**Problem #1:** [4 points]
Compute the Lie brackets $[X, Y]$ of the vector fields:
a)

$$X = e^y \frac{\partial}{\partial x} + \sin(x) \frac{\partial}{\partial z}, \qquad Y = xy \frac{\partial}{\partial y} + (yz)^2 \frac{\partial}{\partial z}$$

b)

$$X = e^r \sin(\theta) \frac{\partial}{\partial r}, \qquad Y = (r^3 \cos(\theta) + \theta \ln r) \frac{\partial}{\partial \theta}.$$

**Problem #2:** [6 points]
To any $n \times n$ matrix $A = [a_{ij}] \in \mathbf{Mat}_n(\mathbb{R})$ we may associate the vector field

$$X_A = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x^i \frac{\partial}{\partial x^j}.$$

1) Compute the Lie bracket $[X_A, X_B]$, express it as $X_C$ for some $C$ and explain the relationship between $A, B, C$ in a coordinate-independent fashion.

2) Let $\phi_A^t, \phi_B^t$ be the flows of $X_A, X_B$ respectively. Compute the first nonzero term in the Taylor series expansion of the following function of $t$:

$$F(t) = \phi_A^t \phi_B^t \phi_A^{-t} \phi_B^{-t}.$$

**Problem #3:** [6 points]
Let $A \in \mathbf{Mat}_n(\mathbb{R})$ be skew-symmetric, i.e. $A + A^\top = 0$.
1) Prove that

$$A^L : X \mapsto (X, XA) \quad \text{and} \quad A^R : X \mapsto (X, AX)$$

each defines a vector field on $SO(n)$.

2) Compute the flows of the above vector fields.

3) Let both $A$ and $B$ be skew-symmetric matrices. Compute the Lie brackets $[A^L, B^L]$, $[A^L, B^R]$ and $[A^R, B^R]$.

4) Fix an element $Y \in SO(n)$ and let $L_Y : SO(n) \to SO(n)$ and $R_Y : SO(n) \to SO(n)$ be defined as follows:

$$L_Y(X) = YX, \qquad R_Y(X) = XY.$$

These operations are known as *left multiplication* and *right multiplication* by $Y \in SO(N)$. Prove that both $L_Y$, $R_Y$ are diffeomorphisms.

5) Prove that $A^L$ is $L_Y$-related to itself and that $A^R$ is $R_Y$-related to itself for any $Y \in SO(n)$. This means that $A^L$ is *left-invariant* and $A^R$ is *right-invariant*.

**Problem #4:** [4 points]
1) Give an example of two vector fields $X, Y \in \mathfrak{X}(\mathbb{R}^3)$ such that for *almost all* $p \in \mathbb{R}^3$ the three tangent vectors

$$X_p, \ Y_p, \ [X, Y]_p$$

are a basis, but for some $p$ they are not. However, at those special points $p$ the three tangent vectors

$$X_p,\ Y_p,\ [X,[X,Y]]_p$$

form a basis. (Hint: The fields $X$ and $Y$ may span the *Martinet distribution*.)

2) Prove that any pair of vector fields $X, Y$ with property 1) cannot be tangent to any two-dimensional submanifold $S \subset \mathbb{R}^3$.

**Problem #5:** **Additional problem – will not be graded**
Consider

$$S^3 = \{(x,y,z,w) \in \mathbb{R}^4 |\ x^2+y^2+z^2+w^2 = 1\}.$$

a) Show that

$$X = w\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} - x\frac{\partial}{\partial w}$$

is tangent to $S^3$.

b) Find another vector field $Y$ (given by a similar formula) that is also tangent to $S^3$, and such that $X, Y$ and $Z := [X,Y]$ span the tangent space $T_pS^3$ for all $p \in S^3$.

*Note: Only for $n = 1,3,7$ is it possible to find $n$ vector fields spanning the tangent space to $S^n$ for all $p \in S^n$. For instance the 2-sphere $S^2$ does not even admit a vector field that is non-zero at all points of $S^2$.*

**Problem #6:** **Additional problem – will not be graded**
Consider $\mathbb{R}^3$ with coordinates $x, y, z$. Introduce new coordinates $u, v, w$ by setting

$$x = e^u\, v, \quad y = e^v, \quad z = u\, v^2\, w,$$

valid on the region where $x > y \geq 1$.

a) Express $u, v, w$ in terms of $x, y, z$.

b) Express the coordinate vector fields $\frac{\partial}{\partial u},\ \frac{\partial}{\partial v},\ \frac{\partial}{\partial w}$ as a combination of the coordinate vector fields $\frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z}$, where the coefficients are functions of $x, y, z$.

**Problem #7:** **Additional problem – will not be graded**
Let $\pi\colon M \to N$ be a surjective submersion. A vector field $X \in \mathfrak{X}(M)$ is called a *lift* of a vector field $Y \in \mathfrak{X}(N)$ if

$$(T_p\pi)(X_p) = Y_{\pi(p)}$$

for all $p \in M$.

Suppose that $X_1$ is a lift of $Y_1$ and $X_2$ is a lift of $Y_2$.

1) Show that $[X_1, X_2]$ is a lift of $[Y_1, Y_2]$.

2) Show that $[X_1, X_2]$ is tangent to all fibers $\pi^{-1}(q),\ q \in N$, if and only if the vector fields $Y_1, Y_2$ commute, i.e, $[Y_1, Y_2] = 0$.

(Hint: What are the tangent spaces to the fibers in terms of $T_p\pi$?)