MAT 367S – Midterm #2, March 26, 2020
Solutions

Problem #1:[5+3=8 points]

a) Find the integral $\int_\gamma \alpha$ of

$$\alpha = e^{x+2y}(dx + 2dy) \in \Omega^1(\mathbb{R}^2)$$

along the path

$$\gamma\colon [0,1] \to \mathbb{R}^2, \quad \gamma(t) = (-3t, \cos \pi t).$$

b) Is α closed? exact? Explain **briefly**.

Solution: a)

$$\alpha = e^{x+2y}(dx + 2dy) = d(e^{x+2y}).$$

So

$$\int_\gamma \alpha = \int_\gamma d(e^{x+2y}) = e^{-3t+2\cos \pi t}|_0^1 = e^{-5} - e^2.$$

b) The form α is exact, and hence it is closed.

Problem #2: [3+3+2=8 points]

a) Show that

$$\Phi_t\Big(x, y\Big) = \Big(e^{-t}x,\, \ln(e^y + t)\Big)$$

is the flow of a vector field X on plane $\mathbb{R}^2$.

b) Find the vector field X on $\mathbb{R}^2$ having the flow $\Phi_t(x, y)$ from part a).

c) Is this vector field complete? Explain **briefly**.

Solution: a) We have $\Phi_0(x) = \Big(1x, \ln(e^y + 0)\Big) = \Big(x, y\Big)$, i.e. $\Phi_0 = id$ and

$$\Phi_{t_2} \circ \Phi_{t_1}\Big(x, y\Big) = \Phi_{t_2}\Big(e^{t_1}x, \ln(e^y + t_1)\Big) = \Big(e^{t_2}e^{t_1}x, \ln(e^{\ln(e^y+t_1)} + t_2)\Big) =$$
$$= \Big(e^{(t_1+t_2)}x, \ln(e^y + (t_1 + t_2))\Big) = \Phi_{t_1+t_2}\Big(x, y\Big).$$

hence Φ_t is a flow.

b) $X := \frac{d}{dt}|_{t=0}\Phi_t(x, y) = \Big(-e^{-t}x, \frac{1}{e^y+t}\Big)|_{t=0} = (-x, \frac{1}{e^y}) = (-x, e^{-y})$.

c) The y-component $y(t)$ of the solution $\big(x(t), y(t)\big)$ with initial condition $\big(x(0), y(0)\big) = \big(x_0, y_0\big)$ is $y(t) = \ln(e^{y_0} + t)$ and it does not exist for $t < -e^{y_0}$. Hence the solution is not defined for all $t \in \mathbb{R}$, and the field X is incomplete.

Problem #3: [4+4=8 points]

Consider the coordinate transformation, for $x > 0,\ y > 1$,

$$(u, v) = F(x, y) = (xe^y,\ 2xy)$$

a) Express the differentials $du,\ dv$ in terms of dx and dy (with coefficients that are functions of x, y).

b) Express the coordinate vector fields $\frac{\partial}{\partial u},\ \frac{\partial}{\partial v}$ in terms of $\frac{\partial}{\partial x},\ \ \frac{\partial}{\partial y}$ (with coefficients that are functions of x, y).

Solution: a)

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = e^y dx + xe^y dy$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = 2ydx + 2xdy$$

b)

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = e^y\frac{\partial}{\partial u} + 2y\frac{\partial}{\partial v}$$

$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = xe^y\frac{\partial}{\partial u} + 2x\frac{\partial}{\partial v}$$

Hence

$$x\frac{\partial}{\partial x} - \frac{\partial}{\partial y} = 2x(y-1)\frac{\partial}{\partial v}$$

$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = x(1-y)e^y\frac{\partial}{\partial u}$$

and therefore

$$\frac{\partial}{\partial v} = \frac{1}{2x(y-1)}\left(x\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)$$

$$\frac{\partial}{\partial u} = \frac{1}{x(1-y)e^y}\left(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)$$

Problem #4: [4+4=8 points]

Compute the Lie brackets $[X, Y]$ and $[[X, Y], Y]$ of the following two vector fields on $\mathbb{R}^3$.

$$X = x\frac{\partial}{\partial y} - z\frac{\partial}{\partial x}, \quad Y = x\frac{\partial}{\partial z} - y\frac{\partial}{\partial x}.$$

Solution:

$$[X, Y] = y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}.$$

$$[[X, Y], Y] = x\frac{\partial}{\partial z} - y\frac{\partial}{\partial x} = Y.$$

Problem #5: [8 (+4 bonus) points]

a) [8 points] Let $S \subset M$ be a submanifold. A vector field $X \in \mathfrak{X}(M)$ is said to *vanish along* S if $X_p = 0$ for all $p \in S$.

Show that if $X, Y \in \mathfrak{X}(M)$ are two vector fields such that X vanishes along S, and Y is tangent to S, then $[X, Y]$ vanishes along S.

b) (Bonus problem [4 points]) Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$. Express

$$[fX, gY] \in \mathfrak{X}(M)$$

as linear combination of X, Y, $[X, Y]$ with coefficients in $C^\infty(M)$.

Solution:

a) Since both fields X and Y are tangent to S their commutator $[X, Y]$ is also tangent to S and $[X, Y]|_S = [X|_S, Y|_S] = [0, Y|_S] = 0$.

b)

$$\begin{aligned}[fX, gY] &= (fX)(g)Y + g[fX, Y] = fX(g)Y - g[Y, fX] = \\ &= fX(g)Y - g(Y(f)X + f[Y, X]) = fX(g)Y - gY(f)X + fg[X, Y].\end{aligned}$$