Problem 1: [4 points] Let M be the closed subset of the projective plane defined by

$$M = \{(x : y : z) \in \mathbb{R}P^2 \mid \quad x^2 \leq 2yz - z^2\}.$$

a) Determine the images of M in all three charts of the standard atlas of $\mathbb{R}P^2$.

b) Sketch the images of M in all three charts.

Problem 2: [4 points] Consider the family of functions $f_{a,b} : \mathbb{R}^2 \to \mathbb{R},\ f_{a,b}(x, y) = y^2+x^3+bx+a$ depending on two real parameters a, b. Describe the set $\Sigma \subset \mathbb{R}^2 = \{(a, b)\}$ of all parameters (a, b) for which 0 is a critical value of $f_{a,b}$.

Remark: This set is called the bifurcation diagram of the singularity A_2.

Problem 3: [2+1+1=4 points]
For a real 2×2-matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Mat}_{\mathbb{R}}(2),$$

let $\|A\|$ be the norm, defined as $\|A\|^2 = \mathrm{tr}\,(A^T A)$.

a) Use the regular value theorem to show that the set

$$S = \{A \in \mathrm{Mat}_{\mathbb{R}}(2) \mid \|A\| = 1,\ \ \det(A) = 0\}$$

is a 2-dimensional submanifold of $\mathrm{Mat}_{\mathbb{R}}(2) \simeq \mathbb{R}^4$.

b) Show that the map

$$\pi :\ S \to \mathbb{R}P^1$$

taking $A \in S$ to its (1-dimensional) kernel $\ker(A) \subset \mathbb{R}^2$ is smooth, and determine its fibers $\pi^{-1}(u : v)$, for $u^2 + v^2 = 1$.

c) Show that S is a 2-torus.

Remark: This gives another embedding of a 2-torus as a submanifold of S^3, where the 3-sphere is realized as the set of A such that $\|A\| = 1$. Again, one finds that the two regions bounded by this 2-torus are solid 2-tori.

Problem 4: [2+2=4 points]
Let $B \in \mathrm{Sym}_{\mathbb{R}}(n)$ be a given symmetric $n \times n$ matrix, $B^\top = B$, with $\det(B) \neq 0$. Let $\mathcal{G} \subset \mathrm{Mat}_{\mathbb{R}}(n)$ be the set of matrices A satisfying

$$A^\top BA = B$$

Thus $\mathcal{G} = F^{-1}(B)$ where $F(A) = A^\top BA$.

a) Find the differential $D_A F\colon\ \mathrm{Mat}_{\mathbb{R}}(n) \to \mathrm{Sym}_{\mathbb{R}}(n)$.

b) Show that for all A satisfying $A^\top BA = B$, the differential is surjective, so that $\mathcal{G}$ is a submanifold. What is its dimension?

Problem 5: [2+2=4 points]

Let $\pi : S^3 \to S^2$ be the Hopf map. Viewing S^3 as the unit sphere inside $\mathbb{C}^2 = \mathbb{R}^4$, and S^2 as $\mathbb{C}P^1$, it is given by the map $(z, w) \mapsto (z : w)$. Let $(U_0, \phi_0),\ (U_1, \phi_1)$ be the standard atlas for $\mathbb{C}P^1$. We suggest writing elements of U_0 in the form $(1 : u)$, and those of U_1 in the form $(v : 1)$.

(a) Give explicit diffeomorphisms

$$F_i : U_i \times S^1 \to \pi^{-1}(U_i),$$

such that $(\pi \circ F_i)(p, e^{i\theta}) = p$ for all $p \in U_i$. (We regard S^1 as the set of complex numbers $e^{i\theta}$ of absolute value 1.)

(b) Each of the two maps $F_0,\ F_1$ restricts to a diffeomorphism

$$(U_0 \cap U_1) \times S^1 \to \pi^{-1}(U_0 \cap U_1).$$

Compute the diffeomorphism

$$K : (U_0 \cap U_1) \times S^1 \to (U_0 \cap U_1) \times S^1$$

such that $F_1 = F_0 \circ K$.

Problem 6: **Additional problem – will not be graded**

Let $\mathrm{SL}^{\pm}(2, \mathbb{R}) \subset \mathrm{Mat}_{\mathbb{R}}(2)$ be the group of 2×2-matrices of determinant $+1$ or -1.

a) Show that $\mathrm{SL}^{\pm}(2, \mathbb{R})$ is a submanifold of $\mathrm{Mat}_{\mathbb{R}}(2)$.

b) Show $\mathrm{SL}^{\pm}(2, \mathbb{R})$ has two components, both of which are diffeomorphic to $S^1 \times \mathbb{R}^2$. In particular, $\mathrm{SL}^{\pm}(2, \mathbb{R})$ is non-compact.

c) Let $q : \mathrm{Mat}_{\mathbb{R}}(2) \backslash \{0\} \to S^3$ be the map $A \mapsto A/\|A\|$, in the notation of Problem 2. Show that the restriction of this map to $\mathrm{SL}^{\pm}(2, \mathbb{R})$ is an inclusion as an open subset, and that the complement is exactly the 2-torus described in problem 2. That is, S^3 is a smooth (!) compactification of the non-compact group $\mathrm{SL}^{\pm}(2, \mathbb{R})$.

Problem 7: **Additional problem – will not be graded**

Sketch the levels $f^{-1}(0)$ for various values of parameters (a, b) off and on $\Sigma \subset \mathbb{R}^2$ in Problem 1. These are nondegenerate and degenerate real cubic curves. Try to trace how the curves change when parameters "pass" through Σ.

Problem 8: **Additional problem – will not be graded**

Let M be a manifold, S a submanifold of M, and R a submanifold of S. Show that R is a submanifold of M.