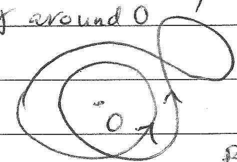
Ex Winding #: $\omega \in \Omega^1(\mathbb{R}^2 \setminus 0)$ Exer (HW) ω is closed $d\omega = 0$

$$\omega = \frac{1}{x^2+y^2}(x\,dy - y\,dx)$$

One can show that in polar coord's $x = r\cos\theta$, $y = r\sin\theta$

$\omega = d\theta$ (but ω is not exact, as θ is not well defined on $\mathbb{R}^2 \setminus 0$)

winding # of γ around 0: $W(\gamma) = \frac{1}{2\pi}\int_\gamma \omega$ — does not change under deform's of γ in $\mathbb{R}^2 \setminus 0$



Before we defined $w(\gamma)$ as the degree of the map γ to S^1 around 0

We have $\frac{d\theta}{2\pi}$ on S^1, then $\int_\gamma \omega = \int_{S^1} i_\gamma^* d\theta = \underbrace{(\deg \gamma)}_{W(\gamma)} \cdot \int_{S^1} d\theta$

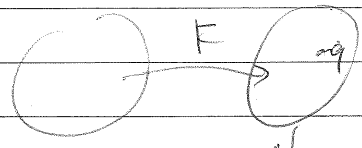
~~"Proof": deform γ to $(\deg\gamma)\cdot S^1$~~ w/o passing 0 $\Rightarrow$ same $\int_\gamma \omega$

Ex. ~~Linking #~~ Degree of F

Let $F : M^m \to N^m$, $\omega \in \Omega^m(N)$, normalize $\int_N \omega = 1$

ω (note: $d\omega = 0$ because of dim)

but $\omega \neq d\alpha$ (since otherwise $\int_N d\alpha = 0$)



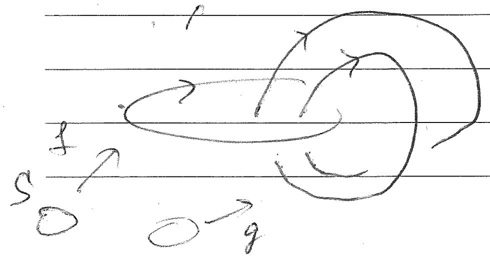
Def. The Degree of $F : M \to N$ is

$$\deg F := \int F^*\omega$$

Exer (hard). Show that it is $= \deg F = \sum_{p \in F^{-1}(q)} \operatorname{sign}(\det dF(p))$

Hint: Change ω by adding an exact form s.t. $\rightsquigarrow \omega_0$ with support near q.

Ex Linking #: $f, g : S^1 \to \mathbb{R}^3$, images do not intersect.



$$F : S^1 \times S^1 \to S^2$$

$$(z, w) \mapsto \frac{f(z) - g(w)}{\|f(z) - g(w)\|}$$

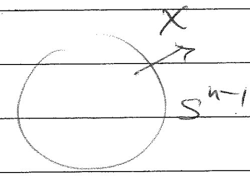
$lk(f,g) = \deg F = \int_{T^2} \frac{1}{4\pi} F^*(\underbrace{x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy}_{\omega}) \in \Omega^2$

$= \int_{T^2} F^*\omega \qquad T^2 = S^1\times S^1 \qquad \Omega^2(\mathbb{R}^3)$

Note $\int_{S^2}\omega = 1$. $\quad \omega = i_X\, dx\wedge dy\wedge dz$ where $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ 16

Rm. A top degree diff form $\Gamma \in \Omega^m(M)$ is called a volume form, if $\Gamma_p \neq 0 \quad \forall p\in M$.

Ex. 1) $\Gamma_0 = dx_1\wedge\ldots\wedge dx_m$ in $\mathbb{R}^m$.



2) $X = \sum x_i \frac{\partial}{\partial x_i}$ Then $\Gamma_{S^{n-1}} := i_X\Gamma_0$ defines a volume form on S^{n-1}. 19

Prop. A mfd M is orientable iff it admits a volume form. Any two vol. forms differ by mult. by a f'n: $\Gamma_1 = f\Gamma_2$, $f>0$ on M.

For a cpt mfd $\mathrm{vol}(M) = \int_M \Gamma$

Rm. Moser thm* $\forall\, \Gamma_1,\Gamma_2$ vol. forms on a cpt M s.t. $\int_M\Gamma_1 = \int_M\Gamma_2$ $\exists\, \varphi: M \circlearrowleft$ diffeo s.t. $\Gamma_1 = \varphi^*\Gamma_2$

Optimal mass transport:

~~Find~~ Given Γ_1, Γ_2 s.t. $\int_M\Gamma_1 = \int_M\Gamma_2$

find "optimal" φ, s.t. $\mathrm{cost}^2(\Gamma_1,\Gamma_2) = \inf_{\substack{\varphi \\ \varphi^*\Gamma_1 = \Gamma_2}} \int_M \mathrm{dist}^2(x,\varphi(x))\,\Gamma_1$ 24

"Pf ~~sketch~~": • volume form defines orient'n

• conversely, if M is oriented, take oriented atlas $(U_\alpha,\varphi_\alpha)$, take a volume form Γ_α in one chart, trans. maps have pos. Jac

$\Gamma_\alpha = \varphi_\alpha^*(dx_1\wedge\ldots\wedge dx_m) \in \Omega^m(U_\alpha) \Rightarrow \Gamma = \sum \chi_\alpha\Gamma_\alpha \in \Omega^m(M)$ – well-defined m-form on M

Rm. Volume forms are always closed: ~~for~~ $d\Gamma \in \Omega^{m+1}(M) = 0$

Cannot be exact for a cpt mfd: if $\Gamma = d\omega$, $\int_M \Gamma = \int_M d\omega = \int_{\partial M}\omega = 0$ but $\int_M \Gamma = \mathrm{vol}(M) \gtrless 0$