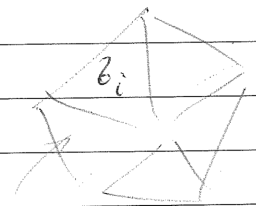
Homology of a mfd *

Let C be an oriented k-dim polyhedron

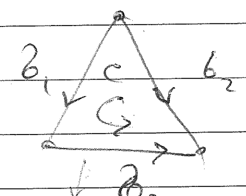
The boundary of C is a linear combination of $k-1$-dim polyhedra as follows:



$$\partial D = \sum b_i$$

↑ oriented by ext. normal

Ex



$$\partial C = b_1 + b_3 - b_2$$

Prop. $\partial^2 = 0$.

Pf True for a Δ ($\forall$ pt appears twice, with $\pm$)

$\Rightarrow$ True $\forall$ dim. $\boxtimes$

Def The group of k-dim chains in M: Suppose M is a (finite) union of polyhedra (triangulated) b_i

$$C_k(M) = \{\mu = \sum_{i=1}^{n} a_i C_i \mid a_i \in \mathbb{Z} \text{ or } \mathbb{R}\}$$

Cycles: $Z_k(M) = \{\mu \in C_k \mid \partial\mu = 0\}$

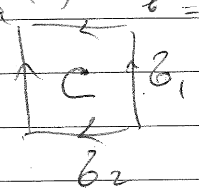
Boundaries $B_k(M) = \{\mu \in C_k \mid \exists \nu \in C_{k+1} \text{ s.t. } \mu = \partial\nu\}$

Prop $\Rightarrow$ $B_k \subset Z_k$

k^{th} the Homology group of M: $H_k(M) = Z_k(M) / B_k(M)$

Def. Euler char. $\chi(M) = \sum (-1)^k b_k = \sum (-1)^k \dim C_k$

Ex 1) Homology of $\mathbb{T}^2$: a, $b = a$



$\partial C = b_1 + b_2 - b_1 - b_2 = 0$

$\partial b_1 = a - b = 0$

$\partial b_2 = 0$, $\partial a = 0$

$H_0 = \{a\} = \mathbb{R}$

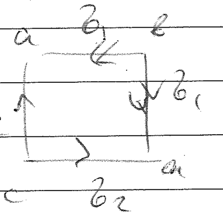
$H_1 = \{b_1, b_2\} = \mathbb{R}^2$

$H_2 = \{C\} = \mathbb{R}$

The Euler char. $\chi(M)=\sum_{k=0}^{m}(-1)^k \dim C_k^{(M)} = \sum_{k=0}^{m}(-1)^k \dim H_k(M)$

$\chi(T^2)=0=1-2+1=1-2+1$

Ex 2) Homology of S^2:



$H_2=\{C\}=\mathbb{R}$

$H_1=\{0\}=0$

$H_0=\{a\}=\mathbb{R}$

$\partial C = b_1-b_1+b_2-b_2$

$\partial b_1 = a-b$, $\partial b_2 = a-c$, $\forall$ comb'n b_1, b_2 $\exists\, \partial b \neq 0$.

$\partial a = \partial b = \partial c = 0$

~~$\chi(S^2)=1-0+1=2$~~

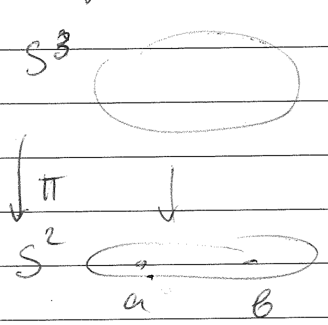
$=3-2+1=2$

$\exists$ pairing of homology & cohomology groups

$H_k(M)$, $H^k(M)$: one can integrate k-forms over k-chains

$$\int_{C^k}\omega^k$$

Hopf invariant: $\pi : S^3 \to S^2$,



1) $\mathrm{Hopf}(\pi) = lk(\pi^{-1}(a), \pi^{-1}(b))$

2) take $\nu \in \Omega^2(S^2)$, $\int \nu = 1$, Note: $d\nu = 0$

Consider $\underbrace{\pi^*\nu}_{=\omega} \in \Omega^2(S^3)$, ω is closed, since $d\omega = d\pi^*\nu = \pi^* d\nu = 0$

$\Rightarrow$ exact $\omega = d\alpha$, $\alpha \in \Omega^1(S^3)$

Then $\mathrm{Hopf}(\pi) = \int_{S^3} \alpha \wedge d\alpha = \int_{S^3} d^{-1}\omega \wedge \omega$

Exer This def'n does not dep. on the choice of ν, of $\overset{\alpha}{d^{-1}\omega}$, of a, b

1) and 2) are equivalent

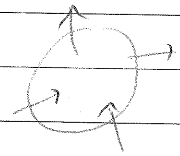
Poincare lemma: (e.s. Locally (in any ball)) ~~$\forall$ closed k-form ω is exact: $\exists \eta$ in U s.t. $\omega = d\eta$~~

On a contractible contin mfd (deformed to a pt)

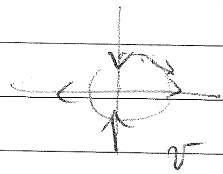
Rm. Index of an isolated zero of a v.f.

v – v.f on $U \subset \mathbb{R}^n$, $v(0) = 0$, regard v as a map $V: S^{n-1}_\varepsilon \to S^{n-1}$, $v(x) \to \frac{v}{\|v\|}$

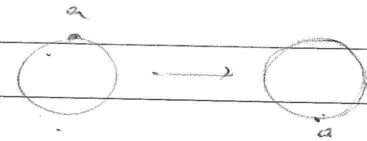
Take a small sphere S^{n-1} around 0, $\mathrm{ind}_q(0) = \deg(V)$



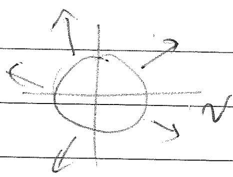
Ex.



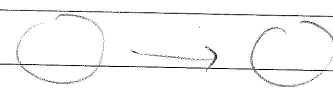
V



$\mathrm{ind}_0 V = -1$



V



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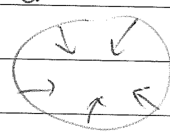
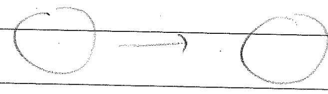
$\mathrm{ind}_0 V = 1$

or



V



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antipodal

Thm (Poincaré-Hopf) $\sum_{\substack{\text{zeros} \\ \text{of } v \in M}} \mathrm{ind}\, v = \chi(M)$

Cor (Hairy ball thm) $\nexists$ smooth v.f on sphere S^2 w/o zeros. (Indeed $\chi(S^2) = 2 \Rightarrow \exists$ zeros)

Cor. $\sum$ ind. of crit. pt $= \sum \mathrm{ind}(\nabla f) = \chi(M)$
$\forall f \in C^\infty(M)$

Morse f'n: locally near zero $f(x_1 \dots x_n) = \underbrace{-x_1^2 - \dots - x_i^2}_{\text{index} = i} + x_{i+1}^2 + \dots + x_n^2$

$\mathrm{ind}\, \nabla f = (-1)^i$