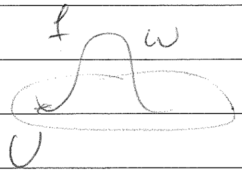
# Integration of diff. forms (6.9–6.10)

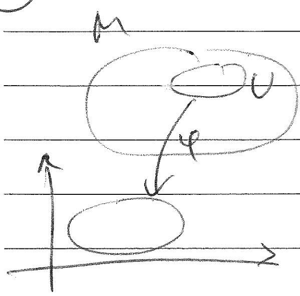
For a top degree form $\omega \in \Omega^m(U \subset \mathbb{R}^m)$, s.t. supp $\omega \subset U$

(here supp $\omega$ = closure of subset where $\omega \neq 0$), $\omega = f(x)\,dx_1 \wedge \dots dx_m$



$$\int_U \omega = \int_{\mathbb{R}^m} f(x)\,dx_1 \dots dx_m$$

(NB) For $U \subset M^m$ (oriented mfd), $\omega \in \Omega^m$ with supp $\omega \subset U$ 16



we consider $(\varphi^{-1})^* \omega = f\,dx_1 \wedge \dots dx_m$

For another chart $(\psi^{-1})^* \omega = g\,dx_1 \wedge \dots \wedge dy_m$

and for $F = \psi \circ \varphi^{-1}$, $y = F(x)$,

$$F^*(dy_1 \wedge \dots \wedge dy_m) = \left(\det \frac{\partial F}{\partial x}\right) dx_1 \wedge \dots \wedge dx_m$$

Then $\displaystyle\int_{\psi(U)} g(y)\,dy_1 \dots dy_m = \int_{\varphi(U)} g(F(x)) \det\frac{\partial F}{\partial x}\,dx_1 \dots dx_m$ — change of variables.

Note: charts must be oriented, $\det \frac{\partial F}{\partial x} > 0$ 19

If supp $\omega \not\subset U$, use partition of unity: $\chi_i \in C^\infty(M)$,

supp $\chi_i \subset U_i$, $\displaystyle\sum_{i=1}^r \chi_i = 1$, $\displaystyle\int_M \omega = \sum_{i=1}^r \int_M \chi_i \omega$

It doesn't depend on the partition: for $\theta_j$, $\sum \theta_j = 1$

$$\int_M \omega = \sum_j \int_M \theta_j \omega = \sum_j \int_M \sum_i \chi_i (\theta_j \omega) = \sum_i \int_M \chi_i \sum_j \theta_j \omega = \sum_i \int_M \chi_i \omega = \int \omega$$ 23

**Def** For $S^k \subset M^m$ – $k$-dim oriented subm., $i: S \hookrightarrow M$ inclusion map

$\displaystyle\int_S \omega = \int_S i^* \omega$, $\omega \in \Omega^k(M)$. Also works for $\forall F: S \to M$

Ex. $\gamma: \mathbb{R} \to M$ – integration of 1-forms $\displaystyle\int_\gamma \alpha$:

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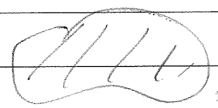
## Stokes' thm (due to Maxwell, 6.11)

~~M - m-dim oriented mfd~~

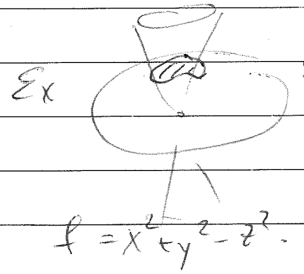
Def. A region w/ smooth boundary in M is a closed subset $D \subseteq M$ s.t. $\exists f \in C^\infty(M)$ where $0$ - reg. value, $D = \{p \in M \mid f(p) \le 0\}$ 3

$\partial D = \{p \in M \mid f(p) = 0\}$



↖ hypersurface in M

Ex



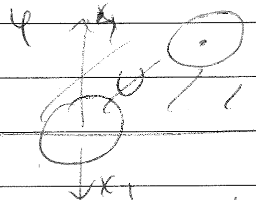
$S^2 = M$

$f = x^2 + y^2 - z^2$. $\quad f|_{\{S^2 = x^2+y^2+z^2=1\}} = 1 - 2z^2$, $f = 0$ is a regular value on $S^2$.

One can give D a structure of a mfd w/ boundary:

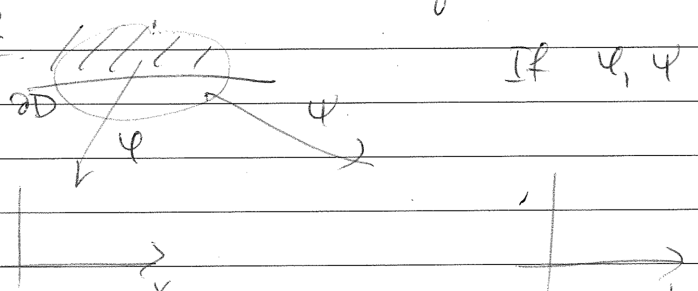
$(U, \varphi)$ - of two types: $\varphi(U) \subset \mathbb{R}^m$, diffeo to $\mathbb{R}^n$

or $\varphi(U \cap D) = \varphi(U) \cap \{x \in \mathbb{R}^n, x_1 \le 0\}$



Furthermore ~~Prop~~ the restriction $(U, \varphi)|_{\partial D}$ gives an oriented atlas for $\partial D$. Indeed,

Pf



If $\varphi, \psi$ map to $x_1 \le 0 \Leftrightarrow y_1 \le 0$, then

$$\left.\frac{\partial y_1}{\partial x_1}\right|_{x_1=0} > 0 \qquad \left.\frac{\partial y_1}{\partial x_j}\right|_{x_1=0} = 0.$$ 5

$$\Rightarrow \det\left(\frac{\partial y_i}{\partial x_j}\right)_{i \ge 2,\ j \ge 2} > 0 \qquad \begin{pmatrix} \frac{\partial y_1}{\partial x_1} > 0 & \\ 0 & * \end{pmatrix}$$

$\det * > 0$. ∎

Cor. $\partial D$ is an oriented mfd.
~~Its orient'n is "induced" by~~
$x_1 \le 0$, i.e. $x_1$ pointing as an exterior normal. Bonus:

Consider $\int_D \omega \; \left( = \sum \int_{D \cap U_i} \chi_i \omega \right)$ 6

↗ D: mfd w/ boundary

~~Thm (Stokes' thm). Let $M$ be an oriented mfd of dim $m$ and $D \subseteq M$ - region of boundary $\partial D$.~~

Let $\alpha \in \Omega^{m-1}(M)$ s.t. supp $\alpha \cap D$ is cpt. Then

$$\int_D d\alpha = \int_{\partial D} \alpha$$ 10

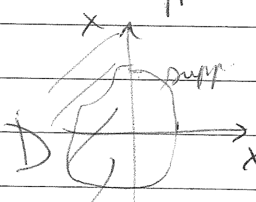
Pf.

Note: $\int_{\partial D} \alpha := \int_{\partial D} i^*\alpha$ for $i: \partial D \to M$

Note: it is suff to prove for supp $\alpha \subset (U, \varphi)$ (in one chart)

(since $\int_D d\alpha = \sum \int_D d(\chi_i \alpha)$; $\int_{\partial D} \alpha = \sum \int_{\partial D} \chi_i \alpha$) 12

It suffices to prove for $\alpha \in \Omega^{m-1}(\mathbb{R}^m)$ and $D = \{x \in \mathbb{R}^m \mid x^1 \le 0\}$




$$\alpha = \sum_{i=1}^m \underbrace{f_i}_{\text{comp supp}} dx_1 \wedge \ldots \wedge \widehat{dx_i} \wedge \ldots \wedge dx_m, \qquad \partial D = \mathbb{R}^{m-1} = \{x_1 = 0\}$$

only 1 term contributes to $\int_{\underset{\partial D}{\overset{\mathbb{R}^{m-1}}{\parallel}}} \alpha = \int f_1(0, x_2, \ldots, x_m)\, dx_2 \ldots dx_m$

$$d\alpha = \sum_{i=1}^m (-1)^{i+1} \frac{\partial f_i}{\partial x_i} dx_1 \ldots dx_m$$ 14

$dx_i$ goes around $(i-1)$ term,

Then $\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_i}{\partial x_i}(x_1 \ldots x_m)\, dx_1 \ldots dx_m = 0 \quad \forall i > 1$

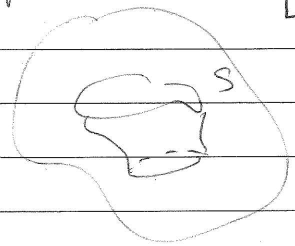
$$\Rightarrow \int_D d\alpha = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{\partial f_1}{\partial x_1} dx_1 \ldots dx_m = \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f_1(0, x_2, \ldots x_m)\, dx_2 \ldots dx_m = \int_{\partial D} \alpha \quad \boxed{}$$

Cor. For $M$ w/o boundary, for $\alpha \in \Omega^{m-1}(M)$ 17

$$\int_M d\alpha = 0$$ - int'l of an exact form over $M$ w/o bound $= 0$.

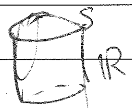
Ex. $\int_\gamma df = 0$ for closed path $\gamma$. 20

**Thm** $S^k$ & $M^m$ - cpt oriented ~~submfd~~ mfds, $\omega \in \Omega^k(M)$, closed $d\omega = 0$




Let $F \in C^\infty(\mathbb{R}\times S, M)$ - smooth map, regarded as a family (deform'n): $F_t = F(t,\cdot): S \to M$
isotopy

Then $\int_S F_t^*\omega$ does not depend on $t$.




**Pf** Consider $D = [a,b]\times S \subset \mathbb{R}\times S$, $\partial D = S_a \sqcup S_b$

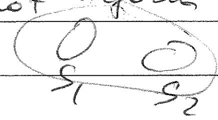
~~orientation of $S_b$ = orient of $S$ = - orien $S_a$ (ext normal)~~

$$\Rightarrow 0 = \int_D F^* \underset{\overset{\|}{0}}{d\omega} = \int_D d F^*\omega = \int_{\partial D} F^*\omega = \int_{S_b} F_b^*\omega - \int_{S_a} F_a^*\omega \Rightarrow \boxed{}$$

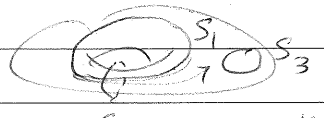
**Cor.** If $F$ shrinks $S$ to a pt (or to $S^{k-1}$), then $\int F_t^* S = 0 \ \forall t$

Indeed $F_1^*\omega = 0$ for dim/degree reasons

If $\int_{S_1}\omega \neq \int_{S_2}\omega$, one cannot deform smoothly $S_1$ to $S_2$ in $M$.



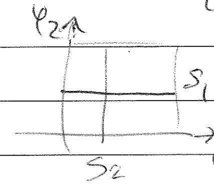

**Ex** $\omega = d\varphi_1$ on $T^2$, closed, not exact as $\varphi_1$ is not a f'n on $T^2$ (not periodic)




$$\int_{S_1}\omega = \int_0^{2\pi} d\varphi_1 = 2\pi$$

$S_1 = \{\varphi_2 = \text{const}\}$

$$\int_{S_2}\omega = \int_0 \underset{\overset{\|}{c}}{d\varphi_2} = \int_0^{2\pi} 0 = 0$$




$\{\varphi_1 = c,\ \varphi_2 = t\}$ mod $2\pi$

$$\int_{S_3}\omega = \int_0^{2\pi} d\varphi_1 = \int_0^{2\pi}\cos t\, dt = \sin t\Big|_0^{2\pi} = 0$$

$\varphi_1 = \frac{1}{2}\cos t$

$\varphi_2 = \frac{1}{2}\sin t$