Tentative Plan:

I. Introducing the Euler equations. Its description as the geodesic flow.
II. Equations on the dual Lie algebra, Lie-Poisson structures, Euler–Arnold equations.
III. The Virasoro algebra and the KdV as an Euler equation.
IV. The Hamiltonian framework for hydrodynamics. Conservation laws for the Euler equations.
V. Geometry of Casimirs: helicity and enstrophies.
VI. Point vortices and vortex filaments.
VII. The Marsden–Weinstein symplectic structure on knots and vortex membranes.
VIII. Geometry of diffeomorphism groups and optimal mass transport.
The Virasoro algebra and the KdV equation.

**Def.** The Virasoro algebra $\mathfrak{vir} = \text{Vect}(S^1) \oplus \mathbb{R}$ is the vector space of pairs $(a, \mathcal{E})$ with the commutator

$$\left[[f(x) a, g(x) b], (h(x) e, k(x) f)\right] := \left((fg' - f'g) a \mathcal{E}, \int_{S^1} f''(x) g'(x) dx\right)$$

for any $(f(x) a, g(x) b) \in \mathfrak{vir}$. Here $\mathcal{E} := \frac{d}{dx}$.

**Run.** The bilinear form $e(f, g) := \int_{S^1} f''(x) g'(x) dx$ is called the Gelfand–Fuchs 2-cocycle.
\textbf{Rm1} It is skew-symmetric, \( c(\mathfrak{f}, \mathfrak{g}) = -c(\mathfrak{g}, \mathfrak{f}) \) and satisfies the cocycle identity: \( \sum c([\mathfrak{f}, \mathfrak{g}], \mathfrak{h}) = 0 \), which is equivalent to \( \text{Vir} \) being a Lie algebra.

\textbf{Note}: \( \alpha, \beta \in \mathbb{R} \) are in the center of \( \text{Vir} \)

\( \iff \text{vir} \) is a central extension of \( \text{Vect}(S') \)

\textbf{Rm2} There exists a \( \text{Virasoro-Bott} \) group 

\( \text{Vir} = \text{Diff}(S') \times \mathbb{R} \) with the multiplication 

\( (\psi, a) \circ (\psi', b) = (\psi \circ \psi', \frac{1}{2} \int_{S'} \log(\psi \circ \psi')' \, d \log \psi') \).

Here \( \psi \in \text{Diff}(S') \), \( \psi(x) = x + \text{periodic}, \psi'(x) > 0 \ \forall x \neq n \).
Def. Fix the $L^2$-energy quadratic form on Vir algebra:

$$E(f(x)\partial,a) := \frac{1}{2} (\int_{S^1} f^2(x) \, dx + a^2)$$

and equip the Virasoro group with the right-inv. Riemann metric.

Consider the corresponding geodesic flow on the Virasoro group.

Thm (Ovsienko - K, 1987) The Euler-Arnold equation corresponding to the geodesic flow for the right-inv. $L^2$-metric on the Virasoro group is a 1-param. family of the Korteweg-de Vries eq's:

$$\left\{ \begin{array}{l}
\partial_t u + 3uu' + cu''' = 0 \\
\partial_t c = 0
\end{array} \right.$$ on a time-dependent function $u$ on $S^1$. Here the constant $c$ is a "depth" of the fluid.
The dual space $\mathfrak{vir}^*$ can be identified with
$$\{ (u(x)(dx)^2, c) \mid u \in C^\infty(S^1), \ c \in \mathbb{R} \}$$

Indeed, the pairing is
$$\langle (u(x)(dx)^2, c), (f(x) \partial, a) \rangle := \int_{S^1} u(x) f(x) \ dx + a \cdot c$$

Let's find $\text{ad}^*$:

$$\langle \text{ad}^*_{(\ell \partial, a)} (u(dx)^2, c), (g \partial, b) \rangle = \langle (u(dx)^2, c), \text{ad}_{(\ell \partial, a)} (g \partial, b) \rangle$$

$$= \langle (u(dx)^2, c), [(\ell \partial, a), (g \partial, b)] \rangle = \langle (u(dx)^2, c), ((\ell g' - \ell' g) \partial, \int_{S^1} g' \ dx) \rangle$$

$$= \int_{S^1} u(\ell g' - \ell' g) \ dx + \int_{S^1} f'' g' \ dx = - \int_{S^1} ((u\ell)' + uf' + cf''') g \ dx$$

Thus
$$\text{ad}^*_{(\ell \partial, a)} (u(dx)^2, c) = -(2uf' + uf' + cf''') (dx)^2, 0)$$
The $L^2$-inner product defines the “identity” inertia operator $\Pi : \text{vir} \rightarrow \text{vir}^*$

$$(u \partial, a) \mapsto (u(\partial x)^2, a)$$

Plug this into the Euler-Arnold equation,

$\dot{m} = \text{ad}^{-1}_\Pi \ m$

Here

$\dot{m} = (u(\partial x)^2, c)$

$\Pi^{-1} \dot{m} = (u \partial, c) \Rightarrow \partial_t (u(\partial x)^2, c) = -((3uu' + cu'')\partial x^2, 0)$

$\Rightarrow \begin{cases} 
\partial_t u + 3uu' + cu''' = 0 \\
\partial_t c = 0 
\end{cases}$

$\uparrow$ the KdV equation

QED.
Exercise: The $H^1$-inner product on $\mathbb{V}$ is

$$E_{\alpha, \beta}(v^2, a) := \frac{1}{2} \left( \int (\alpha v^2 + \beta (v^\prime)^2) \, dx + a^2 \right)$$

leads to the Euler-Arnold equation

$$\alpha (\partial_t u + 3uu^\prime) - \beta (\partial_t u'''' + 2u'u'''' + uu''''') + cu'''''' = 0$$

Note: $\alpha = 1$, $\beta = 0 \Rightarrow L^2$-metric, KdV equation

$\alpha = \beta = 1 \Rightarrow H^1$-metric, Camassa-Holm equation

$\alpha = 0$, $\beta = 1 \Rightarrow H^1$-metric, Hunter-Saxton equation

Here $\Pi := \alpha - \beta \partial^2$
Note: for \( g = \text{Vect}(S^1) = \{ f(x) \in C^\infty(S^1) \mid \int f(x) dx \} \)

\[
\text{ad}_f(g^2) = (fg' - f'g)dx = L_f(g^2), \quad \text{while for}
\]

\[
g^* = \text{QD}(S^1) = \{ u(x)(dx)^2 \mid u \in C^\infty(S^1) \}
\]

\[
\text{ad}_{f'}^* u(dx)^2 = -(2uf' + u') dx^2 = - L_{f'} u(dx)^2
\]

**Exer** Check the signs by using the pairing

\[
< g^2, u(dx^2) > = \int g^2 u dx \quad \text{and relation} \quad L_{f'} < g^2, u(dx^2) > = 0
\]

in the example \( f^2 = \frac{\partial}{\partial x}, \quad g^2 = x^4 \frac{\partial}{\partial x}, \quad u(dx^2) = x^2(dx)^2 \)
It is convenient to identify the dual \( \text{vir}^* \) as
\[
\text{vir}^* = \{ (u(x)(dx)^2, c) \} = \{ c \delta^2 + u(x) \mid u \in C^\infty(S'), c \in \mathbb{R} \},
\]
the space of Hill's operators. Here \( \delta^2 := \frac{d^2}{dx^2} \).

Then the trace of the monodromy matrix for \( c \delta^2 + u(x) \) is a Casimir function. One can classify Virasoro coadjoint orbits in these terms.

KdV, CH, HS are not only Hamiltonian, but also bihamiltonian systems.
The Virasoro coadjoint action is a "reparametrization" of the Hill equation. Namely, for a solution $y(x)$ for $(\frac{d^2}{dx^2} + u(x)) y(x) = 0$ the Virasoro action is

$$\Phi^*: y(x) \mapsto y(\Phi(x)) (\Phi'(x))^{-\frac{1}{2}}$$

Then the Virasoro $\text{Ad}_\Phi^*$ action on the potential $u(x)$ is

$$u(x) \mapsto y_1(x), y_2(x) \mapsto y_1 \circ \Phi, y_2 \circ \Phi \mapsto \text{new } \tilde{u} = \text{Ad}_\Phi^* u$$

This action doesn't change the monodromy of Hill's eq'n, $\text{eq}'n \Rightarrow \Phi(u) := \text{tr}(\text{Mon}(\frac{d^2}{dx^2} + u(x)))$ is a Casimir on $\text{vir}^*$.
Step aside: Bihamiltonian structures

**Def.** Two Poisson structures $\{f\}$ and $\{g\}$ on a mfd $M$ are compatible (or form a Poisson pair) if all their linear combinations $\{f\} + \alpha \{g\}$ are also Poisson structures.

**Rmk.** Bilinearity, skew-symmetry and Leibniz are automatic for any linear combination. The Jacobi identity gives an extra condition

$$\sum_{\sigma \in S_3} \{\{f, g\}_0, h\}_1 + \{\{f, g\}_1, h\}_0 = 0 \quad \forall f, g, h$$

It is sufficient to check for one value $\alpha \neq 0, \infty$, e.g. for $\alpha = 1$.

**Def.** A dynamical system $m = F(m)$ on $M$ is Bihamiltonian if the vector field $F$ is Hamiltonian w.r.t. both $\{f\}$ & $\{g\}$ of a Poisson pair.
The main example  
Consider $\mathfrak{g}^*$ for a Lie algebra $\mathfrak{g}$ and $\{y_i : i \in \mathbb{N}_0\}$ its Lie-Poisson structure, i.e. 

$$\{f, g\}_{L^p}(m) = \langle [d^f|_m, d^g|_m], m \rangle$$

Fix (freeze) a pt $m_0 \in \mathfrak{g}^*$. Associate another Poisson br. to $m_0$.

**Def.** The constant Poisson bracket associated to $m_0 \in \mathfrak{g}$ is defined by 

$$\{f, g\}_{m_0} = \langle [d^f|_{m_0}, d^g|_{m_0}], m_0 \rangle$$

Proposition $\{\cdot\}_{L^p}$ and $\{\cdot\}_{m_0}$ are compatible \(\forall m_0 \in \mathfrak{g}^*\).
\textbf{Pt.} \( \{ J_a := \{ J \}_L \Pi + a f \} \) \( J_m \) is a Poisson bracket, since it is \( \{ J \}_L \Pi \) shifted to the pt \(-J_m\). \( \text{QED} \)

**Rm.** Symplectic leaves of \( \{ J \} \) are the tangent plane to \( O_m \) at \( m \) and all planes parallel to it in \( \mathfrak{g}^* \).
(They depend on the choice of \( m \))

**Rm.-Exer.** The Hamiltonian eq'n for \( \{ J \} \) on \( \mathfrak{g}^* \) and a Hamiltonian \( H \) has the form \( m = \text{ad}^* m \).
Recall, that a Casimir function for a Poisson br $\{ f, g \}$ on $M$ is a function $h$ such that $\{ h, f \} = 0 \forall f \in \mathcal{C}^\infty(M)$. Let $h_2$ be a Casimir function for $\{ h_2, f \} = \{ h_0 + \alpha f \}$ on $M$, i.e. $\{ h_2, f \} = 0 \forall f \in \mathcal{C}^\infty(M)$ $\forall \alpha$. Expand $h_2 = h_0 + \alpha h_1 + \alpha^2 h_2 + \ldots$, $h_i \in \mathcal{C}^\infty(M)$
functions $h_j$, $j=0,1,\ldots$ are Hamiltonians of a hierarchy of bihamiltonian systems. In other words, $H_j$ generates a Hamiltonian vector field $X_j$ w.r.t. $\{\}^1$ (i.e. $L_{X_j} f = \{h_j, f\}^1 \forall f$), which is also Hamiltonian for the other bracket $\{\}^0$ with Hamilton function $-h_{j+1}$ (i.e. $L_{X_j} f = \{-h_{j+1}, f\}^0$). Other coefficients $h_i$, $i\neq j$ are first integrals of the field $X_j$.

In other words, $h_j$ for $j=0,1,\ldots$ are in involution w.r.t. both $\{\}^0$ and $\{\}^1$, i.e. $\{h_i, h_j\}^k = 0$ for $k=0,1$ and $\forall i,j$.

Use the definition of the Casimir condition for $h_j$. 

Thm (Magri, Lenard 1978)
At $\alpha^0, \alpha^1, \alpha^2, \ldots$ we obtain:

1. $\{h_0, f\}_0 = 0$ \hspace{1cm} (1)
2. $\{h_0, f\}_1 + \{h_0, f\}_0 = 0$ \hspace{1cm} (2) \hspace{1cm} $\forall f \in C^0(\Omega)$
3. $\{h_2, f\}_0 + \{h_1, f\}_1 = 0$ \hspace{1cm} (3)

Then (1) $\Rightarrow h_0$ is Casimir for $\{f\}_0$;
(2) $\Rightarrow$ Hamiltonian field $X_0$ for $\{h_0, f\}_1 = \{X_0, f\} = -\{h_1, f\}_0$, etc.

To see that $\{h_i, h_j\}_k = 0$, $k = 0, 1$, $i < j$ we note

\[
\begin{align*}
\{h_i, h_j\}_1 &= -\{h_i, h_{j+1}\}_0 = \{h_{i-1}, h_{j+1}\}_1 = \ldots = \{h_0, h_e\}_{j+1} = 0
\end{align*}
\]

**Exercise:** Prove that for any two Casimirs $h_\lambda, g_\mu$, $\{h_\lambda, g_\mu\}_0 = 0$
Return to KdV:

The KdV equation is bihamiltonian on $\text{vir}^*$. It is Hamiltonian for $\{ J \}_{LP}$ and for $\{ \}^m_o$ with $R_m$.

Exer: Similarly for CH, HS eq's

$m_o = \left( \frac{1}{2} (dx)^2, 0 \right) \in \text{vir}^*$

KdV

$m_o = \left( \frac{1}{2} (dx)^2, 1 \right)$ CH

$m_o = (0, 1)$ HS
Namely, we proved that the KdV is Hamiltonian on $\text{Vir}^*$ for $\{ J_L \}$ and $H_2(U) = \frac{1}{2} \int u^2 dx + \frac{1}{2} a^2$ is irrelevant. For a Poisson structure frozen at $(u_0(\Delta x), a_0) \in \text{Vir}^*$ the Hamilton eq'n for a function $F$ is

$$\partial_t (u(\Delta x)^2, c) = \text{ad}^* (u_0(\Delta x)^2, a_0) = -(2u_f' + u_f' + a_f'')(\Delta x)^2, 0)$$

Obtain for $u_0 = \frac{1}{2}(\Delta x)^2, a_0 = 0$: $\partial_t (u(\Delta x)^2, c) = -(f'(\Delta x)^2, 0)$

**Exer**: For $F(u(\Delta x), c) = \int \left( \frac{1}{2} u^3 - \frac{c}{2} (u')^2 \right) dx$

$$dF = \frac{\delta F}{\delta (u, a)} = \left( \frac{3}{2} u^2 + cu'' \right) \theta, \phi$$
Plug in to the 1st KdV str-re:
\[ \partial_t (u, (dx)^2, a) = -\left( \frac{3}{2} u^2 + cu'' \right)(dx, a) \]
the KdV eq'n again!

Thus the KdV is Hamiltonian w.r.t. \( \{ J \}_{L^p} \) and \( \{ J \}_{m_0} \).

Rm. Set \( h(u) = \log \left( \text{tr} \left( \text{Mon} \left( \frac{d^2}{dx^2} + u(x) \right) \right) \right) \). It is a Casimir for \( \{ J \}_{L^p} \) on \( \text{vir}^* \). Then
\[ h_\lambda (u) = \log \left( \text{tr} \left( \text{Mon} \left( \frac{d^2}{dx^2} + u(x) - \lambda^2 \right) \right) \right) \]
is a Casimir for \( \{ J \}_{L^p} - \lambda^2 \{ J \}_{m_0} \). Expand it:
\[ h_\lambda = 2\pi \lambda - \sum_{n=1}^{\infty} h_{2n-1} \lambda^{1-2n} \]
where
\[ h_1 = \frac{1}{2} \int u \, dx, \quad h_3 = \frac{1}{8} \int u^3 \, dx, \quad h_5 = \frac{1}{16} \int (u^3 - \frac{1}{2} (u')^2) \, dx, \ldots \]
first integrals of the KdV eq'n.