Geometric Fluid Dynamics

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Lecture 1

Tentative Plan:

I. Introducing the Euler equations. Its description as the geodesic flow.

II. Equations on the dual Lie algebra, Lie-Poisson structures, Euler–Arnold equations.

III. The Virasoro algebra and the KdV as an Euler equation.

IV. The Hamiltonian framework for hydrodynamics. Conservation laws for the Euler equations.

V. Geometry of Casimirs: helicity and enstrophies.

VI. Point vortices and vortex filaments.

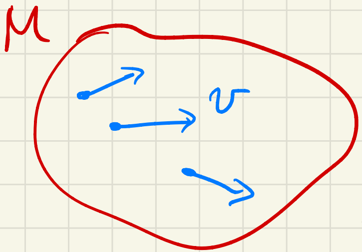
VII. The Marsden–Weinstein symplectic structure on knots and vortex membranes.

VIII. Geometry of diffeomorphism groups and optimal mass transport.

Course website: www.math.toronto.edu/khesin/teaching/henan/geometricfluids21.html

Lecture 1

The hydrodynamic Euler equation. Its description as a geodesic flow.



M, $(,)$ - a Riemannian manifold,

μ - volume form.

The motion of an ideal (inviscid incompressible) fluid filling M is described by the Euler equation

$$\begin{cases} \partial_t v + \nabla_v v = -\nabla p \\ \operatorname{div} v = 0 \quad (\text{and } v \parallel \partial M \text{ if } \partial M \neq \emptyset) \end{cases}$$

on the velocity field v in M

Rm 1. In $\mathbb{R}^n$: $\dfrac{\partial v_i}{\partial t} + \sum v_j \dfrac{\partial v_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i}$ in Euclidean coordinates

Rm 2. The pressure p is defined uniquely modulo an additive constant:

$$\operatorname{div} \Big| \quad \cancel{\operatorname{div} \partial_t v} + \operatorname{div} \nabla_v v = -\operatorname{div} \nabla p, \text{ i.e.}$$

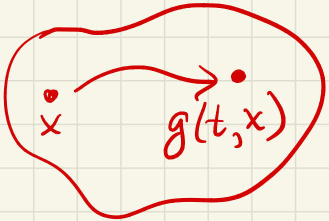
$$\Delta p = -\operatorname{div} \nabla_v v \quad \text{– Poisson eq'n}$$

$$(n, \cdot) \Big| \quad (\partial_t v, n) + (\nabla_v v, n) = -(\nabla p, n), \text{ i.e.}$$

$$\frac{\partial p}{\partial n} = -(\nabla_v v, n) \quad \text{– boundary condition}$$

Thm (V. Arnold 1966) The hydrodynamic Euler equation is the equation of geodesics on the group $\mathrm{Diff}_\mu(M)$ of volume-preserving diffeomorphisms of M with respect to the right-invariant L^2-metric, given at the $\mathrm{id} \in \mathrm{Diff}_\mu(M)$ by the energy $E(v) = \frac{1}{2}\int_M (v,v)\mu$ for $v \in \mathrm{Vect}_\mu(M)$ – a div-free vect. field on M.

A proof sketch for a flat mfd ($\mathbb{R}^n$ or $\mathbb{T}^n$) with coord's $\{x\}$



Let $g(t,x)$ be the motion of the fluid on M (at any moment t a pt $x \in M$ goes to $g(t,x) \in M$, i.e. $g(t,\cdot)$ is a diffeomorphism of M).

Then the fluid velocity is $\partial_t g(t,x) = v(t, g(t,x))$, (i.e. $v = \dot{g} \circ g^{-1}$). Then the fluid acceleration is

$$\partial^2_{tt} g(t,x) = \partial_t(\partial_t g(t,x)) = \partial_t v(t, g(t,x))$$

$$= \Big(\partial_t v + \underset{\underset{\frac{\partial g}{\partial t}}{\shortparallel}}{v} \cdot \frac{\partial v}{\partial x}\Big)(t, g(t,x)) = -\nabla p\,(t, g(t,x))$$

the Euler equation

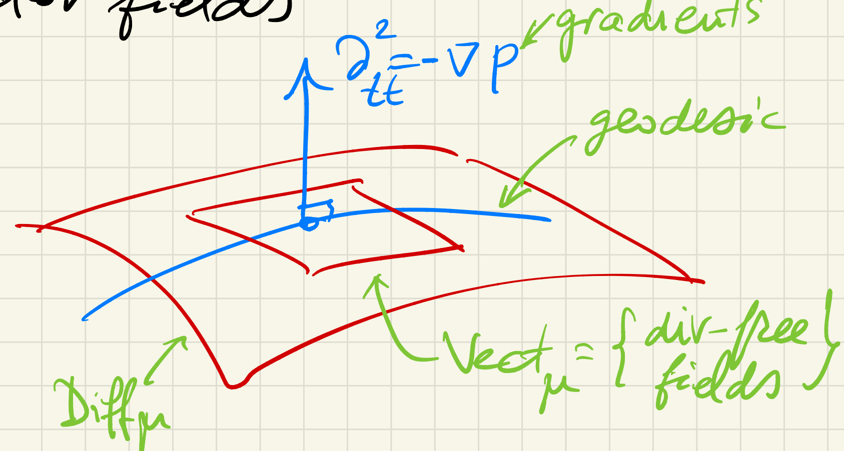
Thus the Euler eq'n $\Longleftrightarrow$ acceleration $\partial^2_{tt} g$ is a gradient field
$\partial_t v + v \cdot \frac{\partial v}{\partial x} = -\nabla p$ equivalent to

Recall the Hodge decomposition:

$$\mathrm{Vect}(M) = \text{Div-free fields} \oplus_{L^2} \text{Gradient fields}$$

In other words, acceleration is L^2-orthogonal to divergence-free vector fields

$\Longleftrightarrow$ this is the definition of a geodesic on a submanifold (accel'n $\perp$ submf'd)



QED

Rm The L^2-metric on $\mathrm{Diff}_\mu(M)$ is right-invariant:

$$\int_M \big(v(y), v(y)\big)\,\mu_y = \int_M \big(v(g(x)), v(g(x))\big)\, g^*\mu_x$$

for $y = g(x)$

$$g^*\mu_x = (\det{}_x g)\,\mu_x, \quad \det{}_x g = 1$$

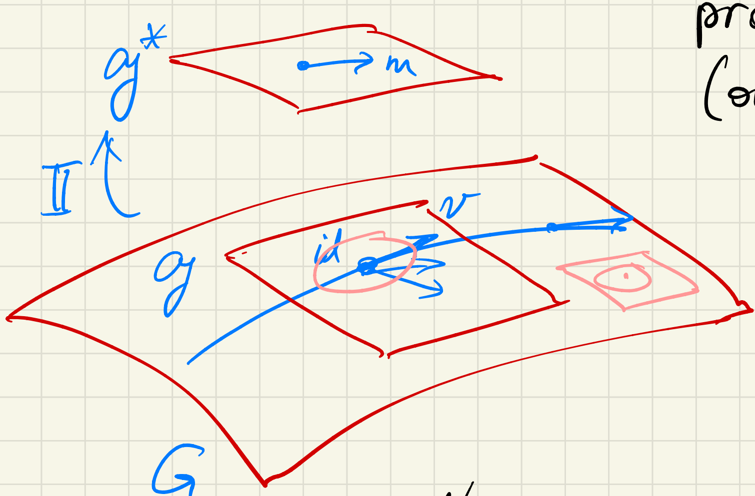
$$= \int_M \big(v(g(x)), v(g(x))\big)\,\mu_x$$

The Riemannian setting of general Euler equations

Let G be a (finite- or infinite-dim) Lie group

$\mathfrak{g} = Lie(G)$ its Lie algebra. Fix some inner product on $\mathfrak{g}$

(or inertia operator $\mathbb{I}: \mathfrak{g} \to \mathfrak{g}^*$)



Define energy $E(v)$ on $\mathfrak{g}$

$$E(v) = \frac{1}{2}\langle \mathbb{I}v, v\rangle$$

$\left(= H(m) \text{ on } \mathfrak{g}^* \text{ for } m = \mathbb{I}v\right)$

Define the left-invar. metric on G:
(or right-inv)

$$(u,v)_{g\in G} := \left(L_{g^{-1}*}u,\ L_{g^{-1}}v\right)_{id}$$

Consider the geodesic flow on G w.r.t this metric
Pull back the velocity vector to the $id \in G$
Obtain an evolution equation $\dot{v} = B(v)$ some nonlinear operator on $\mathfrak{g}$

Definition: The geodesic equation $\dot{v} = B(v)$ is called the (generalized) Euler equation for the group G and inner product $(,)$, corresp. to the inertia op'r $\mathbb{I}$.

Rm a) It can be reformulated as a Hamiltonian equation on $\mathfrak{g}^*$ for the Hamilt. function $H(m) = E(v)$ for $m = \mathbb{I}v$

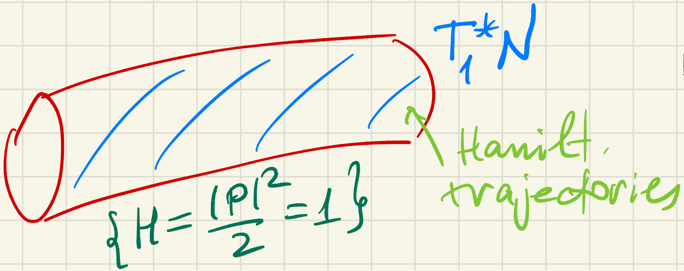
b) The right- and left-invar. geodesic flows differ by $\pm$.

Application: Other groups and energies

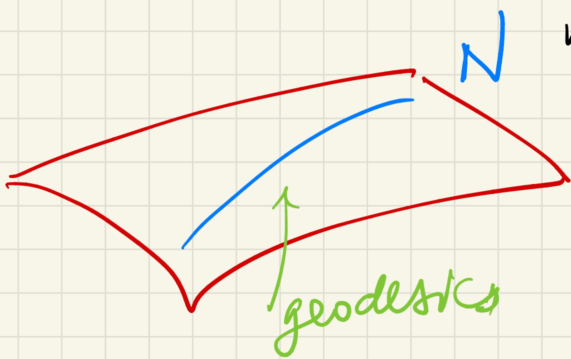
Group	Metric	Equation
$SO(3)$	$\langle \omega, A\omega \rangle$	Euler top
$E(3) = SO(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equation for a body in a fluid
$SO(n)$	Manakov's metrics	n-dimensional top
$\mathrm{Diff}(S^1)$	L^2	Hopf (or, inviscid Burgers) equation
$\mathrm{Diff}(S^1)$	$\dot{H}^{1/2}$	Constantin-Lax-Majda-type equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
$\mathrm{Diff}_\mu(M)$	L^2	Euler ideal fluid
$\mathrm{Diff}_\mu(M)$	H^1	averaged Euler flow
$\mathrm{Symp}_\omega(M)$	L^2	symplectic fluid
Diff(M)	L^2	EPDiff equation
$\mathrm{Diff}_\mu(M) \ltimes \mathrm{Vect}_\mu(M))$	$L^2 \oplus L^2$	magnetohydrodynamics
$C^\infty(S^1, SO(3))$	H^{-1}	Heisenberg magnetic chain

Rm These are Hamiltonian systems on $\mathfrak{g}^*$ with the quadratic Hamiltonian = kinetic energy for the Lie-Poisson bracket.

Rm-preview: Riemannian geometry $\subset$ Symplectic geometry



Namely, geodesics = curves in $T_1 = \{(x,\xi) \in T \mid \frac{|\xi|^2}{2} = 1\}$ on N



minimizing the action f'l $\mathcal{L} = \int_0^1 \frac{|\dot{x}|^2}{2}\,dt$

When rewritten on T^*N (in Hamiltonian form) they are identified by metric on N (i.e. by $\mathbb{I}$) $TN \leftrightarrow T^*N$ with characteristic curves in $T_1^*N = \{(x,p) \mid \frac{|p|^2}{2} = 1\}$ (solutions of the Hamilt. eq'n)

For $N = G$ and left-inv. metric on TG the curves are determined by the energy on $\mathfrak{g}^*$ or by $\mathbb{I}: \mathfrak{g} \to \mathfrak{g}^*$.

Reminder on Poisson structures

<u>Def</u>. A Poisson structure (or P. Bracket) $\{\ \}$ on a mfd M is a bilinear operation on f's on M, $\{\ \}: C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ which a) is skew-symmetric $\{f,g\} = -\{f,g\}$

b) satisfies the Liebniz identity $\{f,gh\} = \{f,g\}h + \{f,h\}g$

c) satisfies the Jacobi identity $\sum_{\circlearrowleft f,g,h} \{\{f,g\},h\} = 0 \quad \forall f,g,h\in C^\infty(M)$

<u>Rm</u> a) & c) $\Rightarrow$ $\exists$ Lie algebra str're on $C^\infty(M)$

b) everything is determined by linear jets $\forall p\in M$.

$\{f,\cdot\}$ is a differentiation of $C^\infty(M)$

<u>Ex</u> a) $\mathbb{R}^2$, $\{f,g\} = \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y} - \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}$

b) $\mathbb{R}^3$, —— " —— — the same bracket (nothing is going on in the z-direction)

Def. A Poisson bracket on M defines an operator

$$\begin{array}{ccc} \Pi : C^\infty(M) & \to & \mathrm{Vect}(M) \\ \Psi & & \Psi \\ H & \longmapsto & \xi_H \end{array}$$

such that $\{H, f\} = L_{\xi_H} f$ for all test f's $f \in C^\infty(M)$

H: Hamiltonian function

ξ_H: Hamiltonian vector field

Ex. $\mathbb{R}^2$, $\{H, f\} = \frac{\partial H}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial f}{\partial x} = L_{\xi_H} f$, where

$$\xi_H = -\frac{\partial H}{\partial y}\frac{\partial}{\partial x} + \frac{\partial H}{\partial x}\frac{\partial}{\partial y} = \left(-\frac{\partial H}{\partial y}, \frac{\partial H}{\partial x}\right)$$

Hamiltonian equations are $\begin{cases} \dot{x} = -\frac{\partial H}{\partial y} \\ \dot{y} = \frac{\partial H}{\partial x} \end{cases}$