Problem 1: [6 points] Let Φ be the diffeomorphism from $\mathbb{R}^2\backslash\{0\}$ to itself, given by

$$\Phi(x, y) = (3x, \frac{1}{3}y).$$

Let $\sim$ be equivalence relation generated by $p \sim \Phi(p)$, and let $M = \mathbb{R}^2\backslash\{0\}/\sim$ be the quotient space, with quotient map

$$\pi : \mathbb{R}^2\backslash\{0\} \to M.$$

Every open subset $V \subseteq \mathbb{R}^2\backslash\{0\}$ for which the restriction $\pi|_V : V \to M$ is injective defines a chart (U, ϕ) for M, where

$$U = \pi(V), \quad \phi \circ \pi|_V = \mathrm{id}_V$$

One can verify (which I don't ask you to do) that the collection of such charts defines an atlas on M; let $\mathcal{A}$ be the maximal atlas containing all these charts.

(a) Show that every point $p \in \mathbb{R}^2\backslash\{0\}$ has an open neighborhood V such that $\pi|_V : V \to M$ is injective.

(b) Give an example of *inequivalent* points $p_1, p_2 \in \mathbb{R}^2\backslash\{0\}$ such that for **all** open subset $V_1, V_2 \subset \mathbb{R}^2\backslash\{0\}$ with $p_1 \in V_1, p_2 \in V_2$, there exists $n \in \mathbb{N}$ such that

$$\Phi^n(V_1) \cap V_2 \neq \emptyset.$$

Justify your claim.

(c) Show that M with the atlas $\mathcal{A}$ does *not* satisfy the Hausdorff property.

** Remark: ** This illustrates how non-Hausdorff manifolds can arise 'in nature'.

Problem 2: [6 points]

(a) Let

$$f : \mathbb{R}\mathrm{P}^2 \to \mathrm{Mat}_{\mathbb{R}}(3)$$

be the map taking a line ℓ in $\mathbb{R}^3$ to the matrix of orthogonal projection onto ℓ. Explain why this map is injective, describe its image, and give a formula for $P = f([x])$ for any given $[x] = (x^0 : x^1 : x^2)$.

(b) Let

$$g : \mathbb{R}\mathrm{P}^2 \to \mathrm{Mat}_{\mathbb{R}}(3)$$

be the map taking a line ℓ in $\mathbb{R}^3$ to the matrix of rotation by π around that line. Explain why this map is injective, describe its image, and give a formula for $Q = g([x])$, for example in terms of $P = f([x])$.

Reminders: A square matrix A is an *orthogonal projection* if $A^\top = A = A^2$. A square matrix A is a *rotation* if $A^\top = A^{-1}$ and $\det(A) = 1$.

Problem 3: [0+3+1 points]

a) A complex number $z = x + iy$ (here $i = \sqrt{-1}$ and x, y are real) defines a complex linear map

$$\mathbb{C} \to \mathbb{C}, \ w \mapsto zw.$$

Identifying $\mathbb{C} = \mathbb{R}^2$, this can be viewed as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. What is the corresponding matrix? **Hint:** This was discussed in class.

b) Show that the map

$$\mathbb{C}\mathrm{P}^1 \to \mathrm{Gr}_{\mathbb{R}}(2,4),$$

sending a complex line ℓ in $\mathbb{C}^2$ to the subspace E which is just ℓ itself regarded as a 2-dimensional real subspace of $\mathbb{C}^2 = \mathbb{R}^4$, is smooth. **Hint:** Use the standard charts, viewing the projective space as $\mathrm{Gr}_{\mathbb{C}}(1,2)$, together with part a).

c) Indicate (together with an idea of proof, not giving details) how this construction generalizes to define a smooth map $\mathrm{Gr}_{\mathbb{C}}(k,n) \to \mathrm{Gr}_{\mathbb{R}}(2k,2n)$.

Problem 4: [1+1+2 points] Let $M = \mathbb{R}\mathrm{P}^3$ with its standard atlas (U_i, ϕ_i) for $0 \le i \le 3$, as explained in class (see also lecture notes, Section 2.3.2).

a) Calculate the determinant of the Jacobian matrix of the transition functions

$$\phi_i \circ \phi_j^{-1}$$

for all $0 \le i < j \le 3$.

b) Use this result to prove that the modified coordinate maps

$$\psi_i = (-1)^i \phi_i \colon U_i \to \mathbb{R}^3$$

define an **oriented** atlas for $\mathbb{R}\mathrm{P}^3$.

c) Indicate (without giving detailed proofs) what parts of this construction generalize to $\mathbb{R}\mathrm{P}^n$ for any n. **Hint:** Distinguish between odd and even n.

Problem 5: **Additional problem – will not be graded**

Let X be the set of all Canadian people. Decide whether the following relations (see Section 2.9.2) are transitive, reflexive, symmetric. Which of these relations are equivalence relations on X?

- $x \sim y$ if and only if both x and y live in the same province or territory.
- $x \sim y$ if and only if x and y are sisters, or are brothers.
- $x \sim y$ if and only if x is a sister of y.
- $x \sim y$ if and only if x is an ancestor of y.
- $x \sim y$ if and only if x has the same last name as y.

Problem 6: **Additional problem – will not be graded**

Construct a diffeomorphism $\mathbb{C}\mathrm{P}^1 \to S^2$ in any of their atlases.

Verify that the map

$$F(w_0 : w_1) = \frac{1}{|w_0|^2 + |w_1|^2} \left(2\mathrm{Re}(w_1 \bar{w}_0), 2\mathrm{Im}(w_1 \bar{w}_0), |w_0|^2 - |w_1|^2\right)$$

defines a diffeomorphism $F : \mathbb{C}\mathrm{P}^1 \to S^2$. Find its inverse $G : S^2 \to \mathbb{C}\mathrm{P}^1$.

Problem 7: **Additional problem – will not be graded**

1) Give an example of an invertible smooth map whose inverse is not smooth.

2) Write out in detail the construction of the 'standard' atlas for $\mathbb{C}\mathrm{P}^n$, parallel to our construction of the standard atlas for $\mathbb{R}\mathrm{P}^n$, and show that the transition maps are smooth.

3) Show that under the identification $\mathbb{R}\mathrm{P}^n = \mathrm{Gr}(1, n+1)$, the standard atlas for $\mathbb{R}\mathrm{P}^n$ is just the same as the standard atlas for $\mathrm{Gr}(1, n+1)$.