Geometric Fluid Dynamics

Henan University, Sept - Oct 2021

Boris Khesin (Univ of Toronto)

Lecture 5
Energy and helicity

Let $M$ be a simply-connected, Riem. nfd

$B$ - a (magnetic) vector field on $M$ (e.g. $M \subset \mathbb{R}^3$)

$\text{div } B = 0$ w.r.t. $\mu = \text{d}^3x$ - volume form on $M$

The energy of $B$ is $E(B) = \|B\|^2_{L^2(M)} = \int_{M} (B, B) \mu$

Consider a volume-preserving diffeomorphism $\Psi : M \to \mathbb{R}^3$

**Question** Given a field $B$ find $a := \inf_{\Psi} E(\Psi \times B)$

Is $a = 0$ or $a > 0$?
Motivation: $B$ - magnetic field of a star (the Sun) "frozen into" the media (plasma), i.e.

$$\frac{\partial B}{\partial t} = -J \times B = -\text{curl}(\mathbf{v} \times \mathbf{B})$$

The star radiates its energy.

Question: Will the star extinguish completely? (will $a$ be positive or $= 0$?)
Topological obstruction:

Assume that \( \text{supp } B = C_1 \cup C_2 \), two solid tori

\[
C_1 \rightarrow \sim \rightarrow C_2
\]

\( E(B) \downarrow \sim \) most orbits shrink

Indeed, length \( \rightarrow \) length/\( \lambda \), time remains the same

\[ \Rightarrow B \rightarrow B/\lambda \Rightarrow E \rightarrow E/\lambda^2 \]

But linking prevents tori from infinite fattening, since transformations are volume-preserving.
Prop (Arnold 1973) \( E(B) \geq c |\text{Hel}(B)| \),

where \( \text{Hel}(B) := \int_M (B, \text{curl}^{-1} B) \mu \) - helicity of \( B \),

and \( c = c(M) \). (Note: “geometry \geq topology”)

Then \( c(M) \) is the max abs. value of its eigenvalues, depends on \( M \).
An important example (Moffatt 1969)

For a vector field $B$ as above, without net twist inside $C_1, C_2$

$\text{Hel}(B) = 2 \text{lk}(C_1, C_2) \cdot \text{Flux}_1 \cdot \text{Flux}_2 \neq 0$

Recall: the linking number of two oriented curves $\Gamma_1$ and $\Gamma_2$ in $M^3$ is

$$\text{lk}(\Gamma_1, \Gamma_2) = \# (\delta^{-1} \Gamma_1) \cap \Gamma_2$$

Note: $\text{lk}$ is symmetric
- does not depend on the choice of $\delta^{-1} \Gamma_1$
- has a higher-dim generalization
An example: the Hopf field in $S^3$

For $S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid \sum_{i=1}^4 x_i^2 = 1\}$

define $\nabla (x_1, x_2, x_3, x_4) = (-x_2, x_1, -x_4, x_3)$

Exer $\nabla$ corresponds
to the max eigenvalue $= \frac{1}{2}$
of curl$^{-1}$ on $S^3$.

$\Rightarrow$ The Hopf field has
the minimal energy
among diffeomorphic ones
(by volume-pres. diffes's)
A metric-free definition of helicity

Let $\mathcal{M}$ be a simply-connected manifold, $\mu$-volume form, and $\xi$ a divergence-free vector field on $\mathcal{M}$, i.e. $\mathcal{L}_\xi \mu = 0$.

$\omega_\xi := \mathcal{I}_\xi \mu$ is a closed (\Rightarrow exact) 2-form on $\mathcal{M}$.

Hence $\omega_\xi = dd\alpha$ for some 1-form $\alpha$.

**Definition.** The helicity $Hel(\xi)$ of $\xi$ on $\mathcal{M}$ is

$$Hel(\xi) = \int_{\mathcal{M}} \alpha \wedge dd\alpha = \int_{\mathcal{M}} d\alpha \wedge \alpha = \int_{\mathcal{M}} \omega_\xi \wedge d^{-1} \omega_\xi,$$

for $dd\alpha = \omega_\xi$.

**Example.** $\xi = \text{curl } \nu$. Then

$$Hel(\xi) = \int_{\mathcal{M}} i_\xi \mu \wedge d^{-1} (i_\xi \mu) = \int_{\mathcal{M}} d\mu \wedge \nu = \int_{\mathcal{M}} (\text{curl } \nu, \nu) \mu.$$
Cor (of coord-free def'n of helicity)

The helicity $\text{Hel}(\xi)$ is preserved under the action on $\xi$ of volume-pres. diffeo's of $M$ (i.e. it is a topological invariant).

Rm. This was an "integral def'n" of helicity. What is its topological meaning?

Helicity as an asymptotic linking

Let $M$ be a simply-connected closed 3D mfld with a volume form $\mu$ (we do not fix metric now).
Def: For a div-free vector field $\zeta$ on $M$ (i.e. $\text{div } \zeta = 0$), introduce the following function $\lambda_\zeta (x,y)$, $x,y \in M$.

Let $g^T(x), g^S(y)$ be pieces of $\zeta$-trajectories for times $T, S$ respectively.

Assume that $\Delta$ is a "system of short paths" (joining any pair of pts on $M$ and chosen a priori, e.g. geodesics)

Close up the trajectory pieces by $\Delta$.

Then define

$$\lambda_\zeta (x,y) := \lim_{T,S \to \infty} \frac{1}{T \cdot S} \text{lk} (g^T(x), g^S(y), \Delta)$$
The limit exists for almost all \( x, y \in \mathcal{M} \) and doesn't depend on \( \Delta \) under some cond's (Arnold).

Better: \( \Delta \) is a system of geodesics, the limit exists in \( L^1(\mathcal{M} \times \mathcal{M}) \) (T. Vogel).

It is based on the Birkhoff or \( L^1 \)-ergodic theorems.

**Thm (V. Arnold 1973)** Helicity \( \text{Hel}(\mathfrak{z}) := \iint (i_z \mu) \, d^{-1} (i_z \mu) \)

is equal to the averaged linking:

\[
\text{Hel}(\mathfrak{z}) = \iint \mathcal{A}_z (x, y) \, \mu_x \, \mu_y
\]

\[\mathcal{M} \times \mathcal{M} \]

It is natural to call it the asymptotic Hopf invariant.
Consider the Biot-Savart integral for vector potential $A = \text{curl}^{-1} \hat{z}$:

$$A(y) = -\frac{1}{4\pi} \int \frac{\hat{z}(x) \times (x-y)}{||x-y||^3} \mu_x$$

Then the helicity is

$$\text{Hel}(3) = \int (3, A) = \frac{1}{4\pi} \iint \frac{(\hat{z}(x), \hat{z}(y), x-y)}{||x-y||^3} \mu_x \mu_y$$

On the other hand, recall the explicit formula for the linking number.
Digression on the Gauss formula

\[ \text{Gauss Theorem:} \] The linking number of closed curves \( \gamma_1(S'), \gamma_2(S') \subset \mathbb{R}^3 \) is given by

\[
\text{lk} (\gamma_1, \gamma_2) = \frac{1}{4\pi} \int_0^{T_1} \int_0^{T_2} \frac{(\dot{\gamma}_1(t), \dot{\gamma}_2(t), \gamma_1(t) - \gamma_2(t))}{\| \gamma_1(t) - \gamma_2(t) \|^3} \, dt_1 \, dt_2
\]

Note: \( \text{lk} (\gamma_1, \gamma_2) = \deg (f : T^2 \to S^2) \), where \( f = F / \| F \| \) for the map \( F(t_1, t_2) := \gamma_1(t_1) - \gamma_2(t_2) \).
Hence for 2 pieces of \( \xi \)-trajectories their lk is
\[
\Lambda_\xi(x, y) = \lim_{T, S \to \infty} \frac{1}{4\pi T S} \int_0^T \int_0^S \frac{(x(t), y(s), x(t) - y(s))}{\|x(t) - y(s)\|^3} \, dt \, ds
\]
where \( x(t) = g^t(x) \), and we neglect the integrals over short paths \( \Delta \).

Now the result follows from the Birkhoff ergodic thm: the time average of \( \Lambda_\xi(x, y) \) along the measure-pres. flow of \( \xi \) coincides with the space average, given by the integral expression of \( \text{Hel}(\xi) \). Q.E.D.
Return to energy estimates

**Cor.** If a div. free field has nonzero helicity, its energy cannot be made arbitrarily small.

But what if $\text{Hel}(\xi) = 0$?

For instance, two pairs of solid tori linked in opposite directions?

**Thm.** (Freedman-He 1991) Suppose a vector field $\xi$ in $\mathbb{R}^3$ has an invariant torus $T$ forming a nontrivial knot of type $K$. Then

$$E(B) \geq \left( \frac{16}{\pi \cdot \text{Vol}(T)} \right)^{\frac{1}{3}} |\text{Flux } B|^{\frac{2}{3}} (2\text{ genus}(K) - 1)$$

**Cor.** For a nontrivial $K$, $E(\Phi \times B) > 0$
Cor If a field $B$ has at least one closed linked trajectory of an elliptic type $\Rightarrow E(\pi_x B) > 0$.

The Sakharov–Zeldovich problem
Let $B$ be the rotation field of a ball in $\mathbb{R}^3$.

Problem: Is $\inf_y E(\pi_y B) = 0$?

Thm (Freedman 1980) There exists a sequence of volume-preserving diffeos $\psi^{(n)} : M \to M$ such that $E(\pi_x \psi^{(n)} B) \to 0$ as $n \to \infty$. 
Stretch a subball to shorten trajectory and put the snake obtained inside the sphere. Use Moser's lemma on existence of volume-pres. diffeom to estimate the energy in the shell image.
Another direction: Fast dynamo problem

**Def.** The kinematic dynamo equation is

\[
\begin{cases}
\partial_t B = -\mathbf{L}_v B + \eta \Delta B \\
\text{div } B = 0
\end{cases}
\]

The unknown magnetic field \( B(t) \) is stretched by the fluid flow with velocity \( \mathbf{v} \), while a low diffusion dissipates the magnetic energy \( E(B) \).

**Problem** Does there exist a div-free vector field \( \mathbf{v} \) in \( M \) s.t. \( E(B(t)) \) grows exponentially in time (for some initial \( B(0) \)) as \( \eta \to 0 \) or \( \eta = 0 \)?
Look for solutions $B = e^{\lambda(t)}B(0)$ such that $\text{Re} \lambda(\eta) \geq \lambda_0 > 0$ as $\eta \to 0$ or $\eta = 0$.

A non-dissipative dynamo ($\eta = 0$) corresponds to a frozen magnetic field.

There are many works constructing explicit dynamos and proving a necessity of chaotic behavior of $\nu$ for $\eta \neq 0$. One of popular examples is the ABC-flow.

The ABC flows (Arnold–Beltrami–Childress) are

$$\nu = (A \sin z + C \cos y) \frac{\partial}{\partial x} + (B \sin x + A \cos z) \frac{\partial}{\partial y} + (C \sin y + B \cos x) \frac{\partial}{\partial z}$$

on 3D torus $T^3 = \{(x, y, z) | \text{mod } 2\pi\}$. They are eigen for curl: $\text{curl } \nu = \nu$. 
**Bonus: Why Hopf?**

Recall: The Hopf invariant of a map $\pi: S^3 \to S^2$ has 2 defs:

- **a) geometric/topological:**
  \[ \text{Hopf}_1(\pi) = \text{lk}(\pi^{-1}(a), \pi^{-1}(b)) \]
  It doesn't depend on $a, b \in S^2$

- **b) integral:** take $\nu \in \pi^2(S^2), \int_{S^2} = 1$, then $\pi^*\nu \in \pi^2(S^3)$ is closed $\Rightarrow$ exact on $S^3$. Then
  \[ \text{Hopf}_2(\pi) = \int_{S^3} \pi^*\nu \wedge d^{-1}(\pi^*\nu) \]

Why $\text{Hopf}_2$ is an integer?
Note: in the formula for Hopf the closed 2-form $\tilde{\nu}$ can be replaced by a cohomological one, $\tilde{\nu}$ on $S^2$, since their difference is exact, $\tilde{\nu} = d\alpha$, $\alpha \in \Omega^1(S^2)$, and

$$\int_{S^3} \pi^* \nu \land \tilde{\nu}^\vee - \int_{S^3} \pi^* \nu \land \tilde{\nu} = \int_{S^3} \pi^* \nu \land \tilde{\nu} = \int_{S^3} \nu \land \alpha = 0 \quad \text{on } S^3 = \pi^{-1}(S^2) \quad S^2$$

Now take $\nu = \delta(a)$, $\tilde{\nu} = \delta(b)$ - the $\delta$-type 2-forms in $S^2$, supported at pts $a$ and $b$. Then $\pi^* \nu = \delta(\pi^{-1}(a))$ and $\pi^* \tilde{\nu} = \delta(\pi^{-1}(b))$, $\delta$-type 2-forms in $S^3$ supported on $\pi^{-1}(a)$, $\pi^{-1}(b)$, and $\pi^{-1}(\tilde{\nu}) = \delta(\pi^{-1}(\tilde{\nu}))$ - 1-form supported on a Seifert surface $\pi^{-1}(a)$ in $S^3$. 

**Note:**
Hence \( \text{Hopf}_2 (\pi) = \int_{S^3} \pi^* \gamma \wedge d^{-1}(\pi^* \gamma) = \int_{S^3} \pi^* \tilde{\gamma} \wedge d^{-1}(\pi^* \gamma) \)

\[
= \int_{S^3} \delta (\pi^{-1}(b)) \wedge \delta (\pi^{-1}(a)) = \# \pi^{-1}(a) \wedge \pi^{-1}(b)
\]

This proves the equivalence of 2 definitions. Arnold's thin is an asymptotic version of this equivalence. Namely, instead of a map \( \pi : S^3 \to S^2 \) where the fibers are closed (which corresponds to a field \( \xi \) whose all trajectories are closed), consider a div.-free v.f. \( \tilde{\xi} \) with arbitrary trajectories.