Exterior differential $\Omega^0(M)$

Recall: $(df)(X) = X(f)$, i.e. $d: C^\infty(M) \to \Omega^1(M)$

Thm-def d extends uniquely to linear maps $d: \Omega^k(M) \to \Omega^{k+1}(M)$ s.t. $d(df) = 0$ and $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$, $\forall \alpha \in \Omega^k, \beta \in \Omega^l$

d is called the *exterior differential*

Pf If exists, such d is local: the restrict'n $(d\alpha)|_U$ depends on $\alpha|_U$ or $\alpha|_U = 0 \Rightarrow d\alpha|_U = 0$. Indeed, if $p \in U$, take $f \in C^\infty(M) = \Omega^0(M)$ s.t. $f|_{M \setminus U} = 0$, $f|_p = 1 \Rightarrow f\alpha = 0$ and by product rule

$$0 = d(f\alpha) = \underbrace{df \wedge \alpha}_{0} + \underbrace{f}_{1}\, d\alpha \Rightarrow d\alpha|_p = 0.$$

(!) Then work in local coord's:

$$\alpha = \sum_{i_1 < \dots < i_k} \alpha_{i_1 \dots i_k} \, dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

By product rule and $d(dx_i) = 0 \Rightarrow d\alpha := \sum \underbrace{d\alpha_{i_1 \dots i_k}}_{\sum_{l=1}^{n} \frac{\partial \alpha_{\dots}}{\partial x_l} dx_l} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$

Conversely, we can define $d\alpha$ by this f'la in $U \subset M$

By uniqueness, in different charts def's agree, and $d(d\omega) \overset{\circ}{=}$ locally as follows from $\frac{\partial^2 \omega_I}{\partial x_i \partial x_j} = \frac{\partial^2 \omega_I}{\partial x_j \partial x_i}$ ☐

Ex $\omega = e^{x^2 y} dx \wedge dz + \sin(xyz)\, dx \wedge dy$

$d\omega = x^2 e^{x^2 y} dy \wedge dx \wedge dz + xy \cos(xyz)\, dz \wedge dx \wedge dy = (-x^2 e^{x^2 y} + xy \cos(xyz))\, dx \wedge dy \wedge dz$

Def. A k-form $\omega \in \Omega^k(M)$ is called *exact* if $\exists \alpha \in \Omega^{k-1}(M)$ s.t. $\omega = d\alpha$. ω is called *closed* if $d\omega = 0$.

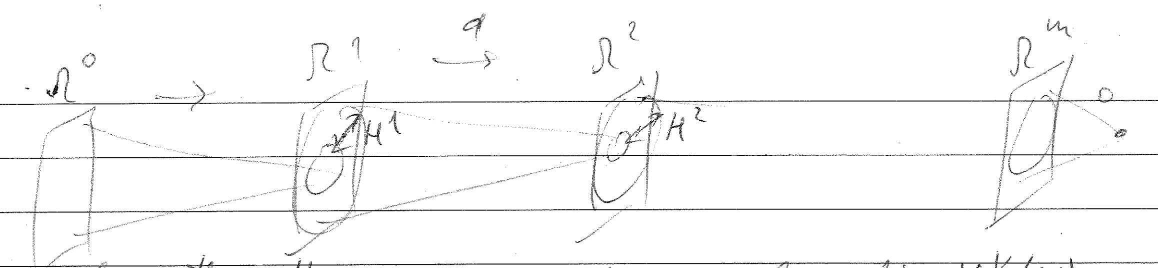
Rm Since $d \circ d = 0$, all exact forms are closed. These are linear spaces

E.g. exact 1-forms: $\alpha = df$

Exer But $\alpha = \frac{x\,dy - y\,dx}{x^2 + y^2} \in \Omega^1(\mathbb{R}^2 \setminus 0)$ is closed, not exact. (PS)

Def The k^{th} (de Rham) cohomology group of a mfd M is

$$H^k(M) = \frac{\{\alpha \in \Omega^k(M) \mid \alpha \text{ is closed}\}}{\{\alpha \in \Omega^k(M) \mid \alpha \text{ is exact}\}}$$



The k^{th} Betti number of M is $b_k = \dim H^k(M)$

Ex. S^2: $b_0 = b_2 = 1$, $b_1 = 0$, S^k: $b_0 = b_k = 1$, $b_l = 0$.

T^2: $b_0 = b_2 = 1$, $b_1 = 2$

$\mathbb{C}P^n$: $b_{2k+1} = 0$, $b_{2k} = 1$. 6

Def The Euler charact. $\chi(M) = \sum_{k=0}^{m} (-1)^k b_k$.

Ex T^2: $H^0(M) = \dfrac{\{f \mid df = 0\}}{\{f \text{ exact}\} = \{0\}} = \mathbb{R}$, $b_0 = 1$ | $H^2(M) = \dfrac{\{\forall\omega \in \Omega^2\}}{\{\omega \in \Omega \mid \int \omega = 0\}} = \mathbb{R}$, $b_2 = 1$ (closed)

$H^1(M) = \dfrac{\{\sum a_1 d\varphi_1 + a_2 d\varphi_2 + df\}}{df} = \mathbb{R}^2$, $b_1 = 2$ | $\chi(T^2) = 1 - 2 + 1 = 0$.

5

Lie derivative and contractions (Cartan calculus, 6.8)

Recall, given $\alpha \in \Omega^1(M)$, $X \in \mathfrak{X}(M)$, $\alpha(X) = \langle \alpha, X \rangle \in C^\infty(M)$

(e.g. $\langle df, X \rangle = \cancel{\mathcal{L}_X}\, X(f)$)

Given $\alpha \in \Omega^k(M)$, $X \in \mathfrak{X}(M)$ define $i_X \alpha \in \Omega^{k-1}(M)$.

~~by~~ contraction (or inner product):

$$(i_X\alpha)(X_1 \ldots X_{k-1}) = \alpha(X, X_1 \ldots X_{k-1}) = \begin{vmatrix} b_1 & \cdots & b_m \\ (X_1) & & \\ \vdots & & \\ X_{k-1} & & \end{vmatrix}$$

$$X = \sum_{k=1}^{m} b_k \frac{\partial}{\partial x_k},$$

8

Property: $i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^k \alpha \wedge i_X \beta$

exer (determinant prop') Thm-def: Given $X \in \mathfrak{X}(M)$ there is a unique collection of linear maps $L_X : \Omega^k(M) \to \Omega^k(M)$ s.t.

$$L_X(f) = X(f), \quad L_X(df) = d L_X(f) = d X(f)$$

and $L_X(\alpha \wedge \beta) = L_X \alpha \wedge \beta + \alpha \wedge L_X \beta$ for $\alpha \in \Omega^k(M)$, $\beta \in \Omega^l(M)$

Pf. Similarly show that L_X is local ($L_X\alpha|_U = 0$ for $\alpha|_U = 0$)

$L_X\alpha$ depends only on $\alpha|_U$, $X|_U$.

and it is determined by its action on f's and diff's df

$L_X f = X(f)$, define it as $L_X(df)$ as $dX(f)$.

Rm Existence is based on the Cartan f'la: $L_X = di_X + i_X d$ 13

Then $L_X df = (di_X df + i_X \underbrace{ddf}_{0}) = dL_X f = dX(f)$

Ex. $X = e^x \dfrac{\partial}{\partial y} + (y^3 + \sin x)\dfrac{\partial}{\partial z}$ $\quad \alpha = (x^2 + y^2 + z^2)\, dx \wedge dz$

$L_X\alpha = (i_X d + di_X)\alpha = i_X(-2y\, dy \wedge dx \wedge dz)$

$+ d(-(y^3 + \sin x)(x^2+y^2+z^2)\, dx)$

$= -2ye^x\, dx \wedge dz + (y^3 + \sin x)(-2y)\, dy \wedge dx$

$(-3y^2(x^2+y^2+z^2) - (y^3 + \sin x)2y)\, dy \wedge dx + (-2z(y^3 + \sin x))dz \wedge dx$

Rm $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$

$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^k \alpha \wedge i_X\beta$ odd deriv's

$L_X(\alpha \wedge \beta) = L_X\alpha \wedge \beta + \alpha \wedge L_X\beta$ - even deriv'n.

$d \circ d = 0$, Cartan f'la $L_X = i_X \circ d + d \circ i_X$

More relations: $L_X L_Y - L_Y L_X = L_{[X,Y]}$

$L_X i_Y - i_Y L_X = i_{[X,Y]}$

Totals: $[d,d] = 0$

graded commutator → $[L_X, L_Y] = L_{[X,Y]}$ | the Cartan calculus

$[i_X, i_Y] = 0$

even. even ⇝ + odd $[d, L_X] = 0$

odd, odd ⇝ + $[L_X, i_Y] = i_{[X,Y]}$

$[d, i_X] = L_X$

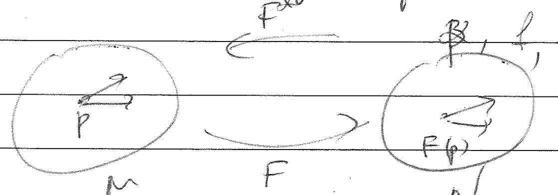
~~Cor-exer.~~ $d\alpha(X,Y) = L_X\alpha(Y) - L_Y\alpha(X) - \alpha([X,Y])$

~~Indeed,~~ $\underbrace{i_Y i_X d\alpha}_{=} = L_X i_Y - L_Y i_X - i_{[X,Y]} \quad \alpha$

$i_Y(L_X\alpha - di_X\alpha) \qquad -d\cancel{i_Y i_X \alpha} \overset{=0 \text{ since } \alpha \text{ 1-form}}{} \quad = -\cancel{L_X i_Y} + L_Y L_X$

$- i_Y d i_X \qquad = \qquad - i_Y d i_X \alpha$

Pull-back for forms: $F^*: \Omega^k(N) \to \Omega^k(M)$



$\beta, f,$ By def

$F^*\beta(v_1, \dots v_k) = \beta_{F(p)}(T_pF(v_1) \dots T_pF(v_k))$

Pull back respects $\wedge$, d: $F^*(\beta_1 \wedge \beta_2) = F^*\beta_1 \wedge F^*\beta_2$

~~In local coord's:~~

$U \xrightarrow{F} V$ $\qquad F^*\beta = \beta(F(x))\, dF_1(x) \wedge \dots \wedge dF_k(x)$ 4

$\{x\} \quad \{y\}$ $\qquad \beta = \beta(y)\, dy_1 \dots \wedge dy_k$

$y = F(x)$

Ex. $F: \mathbb{R}^3 \to \mathbb{R}^2$ $\qquad F^*(e^u du \wedge dv) = e^{y^2 z}\, d(y^2 z) \wedge d(xy)$

$(x,y,z) \mapsto (u,v) = (y^2 z, xy)$ $\qquad = e^{y^2 z}(2yz\,dy + y^2 dz) \wedge (x\,dy + y\,dx)$

$= e^{y^2 z}(-2y^2 z\, dx \wedge dy + xy^2\, dy \wedge dz - y^3\, dx \wedge dz)$ 8

Rm. $L_X \alpha := \left.\frac{d}{dt}\right|_{t=0} \varphi_t^* \alpha$. Change of coord's (i.e. φ^*) respects $\wedge$, d,

$\Rightarrow L_X$ respects them. (invar. defined)

Prop. For top degree form $F: U \subset \mathbb{R}^m \to V \subset \mathbb{R}^{n}$, $m=n$ 10

~~$y = F(x)$ $\quad F^*(dy^1 \wedge \dots \wedge dy^m) = \det\left(\frac{\partial F}{\partial x}\right) dx_1 \dots \wedge dx_m$~~

Geom. meaning for $dx_1 \wedge \dots dx_m$ $\qquad \overset{=}{} \sum_{i_1 \dots i_m} \frac{\partial F_1}{\partial x_{i_1}} \dots \frac{\partial F_m}{\partial x_{i_m}} \underbrace{dx_{i_1} \dots dx_{i_m}}_{=\ \operatorname{sign} \sigma\, dx_1 \dots dx_n}$

= volume of parall. spanned by $X_1, \dots X_m$ $\qquad \det \frac{\partial F}{\partial x}$