Problem 1: [4 points]
On $\mathbb{R}^3$ let

$$\alpha = x^2 \mathrm{d}y - 3y\mathrm{d}x, \quad \beta = y^2 e^z \mathrm{d}x \wedge \mathrm{d}z, X = -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}.$$

Calculate (without sign errors!):
a) $\alpha \wedge \beta$
b) $i_X \beta$
c) $\mathrm{d}\beta$
d) $i_X(\alpha \wedge \beta)$

Problem 2: [4 points]
Consider the 1-form

$$\alpha = e^{xy}(y\mathrm{d}x + x\mathrm{d}y) \in \Omega^1(\mathbb{R}^2).$$

Let $\gamma\colon [0,1] \to \mathbb{R}^2$ be the path

$$\gamma(t) = \left(\sin^3(\frac{\pi t}{2}), 2\cos(\pi t)\right).$$

Find the integral $\int_\gamma \alpha$.

Problem 3: [6 points]
Let M be a 3-dimensional manifold, and $\alpha \in \Omega^1(M)$ a 1-form such that

$$\omega = \mathrm{d}\alpha \wedge \alpha \in \Omega^3(M)$$

is nonzero everywhere. In particular, for all p the covector α_p is nonzero, and hence has a 2-dimensional kernel.

a) Suppose $X, Y \in \mathfrak{X}(M)$ are two vector fields with $\alpha(X) = \alpha(Y) = 0$. Show that if X_p, Y_p are linearly independent at some $p \in M$, then the function

$$\omega(X, Y, [X, Y])$$

does not vanish at p. (Thus $X_p, Y_p, [X, Y]_p$ are linearly independent at p.)

b) Give an example of a 1-form $\alpha \in \Omega^1(\mathbb{R}^3)$ with the property

$$\mathrm{d}\alpha \wedge \alpha = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

Problem 4: [6 points]
Consider the 1-form

$$\alpha = \frac{1}{x^2 + y^2}(x\mathrm{d}y - y\mathrm{d}x) \in \Omega^1(\mathbb{R}^2 \setminus \{0\}).$$

a) Find $\int_\gamma \alpha$ for the path $\gamma(t) = (\cos(2\pi kt),\ \sin(2\pi kt)), \quad k \in \mathbb{Z}, \quad t \in [0, 1]$

b) Let $\gamma\colon [0,1] \to \mathbb{R}^2 \setminus \{0\}$ be a closed path (i.e. $\gamma(0) = \gamma(1)$), parametrized by polar coordinates: $\gamma(t) = (r(t), \theta(t))$ with $r \in \mathbb{R}_{>0}$ and $\theta \in \mathbb{R}$. Express $\int_\gamma \alpha$ in terms of the *winding number*

$$w(\gamma) = \frac{1}{2\pi}(\theta(1) - \theta(0)).$$

c) Prove that α is closed.

Extra questions (*Do not hand in.*)

Problem 5:
Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be given by $F(x, y) = (x^2, xy, y^2)$, and let (u, v, w) be coordinates on the codomain of F. Compute the pullback by F of the 1-forms du, dv, dw as well as the 2-forms $du \wedge dv, du \wedge dw, dv \wedge dw$ and finally the 3-form $du \wedge dv \wedge dw$.

Problem 6:
Let $\alpha \in \Omega^2(\mathbb{R}^3 \backslash \{0\})$ be the closed 2-form

$$\alpha = \frac{1}{||x||^3}(x^1 \mathrm{d}x^2 \wedge \mathrm{d}x^3 - x^2 \mathrm{d}x^1 \wedge \mathrm{d}x^3 + x^3 \mathrm{d}x^1 \wedge \mathrm{d}x^2).$$

Let $f, g\colon S^1 \to \mathbb{R}^3$ be two embeddings, with $f(z) \neq g(w)$ for all $z, w \in S^1$. Define a smooth map

$$F\colon S^1 \times S^1 \to \mathbb{R}^3, \quad (z, w) \mapsto f(z) - g(w).$$

By assumption, this map takes values in $\mathbb{R}^3 \backslash \{0\}$; hence the *linking number*

$$L(f, g) := \int_{S^1 \times S^1} F^* \alpha \in \mathbb{R}$$

is defined. Verify:

a) $L(f, g)$ is is invariant under smooth deformations (isotopies) of f, g (preserving the condition that $f(z) \neq g(w)$ for all z, w). In particular, $L(f, g) = 0$ if one of f, g can be deformed to a constant map (while preserving the condition $f(z) \neq g(w)$).

b) The linking number $L(f, g)$ of

$$f(e^{i\theta}) = (\cos\theta, \sin\theta, 0), \qquad g(e^{i\phi}) = (0, -1 + \cos\phi, \sin\phi)$$

for $\theta, \phi \in [0, 2\pi]$ is non-zero.