Geometric Fluid Dynamics

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Lecture 4

Tentative Plan:

I. Introducing the Euler equations. Its description as the geodesic flow.

II. Equations on the dual Lie algebra, Lie-Poisson structures, Euler–Arnold equations.

III. The Virasoro algebra and the KdV as an Euler equation.

IV. The Hamiltonian framework for hydrodynamics. Conservation laws for the Euler equations.

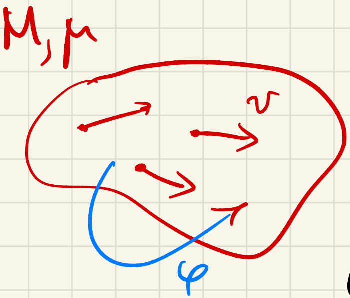
V. Geometry of Casimirs: helicity and enstrophies.

VI. Point vortices and vortex filaments.

VII. The Marsden–Weinstein symplectic structure on knots and vortex membranes.

VIII. Geometry of diffeomorphism groups and optimal mass transport.

Lecture 4: The Hamiltonian framework for hydrodynamics. Conservation laws.



Let M, $(\,,\,)$ be a Riemannian mfd, μ - volume form

Consider the Lie group

$G = \mathrm{Diff}_\mu(M) = \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\mu = \mu\}$ of volume preserving diffeom's. Its Lie algebra is

$\mathfrak{g} = \mathrm{Lie}(G) = \mathrm{Vect}_\mu(M) = \{v \in \mathrm{Vect} \mid L_v\mu = 0 \text{ and } v \parallel \partial M\}$

divergence-free vector fields tangent to ∂M.

Thm The (regular part of the) dual space $\mathfrak{g}^*$ is naturally identified with

$$\mathfrak{g}^* \simeq \Omega^1(M)/d\Omega^0(M) \ni [u] = \{u + df \mid \forall f \in C^\infty(M)\}$$

(all 1-forms modulo exact) $\nwarrow$ cosets of 1-forms

The pairing is

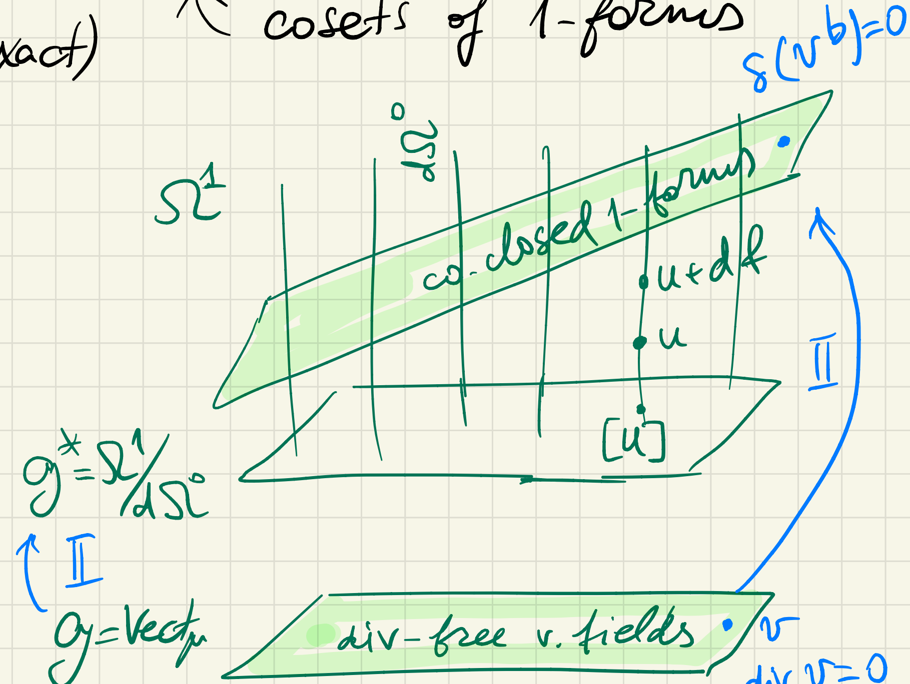
$$\langle [u], v \rangle := \int_M (i_v u)\,\mu$$

The Lie algebra action is

$$\mathrm{ad}_v w = L_v w$$

The coadjoint action is

$$\mathrm{ad}^*_v [u] = -L_v [u]$$



"Pf" The pairing of 1-forms and vect. fields (i.e. Ω^1 & $Vect(M)$) is nondegenerate, while $\langle d\Omega^0, Vect_\mu(M)\rangle = 0$, since

$$\langle df, v\rangle = \int_M (i_v df)\mu = \int_M (L_v f)\mu = -\int_M f\cdot \underbrace{L_v \mu}_{\text{"}0\text{"}} = 0$$

Exer Check for M with boundary

The group adjoint and coadjoint action is geometric, i.e. changing coordinates in vect. fields and 1-forms by a volume-preserving diffeo'm:

$$Ad_\varphi v = \varphi_* v$$

$$Ad^*_\varphi [u] = \varphi^*[u]$$

while the pairing is invariant (change coord's in integral).

QED.

Rm The inertia operator corresponds to the L^2-energy:

$$E(v) = \frac{1}{2}\langle v, \mathbb{I} v\rangle = \frac{1}{2}\int_M (v,v)\,\mu.$$ Hence

$$\mathbb{I}: \begin{array}{ccc} \mathrm{Vect}_\mu & \longrightarrow & \Omega^1/d\Omega^0 \\ \Psi & & \Psi \\ v & \longmapsto & [v^\flat] \end{array}$$

, i.e. $\Omega^1 \ni v^\flat(w) = (v,w)$ $\forall w \in \mathrm{Vect}(M)$

Here $v \to v^\flat$ is the Riemannian metric isomorphism.

It sends a div-free vect. field to a co-closed 1-form $v^\flat$.

(Indeed, $\delta v^\flat = * d i_v \mu = 0$)

The Hamiltonian function on the dual space is

$$H([u]) = \frac{1}{2}\langle [u],[u]\rangle = E(v) \text{ for } u = v^\flat$$

Then the Euler-Arnold equation, i.e. the Hamiltonian eq'n for $H([u])$ and the Lie-Poisson structure on Vect_μ^* is

$$\dot{m} = \mathrm{ad}^*_{\mathbb{I}^{-1}m} m, \text{ i.e. } \partial_t [u] = -L_v [u] \text{ for } v^\flat = u$$

Here $[u] \in \Omega^1 / d\Omega^0 (M)$

For a representative 1-form $u \in \Omega^1$,

$\partial_t u = -L_v u + df$ or $\partial_t u + L_v u = df$ $\rightsquigarrow$

the Euler equation $\partial_t v + \nabla_v v = -\nabla p$,

once we use the Riem. metric identification

Exercise: Prove the identity

$$L_v(v^\flat) = (\nabla_v v)^\flat + \frac{1}{2} d(v,v).$$

It implies that the metric identification sends $\partial_t u + L_v u = df$ for $u = v^\flat$ to $\partial_t v + \nabla_v v = -\nabla p$

for $p = \frac{1}{2}(v,v) - f$

A variation: Infinite conductivity equation

Def The equation of infinite conductivity in $\mathbb{R}^3$ is

$$\begin{cases} \partial_t v + \nabla_v v + \underbrace{v \times B}_{\text{Lorentz force}} = -\nabla p \\ \operatorname{div} v = 0, \end{cases}$$

where B is a fixed magnetic field, v – velocity of the electron gas

Prop This eq'n is equivalent to the Hamilt. eq'n on $\mathfrak{g}^* = \Omega^1/d\Omega^0(\mathbb{R}^3)$ $\partial_t[u] = -L_v[u+\alpha]$, where $u = v^\flat$, $[u] \in \Omega^1/d\Omega^0$ and $\alpha \in \Omega^1(\mathbb{R}^3)$ is defined by $d\alpha = -i_B\mu$ modulo $df \in d\Omega^0$.

Pf - exer: Prove the relation of $v \times B$ and $L_v\alpha$.

Rm It is equivalent to $\partial_t[u+\alpha] = -L_v[u+\alpha]$, i.e. it is Hamiltonian for $\{\,\}_{LP}$ on $\mathfrak{g}^*$ and the "shifted" Hamilt. f'n $H([u]) = \frac{1}{2}\int_{\mathbb{R}^3}([u+\alpha],[u+\alpha])\mu$.

Generalizations: Semi-direct product groups

Ex. $E(3) = SO(3) \ltimes \mathbb{R}^3$, motions of Euclidean $\mathbb{R}^3$

Composition of maps $x \mapsto Ax + b$ gives the group product $(A_2, b_2) \circ (A_1, b_1) = (A_2 A_1, A_2 b_1 + b_2)$

Thm (Vishik–Dolzhansky 1978) The Euler eq'n for $e(3)^*$ and a quadratic Hamiltonian $H = \frac{1}{2}\left(\sum a_i m_i^2 + \sum c_{ij} p_i p_j + \sum b_{ij}(p_i m_j + p_j m_i)\right)$ gives the Kirchhoff equations

$$\begin{cases} \dot{m} = m \times \omega + p \times u \\ \dot{p} = p \times \omega \end{cases} \quad \text{for } (p, m) \in e(3)^*$$

Similarly, for the magnetohydrodynamics (MHD) eq's in $\mathbb{R}^3$

$$\begin{cases} \partial_t v + \nabla_v v = (\operatorname{curl} B) \times B - \nabla P \\ \partial_t B = -L_v B \\ \operatorname{div} v = \operatorname{div} B = 0 \end{cases}$$

Lorentz force

B - magnetic field

(on a unit charge with velocity j in the magn. field B) acts the Lorentz force $j \times B$, while $j = \dfrac{\operatorname{curl} B}{4\pi}$ - Maxwell's eq'n

<u>Thm</u> (Vishik-Dolzhansky, Holm-Kupershmidt, Marsden-Ratiu-Weinstein, ...)

The MHD eq's are Hamiltonian on $\tilde{\mathfrak{g}}^*$ for the group $\tilde{G} = \mathrm{Diff}_\mu(M) \ltimes \mathrm{Vect}_\mu(M)$ with respect to the right-inv $L^2 \oplus L^2$-metric and have the form

$$\begin{cases} \partial_t [u] = -L_v [u] + L_B [b] \\ \partial_t B = -L_v B \end{cases} \quad \text{for} \quad \begin{matrix} u = v^\flat \\ b = B^\flat \end{matrix}$$

Def The vorticity 2-form is defined as $\omega := du \in \Omega^2(M)$ for $u = v^\flat$ for a manifold M of any dim'n.

Note: $\partial_t(du) = -L_v(du) \iff \partial_t \omega + L_v \omega = 0$

This is Kelvin's law: the fluid vorticity is transported (or frozen into) the fluid flow.

Rm For $\mathbb{R}^2$ or any surface M ($\dim M = 2$)
$\omega = \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) dx_1 \wedge dx_2$, i.e. the 2-form ω can be identified with a vorticity function

$$\xi := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = \operatorname{curl} v$$

For $\mathbb{R}^3$ or any 3D mfd M with a volume form μ the 2-form ω can be identified with the vorticity vector field ξ by $i_\xi \mu = \omega$, i.e.

$$\xi = \begin{vmatrix} \xi_1 & \xi_2 & \xi_3 \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\frac{\partial}{\partial x_1} + \circlearrowleft_{1,2,3}$$

The vorticity form of the Euler equation is

$$\partial_t \xi = -L_v \xi \qquad \text{or} \qquad \partial_t \omega = -L_v \omega$$

for $\xi = \operatorname{curl} v$ in $\mathbb{R}^3$

for the curl 2-form for any mfd M.

Invariants of the Euler equations.

Def For 2D and any $k = 1, 2, \ldots$ the quantities

$$h_k(v) = \int_M (\operatorname{curl} v)^k \mu := \int_M \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right) dx_1 dx_2$$ are

called enstrophies.

Thm They are invariants (first integrals) of the Euler equation in 2D, $\forall k = 1, 2, \ldots$

Def. For 3D the quantity

$$\operatorname{Hel}(v) := \int_M (v, \operatorname{curl} v)\, \mu$$ is called helicity.

Thm Helicity Hel is an invariant of the Euler eq'n

Rm 1 Rewrite: $\mathrm{Hel}(v) = \int_M (v, \operatorname{curl} v)\mu = \int_M (i_{\operatorname{curl} v}\, u)\mu$

def of $u = v^\flat$

$$= \int_M u \wedge i_{\operatorname{curl} v}\mu = \int_M u \wedge du$$

Rm 2 Note:

dim M	2	3	4
# first integrals	∞	1	0 ?

No!

Thm (Ovsienko - Chekanov - Kh. (1989))

1) For any $2m$-dim mfd M there exist infinitely-many first integrals of the Euler equation (generalized enstrophies):

$$I_f(v) = \int_M f\left(\frac{(du)^m}{\mu}\right)\mu$$ for any f'n $f: \mathbb{R} \to \mathbb{R}$ and $u = v^\flat$

Ex: $I_k(v) = \int_M \left(\frac{(du)^m}{\mu}\right)^k \mu = \int_{\mathbb{R}^{2m}} \left(\det\left(\frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j}\right)\right)^{k/2} dx^{2m}$

for $f(x) = x^k$ for $M = \mathbb{R}^{2m}$ $\det \omega_{ij}$

2) For any $(2m+1)$-dim mfd M there exists the generalized helicity integral $I(v) = \int_M u \wedge (du)^m$

Ex for $M = \mathbb{R}^{2m+1}$

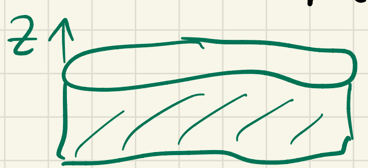
$$I(v) = \int_{\mathbb{R}^{2m+1}} \sum_{(i_1 \dots i_{2m+1})} \varepsilon^{i_1 \dots i_{2m+1}} v_{i_1} \omega_{i_2 i_3} \dots \omega_{i_{2m} i_{2m+1}} d^{2m+1} x$$

for $\omega_{ij} = \frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}$

Cor. Enstrophies in 2D and helicity in 3D are first integrals of the Euler equation.

(Indeed, Euler trajectories $\subset$ coadj. orbits, where Casimirs are const)

Rm - Exer: There are no new integrals for odd dim M^{2m+1} from $\tilde{M}^{2m+2} = M^{2m+1} \times [0,1]$. (Hint: $(d\tilde{u})^{m+1} = 0$ on $\tilde{M}$)



M^{2m+1}

$\tilde{M}$

$d\tilde{u}$ does not include z

$\Rightarrow (d\tilde{u})^{m+1} = 0$

Pf du is well-defined for the whole coset $[u]$, since $d(u+df) = du = d[u]$. Then I_f and I are well-defined on $\mathfrak{g}^* = \Omega^1 / d\Omega^0$

(e.g. $\int_M (u+df)\wedge(d(u+df))^m = \int_M u \wedge (du)^m = \int_M [u]\wedge(d[u])^m$).

Moreover, their definitions are coordinate-free (no metric used) and hence they are invariant w.r.t. coordinate changes $\Rightarrow$ they are Casimirs on $\mathfrak{g}^*$. QED

Prop-exer. The MHD eq's on M^3 admit the first integral $J(v,B) = \int_M (v,B)\,\mu$ (called cross-helicity)

Rm There are more subtle invariants for $[u]$ or du, which give more Casimirs, but not in integral form.

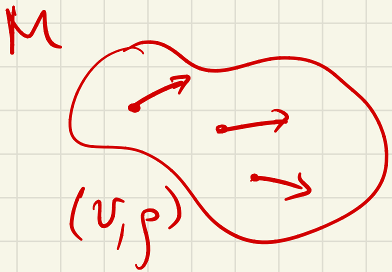
E.g. # of limit cycles or zeros of the vorticity field $\xi = \operatorname{curl} v$ in 3D. They are not necessarily smooth f'ns.

We'll discuss more Casimirs later.

Thm (Kudryavtseva, Enciso-Peralta-Salas-Torres de Lizaur, C. Yang, ...)

Any regular C^1 functional on C^1 (also on $C^{k\geq 4}$, C^∞) exact div-free vector fields on a cpt M^3 and invariant under Diff_μ-action is a C^1-f'n of helicity. In other words helicity is the only regular Casimir for $\mathrm{Diff}_\mu(M^3)$

Equations of a barotropic fluid



v – velocity field

ρ – density of a fluid or gas

$$\begin{cases} \partial_t v + \nabla_v v = -\frac{1}{\rho}\nabla h(\rho) \\ \partial_t \rho + \operatorname{div}(\rho v) = 0 \end{cases}$$

continuity eq'n (ρ is "transported")

Here $p = h(\rho)$ is equation of state

Usually $h(\rho) = c \cdot \rho^a$ in gas dynamics

<u>Rm</u> This eq'n is Hamiltonian on $\mathfrak{g}^*$ for the group $G = \mathrm{Diff}(M) \ltimes C^\infty(M)$ with Hamiltonian $H = \int_M \left(\frac{1}{2}(v,v)\rho + \rho\,\Phi(\rho)\right)\mu$, where $\rho^2\Phi'(\rho) = h(\rho)$