Geometric Fluid Dynamics

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Lecture 2

Reminder on Poisson structures

Def. A Poisson structure (or P. Bracket) $\{\ \}$ on a mfd M is a bilinear operation on f's on M, $\{\ \}: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ which

a) is skew-symmetric $\{f,g\} = -\{f,g\}$

b) satisfies the Liebniz identity $\{f,gh\} = \{f,g\}h + \{f,h\}g$

c) satisfies the Jacobi identity $\sum_{\circlearrowleft f,g,h} \{\{f,g\},h\} = 0 \quad \forall f,g,h \in C^\infty(M)$

Rm a) & c) $\Rightarrow$ $\exists$ Lie algebra str're on $C^\infty(M)$

b) everything is determined by linear jets $\forall p\in M$.

$\{f,\cdot\}$ is a differentiation of $C^\infty(M)$

Ex a) $\mathbb{R}^2$, $\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$

b) $\mathbb{R}^3$, —— " —— — the same bracket (nothing is going on in the z-direction)

Def. A Poisson bracket on M defines an operator

$$\Pi : C^\infty(M) \to \mathrm{Vect}(M)$$
$$H \longmapsto \xi_H$$

Hamiltonian function $\quad$ Hamiltonian vector field

such that $\{H, f\} = L_{\xi_H} f$ for all test f's $f \in C^\infty(M)$

Ex. $\mathbb{R}^2$, $\{H, f\} = \dfrac{\partial H}{\partial x}\dfrac{\partial f}{\partial y} - \dfrac{\partial H}{\partial y}\dfrac{\partial f}{\partial x} = L_{\xi_H} f$, where

$$\xi_H = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y} = \left(-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}\right)$$

Hamiltonian equations are
$$\begin{cases} \dot{x} = -\dfrac{\partial H}{\partial y} \\ \dot{y} = \dfrac{\partial H}{\partial x} \end{cases}$$

Rm • A Poisson structure can be defined by a bivector Π

$$\{f, g\}(m) = \langle \Pi, df \wedge dg \rangle$$

(Axioms of a Poisson str're $\Rightarrow$ conditions on Π : Schouten bracket $[\Pi, \Pi] = 0$)

- Equivalently, $\{\ \}$ defines an operator

$$\Pi : T^*M \to TM$$
$$dH \mapsto \xi_H$$

Then $\operatorname{Im} \Pi$ is a plane distribution on M, a subbundle in TM

Conditions on $\Pi \Longleftrightarrow$ This is an integrable distribution (Frobenius thm)

$\Rightarrow \exists$ integral submanifolds for Π in M

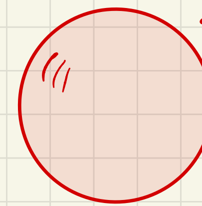
- 2 pts in M are equivalent if there is a path joining them and Hamiltonian at any pt on the way

Integral submfd's $\leftrightarrows$ equivalence classes for Π

Thm (A. Weinstein 1982) For any Poisson manifold its equivalence classes have natural symplectic structures, i.e. any Poisson mfd is (locally) fibered by symplectic leaves. (Moreover, locally there is a splitting into a sympl. mfd and a Poisson mfd of rk 0 at a pt...)

Def A manifold (M^{2n}, ω) is called a symplectic manifold if M is equipped with a closed nondegenerate 2-form ω ($d\omega = 0$ and $\omega^n := \omega \wedge \dots \wedge \omega \neq 0$ on M)

S^2



Ex a) $(\mathbb{R}^2, dx \wedge dy)$ b) $(S^2$, area form$)$

c) S^4 is NOT a symplectic mfd:
(if $\exists\, \omega \in \Omega^2(S^2) \Rightarrow \omega^2$ defines $\underset{\neq 0}{[\omega^2]} \in H^4(S^4) \Rightarrow \underset{\neq 0}{[\omega]} \in \underset{= 0}{H^2(S^4)}$)

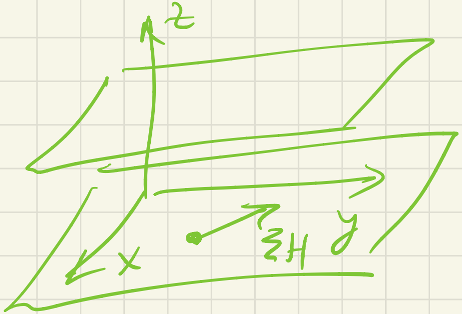
contradiction

Def A Casimir function for $\{\,\}$ on M is a function h such that $\{h, f\} \equiv 0$ $\forall f \in C^\infty(M)$

Ex • $\mathbb{R}^2$, $\{f, g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$

Casimirs are constants only

• $\mathbb{R}^3$, same $\{\,\}$, symplectic leaves are planes $\{z = \text{const}\}$



Casimirs are arbitrary functions $h(z)$

They are constants on symplectic leaves

All Hamiltonian fields are horizontal:

$$\xi_H = \left(-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}, 0\right)$$

Note: If H is a Casimir, $\xi_H = 0$.

Lecture 2

The Lie-Poisson structures. The Euler-Arnold equations.

Let G be a Lie group, $\mathfrak{g} = \mathrm{Lie}(G)$ its Lie algebra

<u>Def</u> On $\mathfrak{g}^*$ there is a linear Poisson bracket (called the Lie-Poisson, Kirillov-Kostant, etc.), i.e. the operation $\{\ \}_{LP} : C^\infty(\mathfrak{g}^*) \times C^\infty(\mathfrak{g}^*) \to C^\infty(\mathfrak{g}^*)$ given by

$$\{\underset{C^\infty(\mathfrak{g}^*)}{f}, \underset{C^\infty(\mathfrak{g}^*)}{g}\}_{LP}(\underset{\mathfrak{g}^*}{m}) := \langle [\underbrace{df|_m, dg|_m}_{\mathfrak{g}}], \underset{\mathfrak{g}^*}{m} \rangle$$

Prop-def'n. The Euler-Arnold (or Euler-Poisson) equation for a Hamiltonian function H with respect to $\{\ \}_{LP}$ is given by

$$\boxed{\dot{m} = \mathrm{ad}^*_{dH|_m} m}$$

Pf $\forall$ test f'n $g \in C^\infty(\mathfrak{g}^*)$

$$\{H, g\}_{LP}(m) = \langle [dH, dg], m \rangle =: \langle \mathrm{ad}_{dH} dg, m \rangle$$

$$= \langle dg, \mathrm{ad}^*_{dH} m \rangle = L_{\xi_H} g(m), \text{ where}$$

$\exists$ 2 def's ad^* with $\pm$

$$\xi_H = \mathrm{ad}^*_{dH|_m} m$$

QED

Rm Recall: for any Lie algebra $\mathfrak{g}$, $[u, v] = \mathrm{ad}_u v$, while the coadjoint operator on $\mathfrak{g}^*$ is defined by

$$\langle [u, v], \ell \rangle =: \langle \underbrace{\mathrm{ad}_u v}_{\mathfrak{g}}, \underbrace{\ell}_{\mathfrak{g}^*} \rangle = \langle \underbrace{v}_{\mathfrak{g}}, \underbrace{\mathrm{ad}^* \ell}_{\mathfrak{g}^*} \rangle \quad \forall \ell \in \mathfrak{g}^*$$

Ex. $G = GL(n) \ni T$, nondeg. $n\times n$ matrices $(\det T \neq 0)$

$\mathfrak{g} = gl(n)$, all $n \times n$ matrices

Group adjoint action = change of coord's, conjugation:

$\forall T \in GL(n) \quad \forall V \in gl(n) \qquad Ad_T V = TVT^{-1}$

Indeed, let $V: x \mapsto Vx$, change coord's $\quad y = Tx$

$T^{-1}y \mapsto VT^{-1}y \qquad\qquad x = T^{-1}y$

$y \mapsto TVT^{-1}y \implies Ad_T V = TVT^{-1}$

For an infinitesimal transf'n $T = I + \varepsilon U$, define

$ad_U: \mathfrak{g} \to \mathfrak{g}$ by $Ad_{(I+\varepsilon U)} \bullet = I + \varepsilon \cdot ad_U \bullet + O(\varepsilon^2)$

Namely, $Ad_{I+\varepsilon U} V = \underbrace{(I+\varepsilon U)}_{=T} V \underbrace{(I+\varepsilon U)^{-1}}_{=T^{-1}\,=\,(I-\varepsilon U)} = V + \varepsilon(UV - VU) + O(\varepsilon^2)$

Thus $ad_U V = UV - VU = [U, V]$

Ex a) $G = SO(3)$ - orthog. 3×3 matrices
$\mathfrak{g} = so(3)$ - skew-sym.

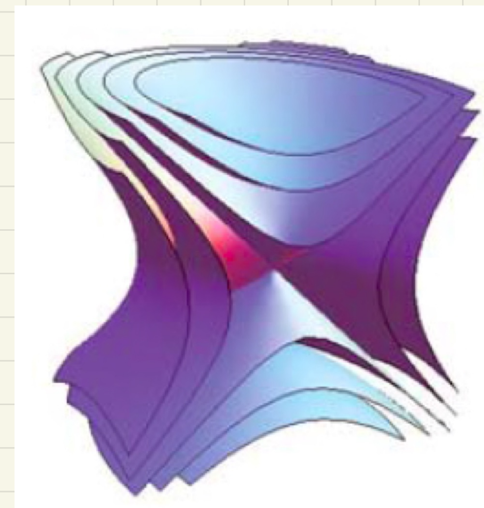
$$so(3) \ni \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \simeq \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \in \mathbb{R}^3$$

Matrix commutator $\simeq$ vector product

group coadjoint orbits $\leftrightarrows$ rotations of vectors $\leftrightarrows$ spheres centered at 0

b) $G = SL(2,\mathbb{R})$ - matrices with $\det = 1$

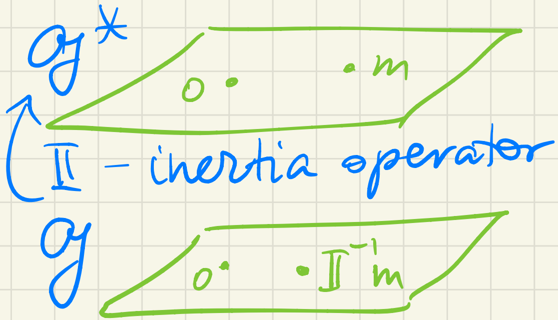
$\mathfrak{g} = sl(2) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}$ - traceless 2×2 matrices



Matrix conjugation $\Rightarrow$ $\Delta = -(a^2 + bc) = \text{const}$

group coadjoint orbits $\leftrightarrows$ hyperboloids, cone, the origin

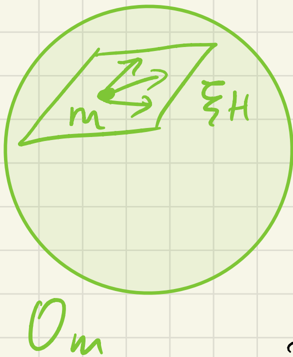
Cor 1 If $H(m) := \frac{1}{2}\langle \mathbb{I}^{-1} m, m \rangle$, a quadratic form for a nondegen. $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^{-1}$



Then $dH|_m = \mathbb{I}^{-1} m$ and the Euler-Arnold equation is

$$\boxed{\dot{m} = ad^*_{\mathbb{I}^{-1} m}\, m}$$

Cor 2 Symplectic leaves of $\{\ \}_{LP}$ - bracket on $\mathfrak{g}^*$ are coadjoint orbits $\mathcal{O}_m = Ad^*_G m$



Indeed, all ξ_H for all f's H have the form $\xi_H(m) = ad^*_{dH} m$, i.e. they are infinitesimal shifts of m by the group coadjoint oper's Ad^*_G,

$$ad^*_{dH} \in T_m (\underbrace{Ad^*_G m}_{\mathcal{O}_m})$$

Thm (V. Arnold)

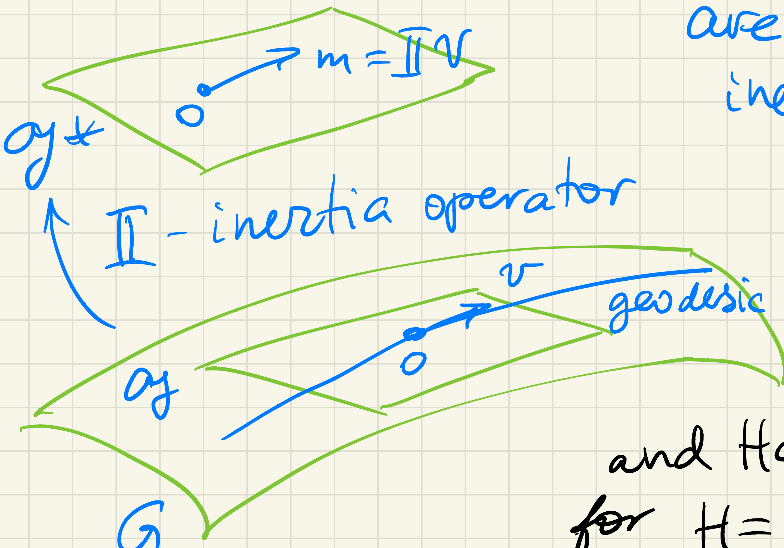
The Euler equation $\dot{m} = \mathrm{ad}^*_{\mathbb{I}^{-1}m}\, m$ on $\mathfrak{g}^*$

and the geodesic equation $\dot{v} = B(v)$ on $\mathfrak{g}$

are related by the

inertia operator $\mathbb{I} : \mathfrak{g} \to \mathfrak{g}^*$

$v \mapsto m = \mathbb{I}v$



"Pf" is the relation of geodesics for $L = \frac{v^2}{2}$ on TG

and Hamiltonian trajectories for $H = \frac{p^2}{2}$ on T^*G.

Example: **Euler top**

Consider the group $G = SO(3)$ and its Lie algebra $\mathfrak{g} = so(3)$

$so(3) \ni \omega$ - angular velocity in the body

The energy $E(\omega) = \frac{1}{2}\langle \omega, \mathbb{I}\omega \rangle = \frac{1}{2}\langle \mathbb{I}^{-1}m, m \rangle = H(m)$,

where $m = \mathbb{I}\omega \in so(3)$ is angular momentum in the body,

$$\mathbb{I} = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix}, \quad \mathbb{I} : so(3) \to so(3)^*$$

$$\mathbb{I} : \omega \mapsto m = \mathbb{I}\omega$$

The Euler top equation is $\dot{m} = ad^*_{\mathbb{I}^{-1}m} m = m \times \mathbb{I}^{-1}m$

$$\Leftrightarrow \quad \dot m_1 = \frac{I_2 - I_3}{I_2 I_3} m_2 m_3 , \quad \dot m_2 = \frac{I_3 - I_1}{I_3 I_1} m_3 m_1 , \quad \dot m_3 = \frac{I_1 - I_2}{I_1 I_2} m_1 m_2$$

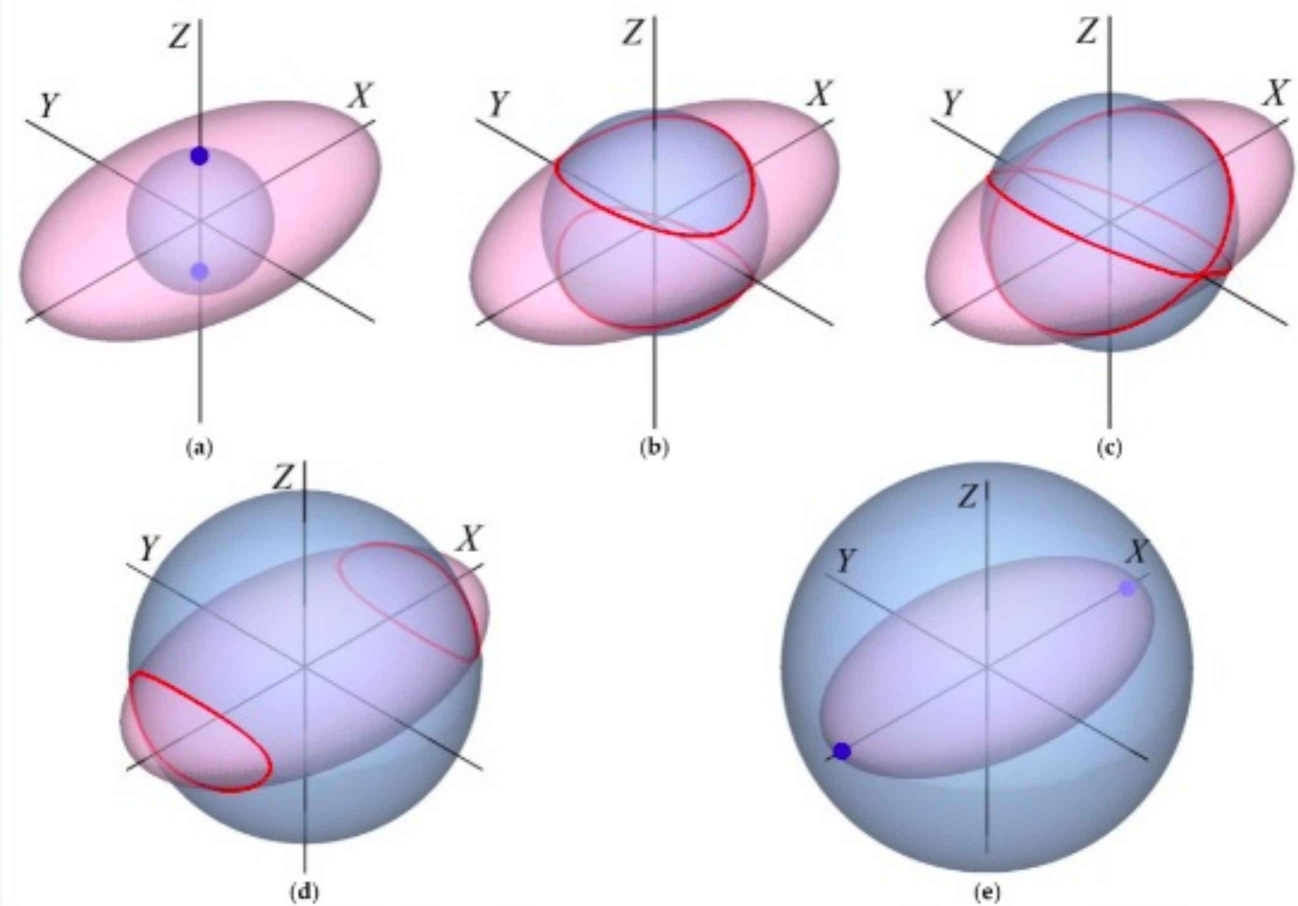
for the angular momentum,
or for the angular velocity

$$I_1 \dot\omega_1 = (I_2 - I_3)\omega_2 \omega_3 , \quad I_2 \dot\omega_2 = \ldots , \quad I_3 \dot\omega_3 = \ldots$$

Hamiltonian picture: symplectic leaves are spheres $\{|m|^2 = \text{const}\} \subset \mathbb{R}^3 = so(3)^*$

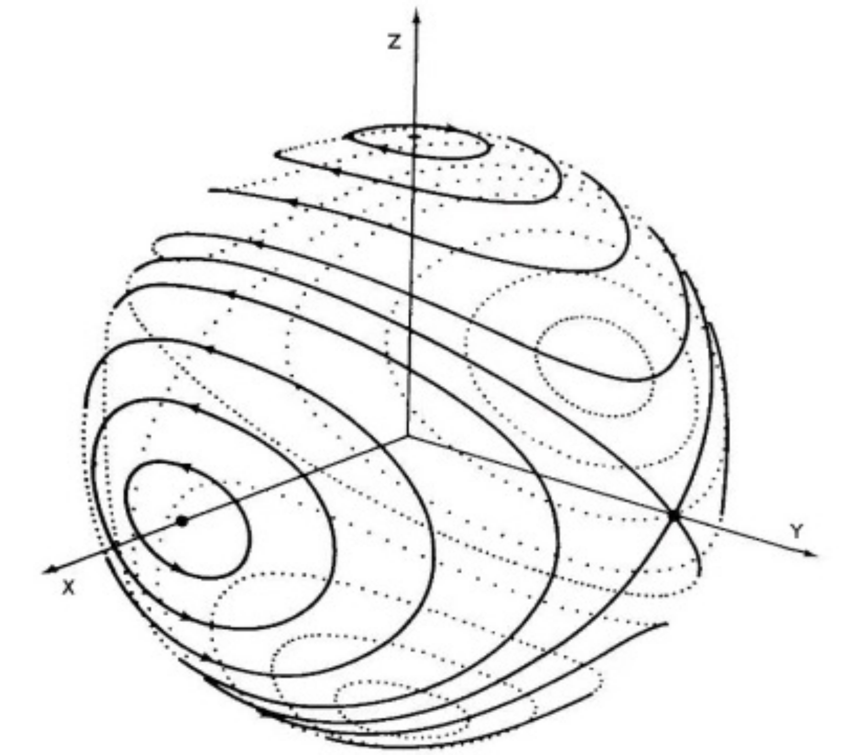
The Hamiltonian $H(m) = \frac{1}{2}\langle \mathbb{I}^{-1} m, m\rangle$, its levels are ellipsoids

trajectories $\subset$ intersections of spheres $|m|^2 = \text{const}$ $\cap$ ellipsoids $H(m) = \text{const}$


Z
Y
X
(a)
Z
Y
X
(b)
Z
Y
X
(c)
Z
Y
X
(d)
Z
Y
X
(e)

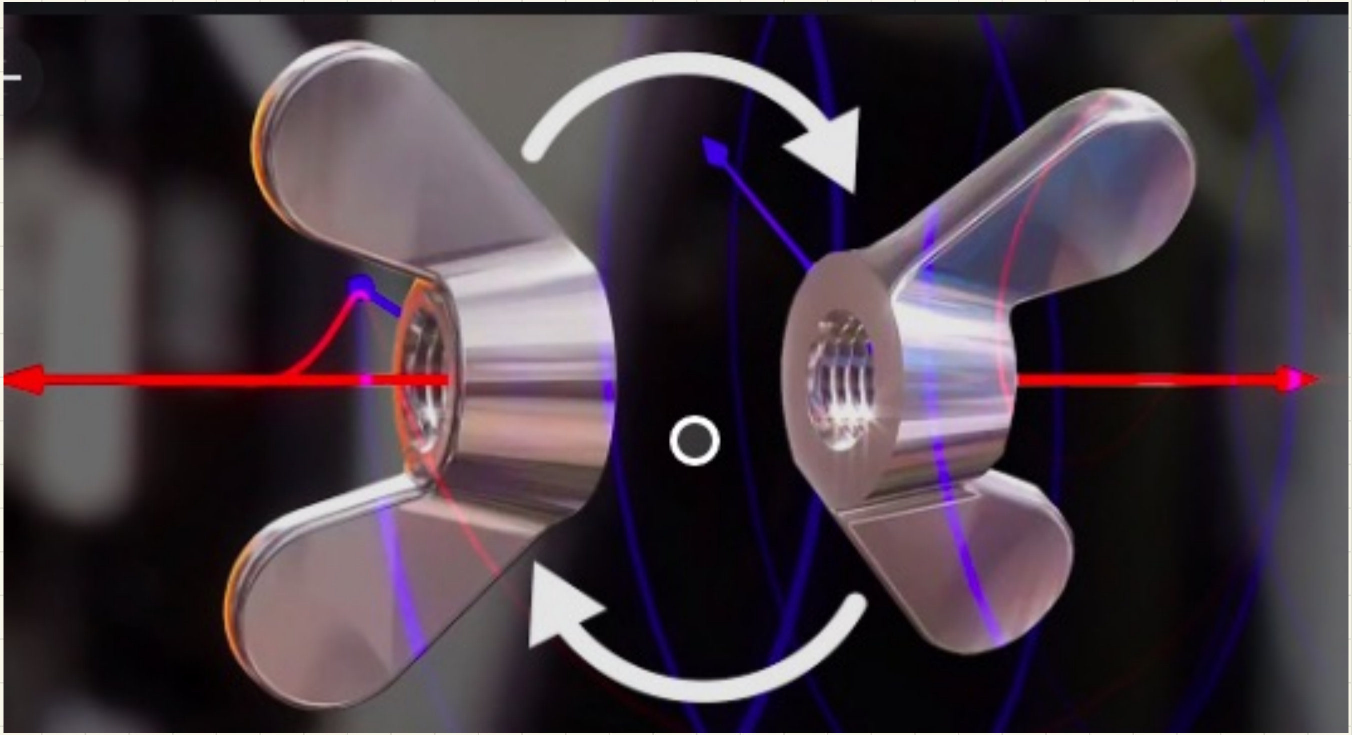
6 stationary rotations:
4 stable (centers)
2 unstable (saddles)

- The Dzhanibekov Effect
- The Tennis Racket Theorem
- The Intermediate Axis Theorem



Trajectories on a phase sphere for a freely rotating rigid body for $I_1 < I_2 < I_3$. Points on the x-axis and z-axis are centres (stable). Points on the y-axis are saddle points (unstable) [image from Bender and Orszag (1978)].

Note: a trajectory close to a saddle pt spends most of the time near two saddles and switches fast between the two.



Higher-dimensional Euler tops

In $n \geq 4$ dimensions a rigid body (with configuration space $SO(n)$) is described as an evolution of its angular velocity $\omega \in so(n)$, skew-sym $n \times n$ matrix.
Its energy is $E(\omega) = -\mathrm{tr}(\omega D \omega)$, where $D = \mathrm{diag}(d_1, \dots, d_n)$ and $d_k = \frac{1}{2} \int_{body} \rho(x) x_k^2 \, dx$ for density ρ at $x \in$ body.

The inertia operator $\mathbb{I}: so(n) \longrightarrow so(n)^*$, $\omega \longmapsto D\omega + \omega D$ is very special:

<u>Thm</u> (Mishchenko $n=4$, Manakov $\forall n \geq 4$) The Euler equation of an n-dim rigid body $\dot{m} = ad^*_{\mathbb{I}^{-1}m} m$ (here $\dot{m} = [\omega, m]$ for $m = D\omega + \omega D$) is a completely integrable system on $so(n)^*$.

Rm Already for $n=4$ stability of steady rotations depend not only on the "semiaxis length", but also on the absolute value $|\omega|$ of the angular velocity.

A. Izosimov associated to a given rotation a "parabolic diagram" (explicitly constructed several parabolas and vertical lines of total degree n)

Thm (Izosimov 2013) A rotation of an n-dim rigid body is stable iff all intersections of the parabolas (and lines) are real and belong to the upper half-plane.