Geometric Fluid Dynamics

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Lecture 3

Tentative Plan:

I. Introducing the Euler equations. Its description as the geodesic flow.

II. Equations on the dual Lie algebra, Lie-Poisson structures, Euler–Arnold equations.

III. The Virasoro algebra and the KdV as an Euler equation.

IV. The Hamiltonian framework for hydrodynamics. Conservation laws for the Euler equations.

V. Geometry of Casimirs: helicity and enstrophies.

VI. Point vortices and vortex filaments.

VII. The Marsden–Weinstein symplectic structure on knots and vortex membranes.

VIII. Geometry of diffeomorphism groups and optimal mass transport.

Lecture 3

The Virasoro algebra and the KdV equation.

Def The Virasoro algebra $Vir = Vect(S^1) \oplus \mathbb{R}$ is the vector space of pairs (a vect. field $f(x)\partial$, a number a) with the commutator

$$[(f(x)\partial, a), (g(x)\partial, b)] := \left((fg' - f'g)(x)\partial, \int_{S^1} f''(x) g'(x) dx\right)$$

for any $(f\partial, a), (g\partial, b) \in Vir$. Here $\partial := \frac{\partial}{\partial x}$

Rm The bilinear form $c(f,g) := \int f''g'(x)dx$ is called the Gelfand-Fuchs 2-cocycle

Rm 1 It is skew-symmetric, $c(f,g) = -c(g,f)$ and satisfies the cocycle identity: $\sum_{\circlearrowleft f,g,h} c([f,g],h) = 0$, which is equivalent to Vir being a Lie algebra.

Note: $a, b \in \mathbb{R}$ are in the center of Vir

$\Leftrightarrow$ vir is a central extension of $\mathrm{Vect}(S^1)$

Rm 2 There exists a Virasoro-Bott group

$\mathcal{V}ir = \mathrm{Diff}(S^1) \times \mathbb{R}$ with the multiplication

$$(\varphi, a) \circ (\psi, b) = \left(\varphi \circ \psi, \frac{1}{2} \int_{S^1} \log(\varphi \circ \psi)' \, d \log \psi'\right).$$

Here $\psi \in \mathrm{Diff}(S^1)$, $\psi(x) = x + \underset{f'n}{\text{periodic}}$, $\psi'(x) > 0 \ \forall x$

Def. Fix the L^2-energy quadratic form on Vir algebra:

$$E(f(x)\partial, a) := \frac{1}{2}\left(\int_{S^1} f^2(x)\,dx + a^2\right)$$ and equip

the Virasoro group with the right-inv. Rieman. metric.

Consider the corresponding geodesic flow on the Vir group.

Thm (Ovsienko-K. 1987) The Euler-Arnold equation corresponding to the geodesic flow for the right-invar. L^2-metric on the Virasoro group is a 1-param. family of the Korteweg-de Vries eq's: on a time-dependent function u on S^1.

$$\begin{cases} \partial_t u + 3uu' + cu''' = 0 \\ \partial_t c = 0 \end{cases}$$

Here the constant c is $\approx$ "depth" of the fluid.

Pf The dual space space vir^* can be identified with $\{(u(x)(dx)^2, c) \mid u \in C^\infty(S^1),\ c \in \mathbb{R}\}$

Indeed, the pairing is

$$\langle (u(x)(dx)^2, c), (f(x)\partial, a)\rangle := \int_{S^1} u(x) f(x)\, dx + a \cdot c$$

Let's find ad^*:

$$\langle \mathrm{ad}^*_{(f\partial, a)}(u(dx)^2, c), (g\partial, b)\rangle = \langle (u(dx)^2, c), \mathrm{ad}_{(f\partial, a)}(g\partial, b)\rangle$$

$$= \langle (u(dx)^2, c), [(f\partial, a), (g\partial, b)]\rangle = \langle (u(dx)^2, c), \Big((fg' - f'g)\partial, \int_{S^1} f'' g' dx\Big)\rangle$$

$$= \int_{S^1} u(fg' - f'g)\, dx + c\int_{S^1} f'' g'\, dx = -\int_{S^1} \big(\underbrace{(uf)'}_{u f' + u' f} + uf' + cf'''\big) g\, dx$$

Thus $\mathrm{ad}^*_{(f\partial, a)}(u(dx)^2, c) = -\big((2uf' + u'f + cf''')(dx)^2, 0\big)$

The L^2-inner product defines the "identity" inertia operator $\mathbb{I}: \mathrm{vir} \to \mathrm{vir}^*$

$$(u\partial, a) \mapsto (u(dx)^2, a)$$

Plug this into the Euler-Arnold equation $\dot{m} = \mathrm{ad}^*_{\mathbb{I}^{-1}m} m$

Here $m = (u(dx)^2, c)$, $\mathbb{I}^{-1}m = (u\partial, c)$ $\Rightarrow \partial_t (u(dx)^2, c) = -((3uu' + cu''')(dx)^2, 0)$

$$\Rightarrow \begin{cases} \partial_t u + 3uu' + cu''' = 0 \\ \partial_t c = 0 \end{cases}$$

the KdV equation

QED.

Exercise: The H^1-inner product on vir

$$E_{\alpha,\beta}(v\partial, a) := \frac{1}{2}\left(\int_{S^1}(\alpha v^2 + \beta (v')^2)\,dx + a^2\right)$$

leads to the Euler-Arnold equation

$$\alpha(\partial_t u + 3uu') - \beta(\partial_t u'' + 2u'u'' + uu''') + cu''' = 0$$

Note: $\alpha=1,\ \beta=0 \rightsquigarrow L^2$-metric, KdV eq'n

$\alpha=\beta=1 \rightsquigarrow H^1$-metric, Camassa-Holm eq'n

$\alpha=0,\ \beta=1 \rightsquigarrow \dot{H}^1$-metric, Hunter-Saxton eq'n

Here $\underline{\mathbb{I}} := \alpha - \beta\partial^2$

Rm Note: for $\mathfrak{g} = \mathrm{Vect}(S^1) = \{ f(x)\partial \mid f \in C^\infty(S^1)\}$

$$\mathrm{ad}_{f\partial}(g\partial) = (fg' - f'g)\partial = L_{f\partial}(g\partial),$$ while for

$$\mathfrak{g}^* = QD(S^1) = \{ u(x)(dx)^2 \mid u \in C^\infty(S^1)\}$$

$$\mathrm{ad}^*_{f\partial}\, u(dx)^2 = -(2uf' + u'f)(dx)^2 = -L_{f\partial}\, u(dx)^2$$

Exer Check the signs by using the pairing

$\langle g\partial, u(dx)^2\rangle = \int g u \, dx$ and relation $L_{f\partial}\langle g\partial, u(dx)^2\rangle = 0$

in the example $f\partial = \frac{\partial}{\partial x}$, $g\partial = x^4 \frac{\partial}{\partial x}$, $u(dx)^2 = x^2 (dx)^2$

Rm It is convenient to identify the dual vir^* as $vir^* = \{(u(x)(dx)^2, c)\} = \{c\partial^2 + u(x) \mid u \in C^\infty(S^1), c \in \mathbb{R}\}$, the space of Hill's operators. Here $\partial^2 := \frac{d^2}{dx^2}$

Then the trace of the monodromy matrix for $c\partial^2 + u(x)$ is a Casimir function. One can classify Virasoro coadjoint orbits in these terms.

Rm. KdV, CH, HS are not only Hamiltonian, but also bihamiltonian systems.

Rm The Virasoro coadjoint action is a "reparametrization" of the Hill equation. Namely, for a solution $y(x)$ for $\left(\frac{d^2}{dx^2}+u(x)\right)y(x)=0$ the Virasoro action is $\varphi^{-1}: y(x) \mapsto y(\varphi(x))(\varphi'(x))^{-1/2}$. Then the Virasoro Ad^*_φ action on the potential $u(x)$ is

$$u(x) \rightsquigarrow y_1, y_2(x) \rightsquigarrow y_1\circ\varphi, y_2\circ\varphi \rightsquigarrow \text{new Hill's eq'n} \quad \tilde{u} = Ad^*_\varphi u$$

potential of $\frac{d^2}{dx^2}+u(x)$ — solutions of Hill's eq'n — change of paramet'n — new Hill's eq'n

This action doesn't change the monodromy of Hill's eq'n $\Rightarrow h(u) := tr\left(Mon\left(\frac{d^2}{dx^2}+u(x)\right)\right)$ is a Casimir on Vir^*.

Step aside: **Bihamiltonian structures**

Def. Two Poisson structures $\{\,\}_0$ and $\{\,\}_1$ on a mfd M are compatible (or form a Poisson pair) if all their linear combinations $\{\,\}_0 + \lambda\{\,\}_1$ are also Poisson structures.

Rm. Bilinearity, skew-symmetry and Leibniz are automatic for any linear combination.
The Jacobi identity gives an extra condition

$$\sum_{\circlearrowleft f,g,h} \{\{f,g\}_0,h\}_1 + \{\{f,g\}_1,h\}_0 = 0 \qquad \forall f,g,h$$

It is sufficient to check for one value $\lambda \neq 0,\infty$, e.g. for $\lambda=1$.

Def. A dynamical system $\dot{m} = F(m)$ on M is Bihamiltonian if the vector field F is Hamiltonian w.r.t. both $\{\,\}_0$ & $\{\,\}_1$ of a Poisson pair.

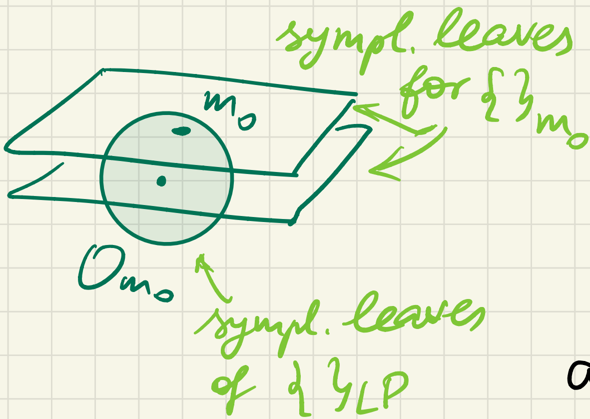
The main example Consider $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$ and $\{\ \}_1 := \{\ \}_{LP}$ its *Lie-Poisson structure*, i.e.

$$\{f, g\}_{LP}(m) := \langle [df|_m, dg|_m], m \rangle$$

Fix (freeze) a pt $m_0 \in \mathfrak{g}^*$. Associate another Poisson br. to m_0

Def The *constant Poisson bracket* associated to $m_0 \in \mathfrak{g}$ is defined by

$$\{f, g\}_{m_0}(m) := \langle [df|_m, dg|_m], m_0 \rangle$$



Proposition $\{\ \}_{LP}$ and $\{\ \}_{m_0}$ are compatible $\forall m_0 \in \mathfrak{g}^*$

Pf $\{\ \}_\lambda := \{\ \}_{LP} + \lambda \{\ \}_{m_0}$ is a Poisson bracket, since it is $\{\ \}_{LP}$ shifted to the pt $-\lambda m_0$. QED

Rm Symplectic leaves of $\{\ \}_{m_0}$ are the tangent plane to O_{m_0} at m_0 and all planes parallel to it in $\mathfrak{g}^*$.
(They depend on the choice of m_0)

Rm-Exer: The Hamiltonian eq'n for $\{\ \}_{m_0}$ on $\mathfrak{g}^*$ and a Ham f'n H has the form $\dot{m} = ad^*_{dH|_m} m_0$

First integrals of bihamiltonian systems

Recall, that a Casimir function for a Poisson br $\{\,\}$ on M is a function h such that $\{h,f\}\equiv 0 \quad \forall f\in C^\infty(M)$

Let h_λ be a Casimir function for $\{\,\}_\lambda = \{\,\}_0 + \lambda\{\,\}_1$ on M, i.e. $\{h_\lambda, f\}_\lambda = 0 \quad \forall f\in C^\infty(M) \quad \forall\lambda$.

Expand $h_\lambda = h_0 + \lambda h_1 + \lambda^2 h_2 + \dots, \; h_i \in C^\infty(M)$

Thm (Magri, Lenard 1978)
Functions h_j, $j = 0, 1, \dots$ are Hamiltonians of a hierarchy of bihamiltonian systems. In other words,
$\forall h_j$ generates a Hamiltonian vect. field X_j w.r.t. $\{\,\}_1$ (i.e. $L_{X_j} f := \{h_j, f\}_1$ $\forall f$), which is also Hamiltonian for the other bracket $\{\,\}_0$ with Hamilton function $-h_{j+1}$ (i.e. $L_{X_j} f = -\{h_{j+1}, f\}_0$). Other coefficients h_i, $i \neq j$ are first integrals of the field X_j

Rm In other words, h_j for $j = 0, 1, \dots$ are in involution w.r.t. both $\{\,\}_0$ and $\{\,\}_1$, i.e. $\{h_i, h_j\}_k = 0$ for $k = 0, 1$ and $\forall i, j$

Pf Use the definition of the Casimir condition for h_λ

$$0 = \{h_\lambda, f\}_\lambda = \{h_0+\lambda h_1+\dots, f\}_0 + \lambda\{h_0+\lambda h_1+\dots, f\}_1$$

At $\lambda^0, \lambda^1, \lambda^2, \dots$ we obtain:

$$\{h_0, f\}_0 = 0 \quad (1)$$

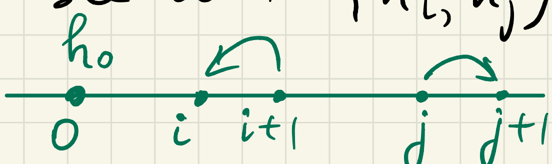
$$\{h_1, f\}_0 + \{h_0, f\}_1 = 0 \quad (2) \qquad \forall f \in C^\infty(M)$$

$$\{h_2, f\}_0 + \{h_1, f\}_1 = 0 \quad (3)$$

Then (1) $\Rightarrow$ h_0 is Casimir for $\{\ \}_0$;

(2) $\Rightarrow$ Hamilton field X_0 for $\{h_0, f\}_1 = L_{X_0} f = -\{h_1, f\}_0$, etc.

To see that $\{h_i, h_j\}_k = 0$, $k = 0,1$, $i<j$ we note



$$\{h_i, h_j\}_1 = -\{h_i, h_{j+1}\}_0 = \{h_{i-1}, h_{j+1}\}_1 = \dots = \{h_0, h_\ell\}_0 = 0$$

QED

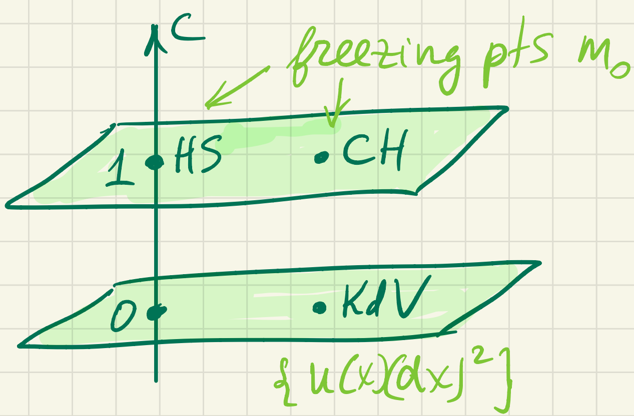
Exercise: Prove that for any two Casimirs h_λ, g_μ, $\{h_i, g_j\} = 0$

Return to KdV:

Thm The KdV equation is bihamiltonian on vir^*:
It is Hamiltonian for $\{\ \}_{LP}$ and for $\{\ \}_{m_0}$ with

$$m_0 = \left(\tfrac{1}{2}(dx)^2, 0\right) \in vir^* \quad \text{KdV}$$

Rm-Exer: Similarly for CH, HS eq's



$$m_0 = \tfrac{1}{2}\left((dx)^2, 1\right) \quad \text{CH}$$

$$m_0 = (0, 1) \quad \text{HS}$$

Namely, we proved that the KdV is Hamiltonian on Vir* for $\{\ \}_{LP}$ and $H_2(u) = \frac{1}{2}\int u^2 dx$ $\left(+\frac{1}{2}a^2\right)$ is irrelevant

For a Poisson structure frozen at $(u_0(dx)^2, a_0) \in$ Vir* the Hamilt. eq'n for a function F is

$$\partial_t\left(u(dx)^2, c\right) = \mathrm{ad}^*_{(f\partial,\, b)}\left(u_0(dx)^2, a_0\right) = -\left(\left(2u_0 f' + u_0' f + a_0 f'''\right)(dx)^2, 0\right)$$

$$\left(\text{cf. } \mathrm{ad}^*_{(f\partial, a)}\left(u(dx)^2, c\right) = -\left(\left(2uf' + u'f + cf'''\right)(dx)^2, 0\right)\right)$$

Obtain for $u_0 = \frac{1}{2}(dx)^2$, $a_0 = 0$:

$$\partial_t\left(u(dx)^2, c\right) = -\left(f'(dx)^2, 0\right)$$

I KdV, II KdV structures

Exer: For $F(u(dx), c) := \int_{S^1}\left(\frac{1}{2}u^3 - \frac{c}{2}(u')^2\right)dx$

$$dF = \frac{\delta F}{\delta(u, a)} = \left(\left(\frac{3}{2}u^2 + cu''\right)\partial, *\right) = (f\partial, a)$$

Plug in to the I^{st} KdV str-re:

$$\partial_t \left(u\,(dx)^2, a\right) = -\left(\left(\tfrac{3}{2}u^2 + cu''\right)'(dx)^2, 0\right) = -\left((3uu' + cu''')(dx)^2, 0\right)$$

the KdV eq'n again!

Thus the KdV is Hamiltonian w.r.t. $\{\,\}_{LP}$ and $\{\,\}_{m_0}$

Rm. Set $h(u) := \log\left(\operatorname{tr}\left(\operatorname{Mon}\left(\frac{d^2}{dx^2} + u(x)\right)\right)\right)$. It is a Casimir for $\{\,\}_{LP}$ on vir^*. Then

$h_\lambda(u) = \log\left(\operatorname{tr}\left(\operatorname{Mon}\left(\frac{d^2}{dx^2} + u(x) - \lambda^2\right)\right)\right)$ is a Casimir for $\{\,\}_{LP} - \lambda^2 \{\,\}_{m_0}$. Expand it: $h_\lambda = 2\pi\lambda - \sum_{n=1}^{\infty} h_{2n-1}\lambda^{1-2n}$

where $h_1 = \frac{1}{2}\int_{S^1} u\,dx$, $h_3 = \frac{1}{8}\int u^2 dx$, $h_5 = \frac{1}{16}\int\left(u^3 - \frac{1}{2}(u')^2\right)dx, \ldots$

first integrals of the KdV eq'n.