Problem 1: [6 points] Let $\Phi$ be the diffeomorphism from $\mathbb{R}^2\{0\}$ to itself, given by

$$\Phi(x, y) = (3x, \frac{1}{3}y).$$

Let $\sim$ be equivalence relation generated by $p \sim \Phi(p)$, and let $M = \mathbb{R}^2\{0\}/\sim$ be the quotient space, with quotient map

$$\pi : \mathbb{R}^2\{0\} \to M.$$

Every open subset $V \subseteq \mathbb{R}^2\{0\}$ for which the restriction $\pi|_V : V \to M$ is injective defines a chart $(U, \phi)$ for $M$, where

$$U = \pi(V), \quad \phi \circ \pi|_V = \text{id}_V.$$

One can verify (which I don’t ask you to do) that the collection of such charts defines an atlas on $M$; let $\mathcal{A}$ be the maximal atlas containing all these charts.

(a) Show that every point $p \in \mathbb{R}^2\{0\}$ has an open neighborhood $V$ such that $\pi|_V : V \to M$ is injective.

(b) Give an example of inequivalent points $p_1, p_2 \in \mathbb{R}^2\{0\}$ such that for **all** open subset $V_1, V_2 \subset \mathbb{R}^2\{0\}$ with $p_1 \in V_1, p_2 \in V_2$, there exists $n \in \mathbb{N}$ such that

$$\Phi^n(V_1) \cap V_2 \neq \emptyset.$$

Justify your claim.

(c) Show that $M$ with the atlas $\mathcal{A}$ does *not* satisfy the Hausdorff property.

**Remark:** This illustrates how non-Hausdorff manifolds can arise ‘in nature’.

Problem 2: [6 points]

(a) Let

$$f : \mathbb{R}P^2 \to \text{Mat}_\mathbb{R}(3)$$

be the map taking a line $\ell$ in $\mathbb{R}^3$ to the matrix of orthogonal projection onto $\ell$. Explain why this map is injective, describe its image, and give a formula for $P = f([x])$ for any given $[x] = (x^0 : x^1 : x^2)$.

(b) Let

$$g : \mathbb{R}P^2 \to \text{Mat}_\mathbb{R}(3)$$

be the map taking a line $\ell$ in $\mathbb{R}^3$ to the matrix of rotation by $\pi$ around that line. Explain why this map is injective, describe its image, and give a formula for $Q = g([x])$, for example in terms of $P = f([x])$.

**Reminders:**

- A square matrix $A$ is an *orthogonal projection* if $A^T = A = A^2$.
- A square matrix $A$ is a *rotation* if $A^T = A^{-1}$ and $\det(A) = 1$.

Problem 3: [0+3+1 points]

(a) A complex number $z = x + iy$ (here $i = \sqrt{-1}$ and $x, y$ are real) defines a complex linear map

$$\mathbb{C} \to \mathbb{C}, \quad w \mapsto zw.$$

Identifying $\mathbb{C} = \mathbb{R}^2$, this can be viewed as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$. What is the corresponding matrix? **Hint:** This was discussed in class.
b) Show that the map
\[ \mathbb{CP}^1 \rightarrow \text{Gr}_{\mathbb{R}}(2,4), \]
sending a complex line \( \ell \) in \( \mathbb{C}^2 \) to the subspace \( E \) which is just \( \ell \) itself regarded as a 2-dimensional real subspace of \( \mathbb{C}^2 = \mathbb{R}^4 \), is smooth. **Hint:** Use the standard charts, viewing the projective space as \( \text{Gr}_{\mathbb{C}}(1,2) \), together with part a).

c) Indicate (together with an idea of proof, not giving details) how this construction generalizes to define a smooth map \( \text{Gr}_{\mathbb{C}}(k,n) \rightarrow \text{Gr}_{\mathbb{R}}(2k,2n) \).

**Problem 4:** [1+1+2 points] Let \( M = \mathbb{RP}^3 \) with its standard atlas \( (U_i, \phi_i) \) for \( 0 \leq i \leq 3 \), as explained in class (see also lecture notes, Section 2.3.2).

a) Calculate the determinant of the Jacobian matrix of the transition functions \( \phi_i \circ \phi_i^{-1} \) for all \( 0 \leq i < j \leq 3 \).

b) Use this result to prove that the modified coordinate maps \( \psi_i = (-1)^i \phi_i: U_i \rightarrow \mathbb{R}^3 \) define an **oriented** atlas for \( \mathbb{RP}^3 \).

c) Indicate (without giving detailed proofs) what parts of this construction generalize to \( \mathbb{RP}^n \) for any \( n \). **Hint:** Distinguish between odd and even \( n \).

**Problem 5:** **Additional problem – will not be graded**

Let \( X \) be the set of all Canadian people. Decide whether the following relations (see Section 2.9.2) are transitive, reflexive, symmetric. Which of these relations are equivalence relations on \( X \)?

- \( x \sim y \) if and only if both \( x \) and \( y \) live in the same province or territory.
- \( x \sim y \) if and only if \( x \) and \( y \) are sisters, or are brothers.
- \( x \sim y \) if and only if \( x \) is a sister of \( y \).
- \( x \sim y \) if and only if \( x \) is an ancestor of \( y \).
- \( x \sim y \) if and only if \( x \) has the same last name as \( y \).

**Problem 6:** **Additional problem – will not be graded**

Construct a diffeomorphism \( \mathbb{CP}^1 \rightarrow S^2 \) in any of their atlases. Verify that the map
\[ F(w_0: w_1) = \frac{1}{|w_0|^2 + |w_1|^2} (2\text{Re}(w_1\bar{w}_0), 2\text{Im}(w_1\bar{w}_0), |w_0|^2 - |w_1|^2) \]
defines a diffeomorphism \( F: \mathbb{CP}^1 \rightarrow S^2 \). Find its inverse \( G: S^2 \rightarrow \mathbb{CP}^1 \).

**Problem 7:** **Additional problem – will not be graded**

1) Give an example of an invertible smooth map whose inverse is not smooth.

2) Write out in detail the construction of the ‘standard’ atlas for \( \mathbb{CP}^n \), parallel to our construction of the standard atlas for \( \mathbb{RP}^n \), and show that the transition maps are smooth.

3) Show that under the identification \( \mathbb{RP}^n = \text{Gr}(1, n+1) \), the standard atlas for \( \mathbb{RP}^n \) is just the same as the standard atlas for \( \text{Gr}(1, n+1) \).