Problem #1: [4 points]
Consider the map

$$F : \mathbb{R}^2 \to \mathbb{R}^2, \quad (r, \theta) \mapsto (x, y) = \left(e^{r+\theta} \cos\theta,\ e^{r+\theta} \sin\theta\right).$$

For $p = (r_0, \theta_0)$, with image point $F(p) = (x_0, y_0)$, find

$$(T_pF)\left(\frac{\partial}{\partial\theta}|_p\right), (T_pF)\left(\frac{\partial}{\partial r}|_p\right).$$

Note: The result should be expressed in terms of $\frac{\partial}{\partial x}|_{F(p)}, \frac{\partial}{\partial y}|_{F(p)}$*; the answer should only contain* x, y *variables, not* r, θ *variables.*

Problem #2: [4 points]
Find the (integer) degree of the map $f : S^n \to S^n$ of the unit sphere, where for $x = (x_0, ..., x_n) \in S^n \subset \mathbb{R}^{n+1}$

a) the map f is the antipodal map $f(x) = -x$.

b) $f(x_0, ..., x_n) = (x_{\sigma(0)}, ..., x_{\sigma(n)})$ for some permutation $\sigma \in S_{n+1}$ of the indices.

Problem #3: [6 points]
Let $S^2 \subset \mathbb{R}^3$ be the 2-sphere, and define

$$TS^2 = \{(p, v) \in \mathbb{R}^3 \times \mathbb{R}^3 |\ \ p \in S^2,\ v \in T_pS^2\}.$$

It may be regarded as a level set of the function $F\colon \mathbb{R}^6 \to \mathbb{R}^2$, $F(p, v) = (p \cdot p,\ p \cdot v)$.

a) Find the differential $(a, w) \mapsto (T_{(p,v)}F)(a, w)$.

b) Show that TS^2 is a 4-dimensional submanifold of $\mathbb{R}^6$.

c) Show similarly that $M = \{(p, v) \in TS^2 |\ v \cdot v = 1\}$ is a 3-dimensional submanifold of $\mathbb{R}^6$.

Remark: The manifold M *has interesting properties: Both maps* $(p, v) \mapsto p,\ (p, v) \mapsto v$ *are surjective submersions* $M \to S^2$*, with fibers diffeomorphic to* S^1*; the map* $(p, v) \mapsto (v, p)$ *restricts to a diffeomorphism of* M *interchanging these two maps.*

Problem #4: [6 points]
Let A, B, C be *distinct* real numbers.

a) Calculate the Jacobian of the map

$$F : \mathbb{R}^3 \to \mathbb{R}^5,\ \ (x, y, z) \mapsto \big(yz, xz, xy, Ax^2 + By^2 + Cz^2,\ x^2 + y^2 + z^2 - 1\big)$$

and show that it has maximal rank except at $(0, 0, 0)$. ** Hints** You may want to consider three cases $x = 0,\ yz \neq 0$, and $x = y = 0,\ z \neq 0$, and $xyz \neq 0$. There are more cases, but those follow 'by symmetry'. Also, you may find it useful to observe that when $Au + Bv + Cw = 0$ and $u + v + w = 0$, then the vanishing of one of u, v, w implies the vanishing of the other two.

b) Describe the tangent space (as a three-dimensional plane in $\mathbb{R}^5$) to the image $F(\mathbb{R}^3) \subset \mathbb{R}^5$ at the point $F(0, 1, 2)$.

c) Using the results from part a), show that the map

$$f\colon S^2 \to \mathbb{R}^4,\ (x,y,z) \mapsto \big(yz, xz, xy, Ax^2 + By^2 + Cz^2\big)$$

is an immersion. (Here $S^2 = \{(x,y,z) |\ x^2 + y^2 + z^2 = 1\}$.)

d) Show that the map

$$g\colon \mathbb{R}\mathrm{P}^2 \to \mathbb{R}^4,\ (x:y:z) \mapsto \frac{1}{x^2+y^2+z^2}\big(yz, xz, xy, Ax^2 + By^2 + Cz^2\big)$$

is an injective immersion. Since $\mathbb{R}\mathrm{P}^2$ is compact, it is an embedding.

** Hint** To show that g is an immersion, use that $f = g \circ \pi$ where $\pi\colon S^2 \to \mathbb{R}\mathrm{P}^2$ is the quotient map.

Problem # 5: **Additional problem – will not be graded**

Let $S \subset \mathbb{R}^3$ be a 2-dimensional submanifold, and let

$$f\colon S \to \mathbb{R}, \quad f(x,y,z) = z.$$

Show that $p \in S$ is a critical point for f if and only if the tangent space $T_pS \subset \mathbb{R}^3$ is the x-y plane.

Hint: View f as the restriction of a function $g\colon \mathbb{R}^3 \to \mathbb{R},\ (x,y,z) \mapsto z$.

Problem #6: **Additional problem – will not be graded**

Let $F\colon M \to N$ be a submersion. Show that for any submanifold $S \subset N$, the pre-image

$$F^{-1}(S) \subset M$$

is a submanifold.

Hint: Use the normal form for submersions.

Problem # 7: **Additional problem – will not be graded**

A planar arm consists of a number n of rigid unit length segments in a plane, joined end-to-end, starting with an end with fixed position and ending with a free end.

a) Define a bijection between the space of configurations X of such a planar arm with n segments and the n-dimensional torus $T^n = (S^1)^n$.

b) If the fixed end is located at $(0,0) \in \mathbb{R}^2$, define the map $f : X \to \mathbb{R}^2$ on any configuration by taking the position of its free end. Describe precisely the critical points and critical values of this map.

c) Consider the configurations $Y \subset X$ where the free end is constrained to lie on the positive x-axis $\{(x,y)\ :\ x > 0, y = 0\}$. Prove that Y is a smooth submanifold.

d) On this submanifold $Y \subset X$, consider the function $g : Y \to \mathbb{R}$ given by the distance of the free end from the fixed end. Describe carefully the critical points and critical values of this map in the cases $n = 2, 3, 4$.

e) In the case $n = 3$, prove that $g^{-1}(2)$ is diffeomorphic to S^1.

f) In the case $n = 4$, prove that $g^{-1}(3)$ is diffeomorphic to S^2. What is $g^{-1}(1)$?