MAT 367S – Midterm #2, March 29, 2019
Solutions

Problem #1:[5+1=6 points]

a) Find the integral $\int_\gamma \alpha$ of

$$\alpha = \mathrm{d}z + y\mathrm{d}x \in \Omega^1(\mathbb{R}^3)$$

along the path

$$\gamma\colon [0, 2\pi] \to \mathbb{R}^3, \quad \gamma(t) = (\cos t,\, \sin t,\, t^2).$$

b) Is α closed? exact? Explain **briefly**.

Solution: a)

$$\int_\gamma \alpha = \int_0^{2\pi} \mathrm{d}(t^2) + \sin t\ \mathrm{d}(\cos t) = \int_0^{2\pi} \mathrm{d}(t^2) - \sin^2 t\ \mathrm{d}t = t^2|_0^{2\pi} - \frac{1}{2}\cdot 2\pi = (2\pi)^2 - \pi\,.$$

b) $\mathrm{d}\alpha = \mathrm{d}y \wedge \mathrm{d}x \neq 0$, so α is not closed, and hence not exact.

Problem #2: [3+2+2=7 points]

a) Show that the function

$$\Phi(t, x) = \left(\sqrt[3]{x} + t\right)^3,$$

is the flow $\Phi_t(x) = \Phi(t, x)$ of a vector field on $\{x|\ x \neq 0\} \subset \mathbb{R}$.

b) Find the vector field on $\{x|\ x \neq 0\} \subset \mathbb{R}$ having the flow $\Phi_t(x)$ from part a).

c) Is this vector field complete? Explain **briefly**.

Solution: a) We have $\Phi_0(x) = (\sqrt[3]{x} + 0)^3 = x$, i.e. $\Phi_0 = id$ and

$$\Phi_{t_2} \circ \Phi_{t_1}(x) = \Phi_{t_2}\big((\sqrt[3]{x} + t_1)^3\big) = \Big(\sqrt[3]{(\sqrt[3]{x} + t_1)^3} + t_2\Big)^3 = \big(\sqrt[3]{x} + t_1 + t_2\big)^3 = \Phi_{t_1+t_2}(x)$$

hence Φ_t is a flow.

b) $X := \frac{d}{dt}|_{t=0}\Phi_t(x) = 3(\sqrt[3]{x} + t)^2|_{t=0} = 3x^{3/2}\,.$

c) The solution $x(t)$ with initial condition $x(0) = x_0$ becomes zero at $t = -\sqrt[3]{x_0}$ and hence leaves the domain $\{x|\ x \neq 0\} \subset \mathbb{R}$. Hence it does not exists for all $t \in \mathbb{R}$.

(Actually, if we extend the domain to $x \in \mathbb{R}$, then the vector field $X(x)$ is not smooth at $x = 0$, and $\dot{x} = 3x^{3/2}$ is an example of a differential equation with non-unique solutions: e.g. for $x(0) = 0$ one has solutions $x(t) = 0$ for all t and $x(t) = t^3$.)

Problem #3: [6 points]
Consider the following coordinate transformation on $\mathbb{R}^2$,

$$u = -2x - 7y, \quad v = x + 5y.$$

Express the coordinate vector fields

$$\partial/\partial x, \ \partial/\partial y$$

for the (x, y) coordinates in terms of the coordinate vector fields

$$\partial/\partial u, \ \partial/\partial v$$

for the (u, v) coordinates.

Solution:

$$\frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} = -2\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$$
$$\frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v} = -7\frac{\partial}{\partial u} + 5\frac{\partial}{\partial v}$$

Problem #4: [6 points] Consider the following vector fields on $\mathbb{R}^2 \setminus \{0\}$,

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Find a 1-form $\alpha \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ such that

$$\alpha(X) = 0, \quad \alpha(Y) = -1.$$

Solution:

Look for α in the form $\alpha = f \ \mathrm{d}x + g \ \mathrm{d}y$, with unknown functions f, g on $\mathbb{R}^2 \setminus \{0\}$. Then they satisfy the system of linear equations

$$-yf + xg = 0$$

$$xf + yg = -1$$

Then e.g. by Cramer's rule $f = \frac{-x}{x^2+y^2}$ and $g = \frac{-y}{x^2+y^2}$, i.e.

$$\alpha = \frac{-1}{x^2 + y^2}(x \ \mathrm{d}x + y \ \mathrm{d}y).$$

Problem #5: [4+3=7 points]
a) Compute the Lie brackets $[X, Y]$, $[[X, Y], Y]$, and $[[[X, Y], Y], Y]$ of the following two vector fields on $\mathbb{R}^3$.

$$X = \frac{\partial}{\partial z} + y^3\frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y}.$$

b) Explain **briefly** if there can be a 2-dimensional submanifold $S \subset \mathbb{R}^3$ such that X, Y are everywhere tangent to S.

Solution: a)

$$[X, Y] = -3y^2 \frac{\partial}{\partial x}, \quad [[X, Y], Y] = 6y \frac{\partial}{\partial x}, \quad [[[X, Y], Y], Y] = -6 \frac{\partial}{\partial x}.$$

b) By contradiction: if there is such a submanifold S so that X and Y are tangent to it, then so are all their brackets. But the fields X, Y, and $Z = [[[X, Y], Y], Y]$ are linearly independent at all points of $\mathbb{R}^3$, hence such a surface cannot exist.