MAT 1126HF
Instructor: B. Khesin

Graduate course
"Lie Groups and Fluid Dynamics"

Fall 2005

Overview:

This course deals with various problems in Lie theory, Hamiltonian systems, topology, geometry and analysis, motivated by hydrodynamics and magneto-hydrodynamics. After defining the necessary notions in Lie groups, we discuss the dynamics of an ideal fluid from the group-theoretic and Hamiltonian points of view. We cover geometry of conservation laws of the Euler equation, topology of steady flows and their stability, relation of the energy and helicity of vector fields, geometry of diffeomorphism groups, as well descriptions of magneto-hydrodynamics and of the Korteweg-de Vries equation in the Lie group framework.

References:


Plan:

I. Main notions in Lie groups, Lie algebras, adjoint and coadjoint representations:
   A Lie group and its Lie algebra, the adjoint representation, group adjoint
   and coadjoint orbits.

II. The Euler equations as equations of the geodesic flow:
    Least action principle, the Euler top, the Euler equation of an ideal fluid.

III. The Hamiltonian framework for the Euler equations:
     Equations on the dual Lie algebra, Poisson structures.

IV. Examples:
    $SO(3)$ (motion of a rigid body), $E(3)$ (Kirchhoff equations: body in
    a fluid), $SDiff(M)$ (ideal hydrodynamics), MHD (magnetohydrodynamics),
    Virasoro algebra (the KdV and Camassa-Holm equations), and Landau-
    Lifschits equation.

V. Conservation laws for fluids:
    First integrals for ideal, barotropic, and compressible fluids. Relation to
    symplectic structures on the spaces of curves and polygons in $\mathbb{R}^3$. Vortex
    approximations in 2D.

VI. Steady solutions in 2D and 3D.
    Bernoulli function, related variational problems, restrictions on topology
    of steady solutions, Arnold's stability criterion.

VII. Topology bounds the energy of a field.
    An ergodic interpretation of helicity (asymptotic Hopf invariant), energy
    estimates, Sakhnovich-Zeldovich problem.

VIII. Other applications (time permitted):
     Differential geometry of diffeomorphism groups: diameter and curva-
     tures; generalized flows, Brenier's model with Coulomb interaction, etc.