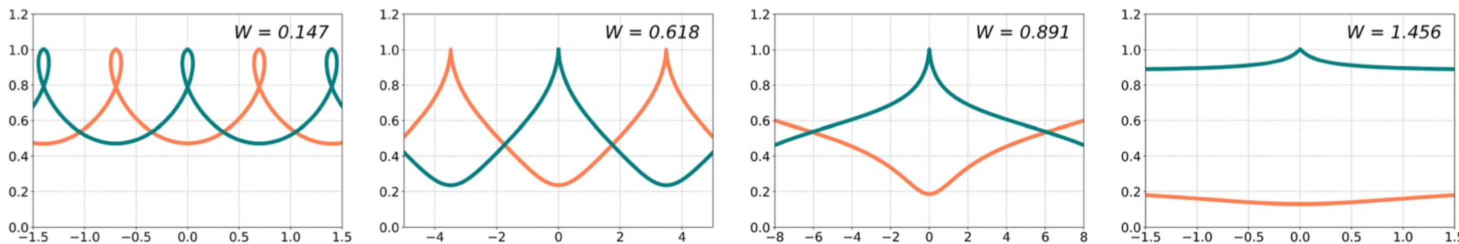
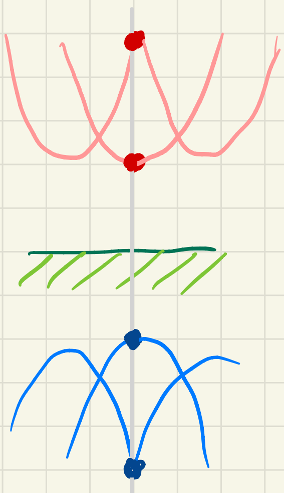



**Figure 7.** For a vortex pair, as the interaction weakens, a leapfrogging motion of the vortices changes to intertwining sinusoidal-like trajectories via a cusp-type motion.

**Thm** (Wang - K. 2021)

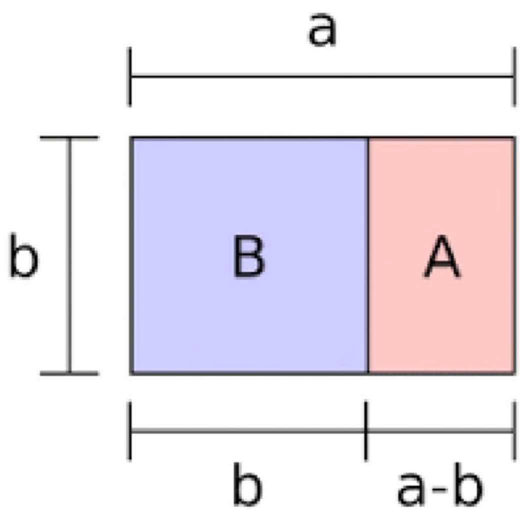
At the moment of cusp bifurcation the two vertices lie on the same vertical. The cross-ratio of four points $CR(z_1, z_2, \bar{z}_2, \bar{z}_1) = \varphi = \frac{\sqrt{5}+1}{2}$, the golden ratio.



**Reminder 1.** The cross-ratio of 4 pts on a line

$$CR(y_1, y_2, y_3, y_4) = \frac{(y_1 - y_4)(y_2 - y_3)}{(y_1 - y_2)(y_3 - y_4)}$$

# Reminder 2




The golden ratio $\phi = a/b$ is the ratio of length to width for the special rectangle $A \cup B$ that preserves this ratio after cutting out the square $B$: $\phi = b/(a-b)$.

$$\varphi = \frac{a}{b} = \frac{b}{a-b}, \text{ i.e.}$$

$\varphi$ is a positive root of

$$\varphi^2 - \varphi - 1 = 0$$

$$\varphi = \frac{\sqrt{5}+1}{2} = 1,618033\ldots$$

The best application of the golden ratio:

to convert miles $\to$ kilometers take the next Fibonacci number:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots,$$


5 mi $\approx$ 8 km

55 mi $\approx$ 89 km

or 130 km $\approx$ 80 mi

Indeed, $\dfrac{F_{n+1}}{F_n} \to \varphi = 1.6180\ldots$ as $n \to \infty$,

while $\dfrac{\text{mi}}{\text{km}} = 1.6093$,

less than 0.5% off $\varphi$!

## Reminder: Invariants of the Euler equations.

**Def** For 2D and any $k=1,2,\dots$ the quantities

$$h_k(v) = \int_M (\operatorname{curl} v)^k \mu := \int_M \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) dx_1 dx_2$$ are

called enstrophies.

**Thm** They are invariants (first integrals) of the Euler equation in 2D, $\forall k=1,2,\dots$

Furthermore, enstrophies are Casimirs, i.e. invariants of the $\mathrm{Diff}_\mu(M)$-action.

Recall: For a manifold $M$ of any dimension with a volume form $\mu$ Casimirs for the group $\mathrm{Diff}_\mu(M)$ are invariants of cosets $[u]$ of 1-forms $u = v^\flat$ or of the (exact) vorticity 2-form $\omega = du$ on $M$.

In 2D the vorticity 2-form $\omega = du$ corresponds to the vorticity function on a symplectic surface

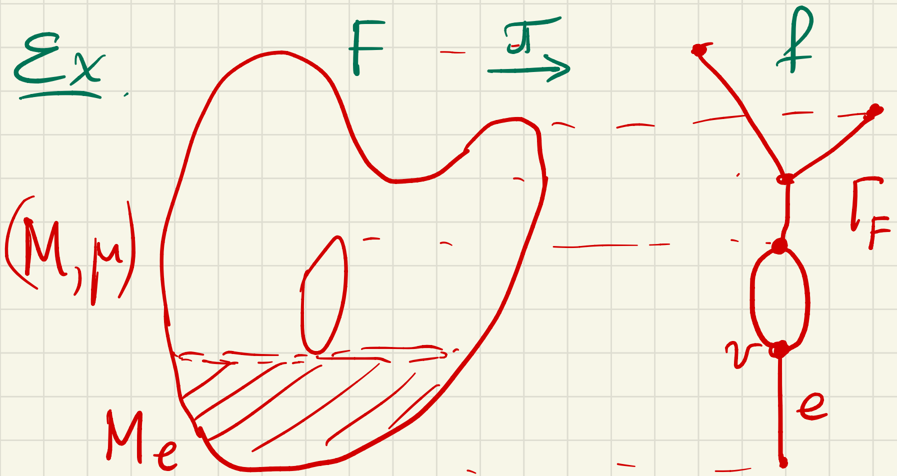
Auxiliary problem:

Find a complete set of invariants of a smooth function on a symplectic surface $(M^2, \mu)$.

**Def** A smooth function $F$ on $M$ is a simple Morse function if all its critical points are nondegenerate and all its critical values are distinct.

**Def** The Reeb graph of $F$ is $\Gamma_F := M / \{F\text{-levels}\}$, the set of $F$-levels.

**Ex**




Properties of $\Gamma_F$:

- crit. pts $\rightsquigarrow$ vertices
- $F$ – natural parameter
- area $\mu$ on $M$ $\rightsquigarrow$ measure $\nu$ on $\Gamma_F$
- genus$(M) \rightsquigarrow b_1(\Gamma_F)$

Furthermore • at min/max ( ) $\nu$ is smooth, $\frac{d\nu}{df} \neq 0$

• at saddles ( ) $\nu$ is log-smooth, i.e.

$$\nu([v,x)) = \varepsilon_i \, \psi(f(x)) \ln|f(x)| + \eta_i(f(x)), \quad \psi(0)=0, \ \psi'(0)\neq 0$$

for $f(v)=0$

**Def** $(\Gamma, f, \nu)$ is *a measured Reeb graph* if

$\Gamma$ is an oriented graph with 1- or 3-valent vertices of types or

$f$ is monotone and $\nu$ is log-smooth.

**Thm** (Izosimov-Kh. 2016) Two simple Morse $f$'s on $(M^2, \mu)$ are in the same $\mathrm{Diff}_\mu$-orbit iff their measured Reeb graphs are isomorphic (i.e. $\exists$ 1-1 correspondence between simple Morse $f$'s and measured Reeb gr's compatible with $M$: $\int_\Gamma \nu = \int_M \mu$, $\mathrm{genus}(M) = b_1(\Gamma)$).

**Cor** The "generalized enstrophies" of $F$ are

$$h_{k,e}(F) := \int_{M_e} F^k \mu$$

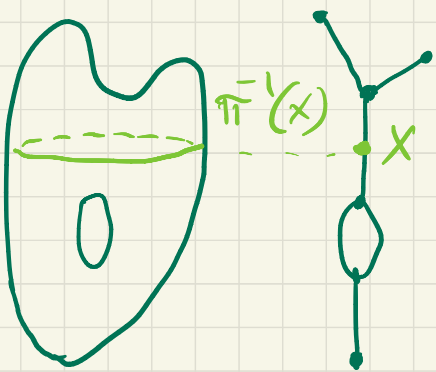
and they form a complete set of Casimirs for $M = S^2$, where $S^2 \supset M_e := \pi^{-1}(e)$ for all edges $e \in \Gamma_F$ and all $k \in \mathbb{Z}_+$.

(Proof follows from the Hausdorff moment problem)

**Rm** For a surface $M$ of higher genus one needs to fix also circulations over handles of $M$, since invariants of cosets $[u] \in \Omega^1/d\Omega^0$ are more subtle than for $d[u]$ (or the vorticity f'n $F := du/\mu$).

Let $M$ be a surface of genus $\varkappa$ (i.e. with $\varkappa$ handles). Consider a coset $[u]$ with vorticity f'n $F = du/\mu$ and the measured Reeb graph $\Gamma_F$. It turns out, one can define "an integral" of a 1-form over a graph (rather that over a segment).

**Def** The circulation space consists of all functions $c$ on the meas. Reeb graph that are antiderivatives of $f$, i.e. satisfying




1) $\forall x \in \Gamma \setminus V(\Gamma)$, $f = \frac{\partial c}{\partial \mu}$ and for $\lim\limits_{x \to v} c(x) = \ell$ one has

2) $\ell = 0$ at and 3) $\ell_0 = \ell_1 + \ell_2$ at

In other words, when we integrate over a branch pt, we take the "space of all possible splittings", satisfying the Kirchhoff rule.

Ex. Given $[u]$ on $M$ (with $F = \frac{du}{\mu}$), define a f'n $c : \Gamma_F \setminus V(\Gamma_F) \to \mathbb{R}$ by $c(x) = \int_{\pi^{-1}(x)} u$. It does not depend on $u \in [u]$ and $f = \frac{\partial c}{\partial \mu}$.

If $u = v^\flat$ for a vect. field $v$ on $M$, then $c(x)$ is the circulation of $v$ over the cycle $\pi^{-1}(x) \subset M$

Prop (Izosimov-Kh, 2016) The meas. Reeb graph $(\Gamma, f, \mu)$ admits a circulation function iff $\int_\Gamma f\mu = 0$.

The space of circulation functions has dim $= \varkappa = b_1(\Gamma)$
$= \text{genus } M$.

**Thm** (Izosimov–Kh 2016)

Coadjoint orbits of $\mathrm{Diff}_\mu(M)$ are in 1-1 correspondence with circulation graphs compatible with $M$ (i.e. for which meas + genus coincide).

**Pf** Based on the lemma de Morse isochore (Colin de Verdier–Vey)

For a Morse f'n $F$ on a surface with area form $\mu$ there is a chart s.t. locally $\mu = dp\wedge dq$ and $F = \lambda \circ S$ with $S = p^2+q^2$ or $pq$ and $\lambda$ is smooth near $0 \in \mathbb{R}$ and $\lambda'(0) \neq 0$.

**Rm** There are versions for the groups $\mathrm{Diff}_{\mu,0}(M)$, $\mathrm{Ham}(M)$, for $M$ with boundary (Izosimov, I. Kirillov, Mousavi, K)