Integration of differential forms (6.9-6.10)

For a top degree form \( w \in \Omega^m(U) \), and \( \text{supp } w \subseteq U \)
(\text{here } \text{supp } w = \text{closure of subset where } w \neq 0), \ w = f(x_1 dx_{m-1} \ldots dx_m

\[ \int w = \int f(x) \, dx_1 \ldots dx_m \]

For \( U \subseteq M \subset \mathbb{R}^m \) with \( \text{supp } w \subseteq U \)

we consider \( (\mathcal{U}_i)^* w = \int f_i dx_{m-1} \ldots dx_m \)

For another chart \( (\mathcal{V}_j)^* w = g_j dx_{m-1} \ldots dx_m \)

and for \( F = \psi \circ \mathcal{U}_i \), \( y = F(x) \),

\[ F^*(dy_1 \ldots dy_m) = \det \frac{\partial F}{\partial x} \, dx_1 \ldots dx_m \]

Then \( \int g(y) \, dy_1 \ldots dy_m = \int g(F(u)) \left| \det \frac{\partial F}{\partial x} \right| dx_1 \ldots dx_m \)

\[ \psi^*(w) = \text{change of variable} \]

Note: charts must be oriented, \( \det \frac{\partial F}{\partial x} > 0 \)

If \( \text{supp } w \neq U \), use partition of unity: \( \chi_i \in C_c^{\infty}(M) \),

\[ \text{supp } \chi_i \subseteq U_i, \quad \sum_{i=1}^M \chi_i = 1, \quad \int w = \sum_{i=1}^M \int \chi_i \, w \]

It doesn't depend on the partition. For \( B_j, \quad \sum B_j = 1 \)

\[ \int w = \sum_{j=1}^M \int \chi_j \, w = \sum_{j=1}^M \sum_{i=1}^M \chi_i \, (\chi_j \, w) = \sum_{j=1}^M \chi_j \, \sum_{i=1}^M \chi_i \, w = \sum_{i=1}^M \chi_i \, w = \int w \]

Def. For \( S \subseteq M \) - k-dim oriented submanifold, \( C: S \rightarrow M \)

\[ \int_S w = \int i^* w \]

Also works for \( V \subset S \rightarrow M \)

\[ \int_S w = \int \pi^* w \]

Ex. \( f: \mathbb{R} \rightarrow M \) - intersection of 1-forms \( f \):
Stokes' Theorem (due to Maxwell, 6.11)

A region \( w \) smooth boundary in \( M \) is a closed subset \( D \subset M \) s.t. \( f \in C^0(w) \) where \( 0 \)-reg. value, \( D = \{ p \in M \mid f(p) < 0 \} \)

\[ \partial D = \{ p \in M \mid f(p) = 0 \} \]

One can give \( D \) a structure of a manifold with boundary.

\( \{(U, \varphi)\} \) of two types:

- \( \varphi(U) \subset \mathbb{R}^n \), differentiable
- \( \varphi(U \cap D) = \varphi(U) \cap \{ x \in \mathbb{R}^n \mid x_n = 0 \} \)

Prop. \( \partial D \) furthermore the restriction \( \varphi|_{\partial D} \) gives an oriented atlas for \( \partial D \).

Indeed, if \( \varphi, \psi \) map to \( x_1 \leq 0, y_1 \leq 0 \), then

\[ \frac{\partial y_1}{\partial x_1} > 0, \quad \frac{\partial y_1}{\partial x_j} = 0, \quad x_1 = 0, \quad y_1 > 0, \quad j > 1 \]

Cor. \( \partial D \) is an oriented manifold.

Its orientation is "induced" by \( x_1 < 0 \), i.e. \( x_1 \) pointing as an exterior normal.

Consider \( \int_{\partial D} \omega = \sum_{i=1}^n x_i \omega \)

\[ \int_D \omega \]
Thus (Stokes' theorem). Let \( M \) be an oriented manifold of dim \( m \), and \( D \subseteq M \) a region w boundary \( \partial D \).

Let \( \alpha \in \Omega^m(M) \) s.t. \( \text{supp } \alpha \cap \partial D \) is cpt. Then,

\[
\int_D d\alpha = \int_{\partial D} \alpha
\]

**Proof.**

Note: \( \int_D \alpha := \int i^*\alpha \) for \( i: \partial D \to M \)

\[
\int_D \alpha = \int_{\partial D} \sum_i f_i^*(x_i) dx_1 \wedge \ldots \wedge dx_n
\]

Note: It is sufficient to prove for \( \text{supp } \alpha \subseteq (0, \infty) \) (in one chart)

\[
\int_D \alpha = \sum_i f_i(0, x_1^i, \ldots, x_n^i) dx_2 \ldots dx_n
\]

It suffices to prove for \( \alpha \in \Omega^{m-1}(\mathbb{R}^m) \) and \( D = \{ x \in \mathbb{R}^m \mid x \leq 0 \} \)

\[
\alpha = \sum_{i=1}^m f_i x_1 \wedge \ldots \wedge \overset{\wedge}{dx_i} \wedge \ldots \wedge dx_m
\]

\[
\int_D \alpha = \int_{\partial D} \sum_{i=1}^m \left( \text{comp. supp. } \right) f_i(0, x_1^i, \ldots, x_n^i) dx_2 \ldots dx_n
\]

Only 1 term contributes to \( \int_{\partial D} \alpha \)

\[
\int_{\partial D} \alpha = \int_{x_1 = 0} \sum_{i=1}^m f_i(0, x_1, \ldots, x_n) dx_2 \ldots dx_n
\]

Then \( \int D \alpha = \int_{x_1}^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f_i(x_1, \ldots, x_n) dx_1 \ldots dx_m = 0 \) if \( i > 1 \)

\[
\Rightarrow \int_D d\alpha = \int_{\partial D} \alpha = \int_{x_1}^\infty \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} f_i(0, x_2, \ldots, x_n) dx_2 \ldots dx_n
\]

**Cor.** For \( M \) w/o boundary, for \( \alpha \in \Omega^m(M) \)

\[
\int_D d\alpha = 0 \quad \text{iff } \text{of an exact form over } M \text{ w/o boundary.}
\]

**Ex.** \( \int_D df = 0 \) for closed pde \( f \).
Thus \( S^k \& M_{\text{o}} \) is orientable, \( \omega \equiv \omega^k_M \) closed.

\[
\omega = \frac{1}{k!} \sum_{\sigma} \text{sign} \sigma \prod_{i=1}^{k} dx_i 
\]

Let \( F \in C^\infty (R \times S^k, M) \) smooth map, regarded as a family (deformation): \( F_t = F(t, \cdot) : S^k \to M \)

Then \( \int F_t^* \omega \) does not depend on \( t \).

**Proof** Consider \( D = [0, b] \times S^k \subset R \times S^k \), \( \partial D = S^k_0 \cup S^k_2 \)

Orientation of \( S^k_0 \) = orientation of \( S^k_2 = \) one of \( S^k_0 \) (ext. normal)

\[
0 = \int_D F^* \omega = \int_D F_t^* \omega = \int_{S^k_0} F_t^* \omega = \int_{S^k_2} F_t^* \omega = 2 \tag{3}
\]

**Cor.** If \( F \) shrinks \( S^k \) to a pt \((0, k+1)\), then \( \int F^* \omega = 0 \) \( \forall t \in [0, b] \).

Indeed \( F_t^* \omega = 0 \) for dim/degree reasons.

If \( \int d\omega \neq \int \omega \) one cannot deform smoothly \( S^k_1 \) to \( S^k_2 \) in \( M \).

\[
\int d\omega = \int d\Phi_1 = 2\pi \tag{4}
\]

\[
\int \omega = \int d\Phi_2 = 2\pi \tag{5}
\]

\[
\int_{S^k_1} \omega = \int_{S^k_2} \omega = 0 \tag{6}
\]

\[
\int_{S^k_2} \omega \int_{S^k_3} \omega = 0 \tag{7}
\]

\[
\omega = \frac{1}{2} \cot \theta \quad \Phi_1 = \frac{1}{2} \cot \theta
\]

\[
\Phi_2 = \frac{1}{3} \sin \theta
\]