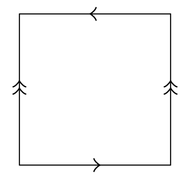
Problem 1 (6 points)**.** Regard the Klein bottle as obtained from a square by the boundary identifications



Using such gluing diagrams, explain that it is possible to cut the Klein bottle along a circle, in such a way that the resulting 'surface with boundary' is either

a) a cylinder (with two boundary circles).

b) a single Möbius strip.

c) a disjoint union of two Möbius strips.

Problem 2 (5 points)**.** The real projective plane $\mathbb{RP}^2$ can be obtained from the 2-sphere S^2, realized as the level set $x^2 + y^2 + z^2 = 1$, by *antipodal identification*, identifying a point $p = (x, y, z)$ with the antipodal point $-p = (-x, -y, -z)$.

What surface is obtained by antipodal identification of a 2-torus, embedded as the set of all $p = (x, y, z)$ with

$$(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$$

for $0 < r < R$? Explain your answer in words and/or pictures; without giving a detailed mathematical proof.

Problem 3 (5 points)**.** Consider a different version of stereographic projection for the 2-sphere $S^2 \subset \mathbb{R}^3$, as follows. As in class, let $\mathsf{n} = (0, 0, 1)$ and $\mathsf{s} = (0, 0, -1)$ be the north and south poles, and put $U = S^2 \backslash \{\mathsf{s}\}, \quad V = S^2 \backslash \{\mathsf{n}\}$.

Let $\phi \colon U \to \mathbb{R}^2$ be the map taking $p = (x, y, z)$ to the unique (u, v) such that $p' = (u, v, 3)$ is on the line through p and s. Let $\psi \colon V \to \mathbb{R}^2$ be the map taking $p = (x, y, z)$ to the unique (u, v) such that $p' = (u, v, -2)$ is on the line through p and n.

a) Explain (e.g., by drawing a picture) what the map ϕ is geometrically. (Of course, for ψ it will be similar.)

b) Give explicit formulas for $\phi(x, y, z)$ and $\psi(x, y, z)$.

c) Compute the transition map $\phi \circ \psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V)$.

Problem 4 (4 points)**.** Let $M \subset \mathbb{R}^2$ be the boundary of a square with vertices at $(\pm 1, \pm 1)$:

$$M = \{(x, y) \in \mathbb{R}^2 \big| \ |x| \le 1 \text{ and } |y| \le 1, \text{ with } |x| = 1 \text{ or } |y| = 1\}.$$

Decide whether or not the charts $(U, \phi), (V, \psi)$ given as

$$U = \{(x, y) \in M|\ y > -1\}, \quad \phi\colon U \to \mathbb{R}, \quad (x, y) \mapsto \frac{x}{1+y}$$

$$V = \{(x, y) \in M|\ y < 1\}, \quad \psi\colon V \to \mathbb{R}, \quad (x, y) \mapsto \frac{x}{1-y}$$

define an atlas on M. Justify your answer.

Problem 5 (Additional problem – will not be graded)**.** Explain the following facts:

a) "Cutting an Möbius strip along its central circle gives a cylinder."

b) "Consider a regular hexagon in which every two opposite sides are identified in a parallel way. The surface obtained this way is a torus."

Problem 6 (Additional problem – will not be graded)**.** Given two surfaces $\Sigma_1,\ \Sigma_2$, one can construct a new surface, denoted $\Sigma_1 \# \Sigma_2$, by taking the 'connected sum'. This is done by removing small disks from Σ_1, Σ_2, and gluing the resulting surfaces with boundary along their boundary circles.

a) Show that $\mathbb{R}P^2 \# \mathbb{R}P^2$ is a Klein bottle.

b) Let Σ be a non-orientable surface. Explain why the connected sum of Σ with a 2-torus is the same as the connected sum with a Klein bottle.

Problem 7 (Additional problem – will not be graded)**.** Show that the set M of affine lines in $\mathbb{R}^3$ is a manifold. Sketch a construction of an atlas for this manifold. What is the dimension?

Problem 8 (Additional problem – will not be graded)**.** Describe an atlas for the set of orbits (phase curves) of the differential equation on the plane:

$$dx/dt = y, \qquad dy/dt = 0.$$