Problem #1: [4 points]
Compute the Lie brackets \([X, Y]\) of the vector fields:

a) \(X = e^y \frac{\partial}{\partial x} + \sin(x) \frac{\partial}{\partial z}, \quad Y = x y \frac{\partial}{\partial y} + (yz)^2 \frac{\partial}{\partial z}\)

b) \(X = e^r \sin(\theta) \frac{\partial}{\partial r}, \quad Y = (r^3 \cos(\theta) + \theta \ln r) \frac{\partial}{\partial \theta} \)

Problem #2: [6 points]
To any \(n \times n\) matrix \(A = [a_{ij}] \in \text{Mat}_n(\mathbb{R})\) we may associate the vector field

\[X_A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x^i \frac{\partial}{\partial x^j}.\]

1) Compute the Lie bracket \([X_A, X_B]\), express it as \(X_C\) for some \(C\) and explain the relationship between \(A, B, C\) in a coordinate-independent fashion.

2) Let \(\phi_A^t, \phi_B^t\) be the flows of \(X_A, X_B\) respectively. Compute the first nonzero term in the Taylor series expansion of the following function of \(t\):

\[F(t) = \phi_A^t \phi_B^t \phi_A^{-t} \phi_B^{-t}.\]

Problem #3: [6 points]
Let \(A \in \text{Mat}_n(\mathbb{R})\) be skew-symmetric, i.e. \(A + A^\top = 0\).

1) Prove that \(A^L : X \mapsto (X,XA)\) and \(A^R : X \mapsto (X,AX)\) each defines a vector field on \(SO(n)\).

2) Compute the flows of the above vector fields.

3) Let both \(A\) and \(B\) be skew-symmetric matrices. Compute the Lie brackets \([A^L, B^L], [A^L, B^R]\) and \([A^R, B^R]\).

4) Fix an element \(Y \in SO(n)\) and let \(L_Y : SO(n) \to SO(n)\) and \(R_Y : SO(n) \to SO(n)\) be defined as follows:

\[L_Y(X) = YX, \quad R_Y(X) = XY.\]

These operations are known as left multiplication and right multiplication by \(Y \in SO(N)\). Prove that both \(L_Y, R_Y\) are diffeomorphisms.

5) Prove that \(A^L\) is \(L_Y\)-related to itself and that \(A^R\) is \(R_Y\)-related to itself for any \(Y \in SO(n)\). This means that \(A^L\) is left-invariant and \(A^R\) is right-invariant.

Problem #4: [4 points]
1) Give an example of two vector fields \(X, Y \in \mathfrak{X}(\mathbb{R}^3)\) such that for almost all \(p \in \mathbb{R}^3\) the three tangent vectors

\[X_p, Y_p, [X,Y]_p\]
are a basis, but for some $p$ they are not. However, at those special points $p$ the three tangent vectors

$$X_p, Y_p, [X,[X,Y]]_p$$

form a basis. (Hint: The fields $X$ and $Y$ may span the Martinet distribution.)

2) Prove that any pair of vector fields $X,Y$ with property 1) cannot be tangent to any two-dimensional submanifold $S \subset \mathbb{R}^3$.

Problem #5: **Additional problem – will not be graded**

Consider $S^3 = \{(x,y,z,w) \in \mathbb{R}^4| x^2 + y^2 + z^2 + w^2 = 1}\$.

a) Show that

$$X = w \frac{\partial}{\partial x} + z \frac{\partial}{\partial y} - y \frac{\partial}{\partial z} - x \frac{\partial}{\partial w}$$

is tangent to $S^3$.

b) Find another vector field $Y$ (given by a similar formula) that is also tangent to $S^3$, and such that $X,Y$ and $Z := [X,Y]$ span the tangent space $T_pS^3$ for all $p \in S^3$.

Note: Only for $n = 1, 3, 7$ is it possible to find $n$ vector fields spanning the tangent space to $S^n$ for all $p \in S^n$. For instance the 2-sphere $S^2$ does not even admit a vector field that is non-zero at all points of $S^2$.

Problem #6: **Additional problem – will not be graded**

Consider $\mathbb{R}^3$ with coordinates $x, y, z$. Introduce new coordinates $u, v, w$ by setting

$$x = e^u v, \quad y = e^v, \quad z = u v^2 w,$$

valid on the region where $x > y \geq 1$.

a) Express $u, v, w$ in terms of $x, y, z$.

b) Express the coordinate vector fields $\frac{\partial}{\partial u}, \frac{\partial}{\partial v}, \frac{\partial}{\partial w}$ as a combination of the coordinate vector fields $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$, where the coefficients are functions of $x, y, z$.

Problem #7: **Additional problem – will not be graded**

Let $\pi: M \to N$ be a surjective submersion. A vector field $X \in \mathfrak{X}(M)$ is called a lift of a vector field $Y \in \mathfrak{X}(N)$ if

$$(T_p\pi)(X_p) = Y_{\pi(p)}$$

for all $p \in M$.

Suppose that $X_1$ is a lift of $Y_1$ and $X_2$ is a lift of $Y_2$.

1) Show that $[X_1, X_2]$ is a lift of $[Y_1, Y_2]$.

2) Show that $[X_1, X_2]$ is tangent to all fibers $\pi^{-1}(q)$, $q \in N$, if and only if the vector fields $Y_1, Y_2$ commute, i.e., $[Y_1, Y_2] = 0$.

(Hint: What are the tangent spaces to the fibers in terms of $T_p\pi$?)