The Pentagram map and its contin limit

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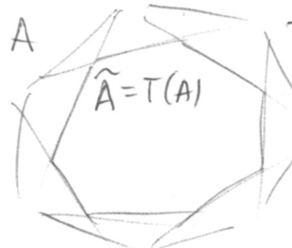
(1)

I. 2D pentagram map

A – convex n-gon in $\mathbb{RP}^2$ $A = (v_i) \in \mathbb{RP}^2$
$i \in \mathbb{Z}$, n-periodic

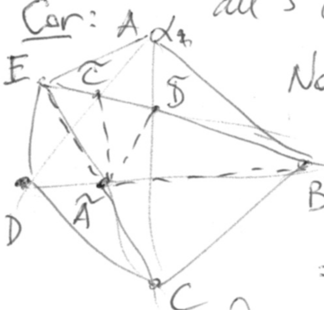


$T : A \mapsto \widehat{A} = T(A)$ mod proj. equiv
pentagram map R. Schwartz 1992

$n=5$ $T = id$
$n=6$ $T^2 = id$
$n \geq 7$ T – quasiperiodic
$\overset{?}{\Rightarrow}$ integrability?

- 2010–2019: invar. Poiss. str's, I^{st} int'ls, Lax form, contin. limit, higher dim, refactorization, clusters

(OST, GSTV, KS, MB, FM, Iz)



Cor: all 5 cross ratios same $\Rightarrow$ 5-gons are equiv.

Note $\alpha(4 \text{ lines}) = \alpha(E, \widetilde{C}, \widetilde{D}, B)$
$= \widetilde{\alpha}(4 \text{ new lines})$
$\Rightarrow$ all cross-ratios are T-invar.
$\Rightarrow T = id$ for $n=5$

Q. $T^2 = id$ for $n=6$?

II. Contin. limit. Consider $n \to \infty$

A – n-gon $(v_i) \in \mathbb{RP}^2$ $\rightsquigarrow$ smooth (param) curve $\gamma(x) \in \mathbb{RP}^2$
loc. convex γ – nondeg ($\gamma' \nparallel \gamma''$)

diag. (v_{i-1}, v_{i+1}) $\rightsquigarrow$ fix $\varepsilon > 0$, chord $(\gamma(x-\varepsilon), \gamma(x+\varepsilon))$
polygon $\widehat{A}$ $\rightarrow$ envelope $\ell_\varepsilon(x)$

Ex
- $m=3$ $L = \partial^3 + u(x)\partial + v(x)$, $m=3$
- $(2,3)$ KdV is

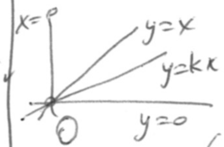
$\partial_t L = [\partial^2 + \frac{2}{3}u(x), L] \Leftrightarrow \begin{matrix}\partial_t u = \dots \\ \partial_t v = \dots\end{matrix} \Leftrightarrow \partial_{tt}^2 u + (u^2)'' + u'''' = 0$
Boussinesq

- $m=2$, $L = \partial^2 + u$,
$(3,2)$-KdV ~~is the~~ standard KdV $\partial_t u + u u' + u''' = 0$

Now in $\mathbb{RP}^d$: $\gamma \subset \mathbb{RP}^d$, nondeg. curve
$(\gamma', \dots, \gamma^{(d)})$ – lin. indep $\forall x$

Fix $\varepsilon > 0$. Take $k_1 < \dots < k_d$, $\sum k_j = 0$
$P_\varepsilon(x) = (\gamma(x + k_1\varepsilon), \dots, \gamma(x + k_d\varepsilon))$
hyperplane $\ell_\varepsilon(x)$ – envelope.
$\ell_\varepsilon(x) = \gamma(x) + \varepsilon^2 b_\gamma(x) + O(\varepsilon^3)$

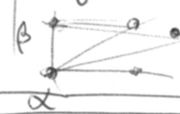
Rm. Why $T = id$ for $n=5$
$PGL_2 \simeq SL_2$ – invar. of 4 lines in $\mathbb{R}^2 \simeq$ inv. 4 pts in $\mathbb{RP}^2$



k – invar. Cross-ratio of 4 pts:
$$(z_1, z_2, z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \simeq k$$
6 possible ways: k, $1-k$, $1 - \frac{1}{k}$ & recipr.

For lines in $\mathbb{R}^3 \simeq$ pts in $\mathbb{RP}^2$, invar's of PGL_3-ac
$\dim PGL_3 = \dim SL_3 = 8$
$\dim \mathcal{P}_5 = 10 - 8 = 2$
5-gons mod. equiv; 5-gons on $\mathbb{R}^2$; $\dim SL_3$

5-gon mod. proj. equiv $\simeq$ fix 2 cross-ratios



4 pts fixed
1 pt free

Expand $\ell_\varepsilon(x) = \gamma(x) + \varepsilon^2 b_\gamma(x) + O(\varepsilon^4)$

Thm (OST 2010) The eq'n $\partial_t \gamma(x) = b_\gamma(x)$ is equiv to the Boussinesq eq'n
$\partial_{tt}^2 u + (u^2)'' + u'''' = 0 \Leftrightarrow (2,3)$ KdV eq'n

Rm on KdV hierarchy. Let $\partial := \frac{\partial}{\partial x}$
$L = \partial^m + u_{m-2}(x)\partial^{m-2} + \dots + u_0(x)$, $u_j \in C^\infty(S^1)$
$\exists Q := L^{1/m} = \partial + \sum_{j=1}^{\infty} a_j(x)\partial^{-j}$, $\partial f = f\partial + f'$ Leibniz

Take $Q_+^k = \underbrace{\partial^k + \dots + P_0(u, x)\partial^0}_{} + \dots$
E.g. $Q_+^2 = \partial^2 + \frac{2}{m} u_{m-2}(x)$

Def. (k,m)-KdV eq'n: $\partial_t L = [Q_+^k, L]$

Thm (K.–Soloviev 2012) The eq'n $\partial_t \gamma(x) = b_\gamma(x)$ is equiv to $(2, d+1)$-KdV eq'n $\forall k_j$

Rm. How to associate $L \rightsquigarrow \gamma$ (mod. proj. equiv)
Take $(d+1)^{th}$ order LDE $L\psi = 0$ on S^1
FSS $\Psi : \mathbb{R} \to \mathbb{R}^{d+1}$ $x \mapsto (\psi_1(x), \dots, \psi_{d+1}(x))$
$\gamma : \mathbb{R} \to \mathbb{RP}^d$, $x \mapsto \gamma = (\psi_1(x) : \dots : \psi_{d+1}(x))$
Wronskian $(\psi_1, \dots, \psi_{d+1}) = const \neq 0 \Leftrightarrow \gamma$ – nondeg. $(\gamma', \dots, \gamma^{(d)})$ lin indep.

Rm. If $\sum k_j \neq 0$, get $(1, d+1)$ KdV
Q How to visualize other $(k, d+1)$ KdV?

III. Lax form ($\approx$ integrability)

If a dynam. system $\leftrightarrows$ $\dot L = [A, L]$

then $L(t) = U(t) L_0 U^{-1}(t)$ for $A = \dot U U^{-1}(t)$

Indeed, $\dot L = \dot U L_0 U^{-1} + U L_0 \underbrace{\dot{U}^{-1}}_{= -U^{-1}\dot U U^{-1}}$

$= \dot U U^{-1} L - L \dot U U^{-1} = [A, L]$.

Cor. spec $L = \{\lambda_1, \dots \lambda_n\}$ is invar., first int.

Not enough? Make $L = L(s)$ $\approx$ spec. param.

$\lambda_j(s)$ - invar $\Rightarrow$ spectral curve

$\Sigma = \{(s, \lambda) \mid \det(L(s) - \lambda \cdot 1) = 0\}$,

coef's are first int'ls

Rm. Zero curvature eq'n

$\left[\frac{\partial}{\partial t} + A, \frac{\partial}{\partial x} + L\right] = 0 \iff \partial_t L - \partial_x A = [L, A]$

Rm KdV eq's have Lax form $\Rightarrow$ integrability

Pf idea: $\gamma \subset \mathbb{RP}^d \rightsquigarrow \Gamma \subset \mathbb{R}^{d+1}$

$\det(\Gamma, \Gamma', \dots, \Gamma^{(d)}) = 1$

Compute: $\Gamma_\varepsilon(x) = \Gamma + \varepsilon^2 Q_+^2 \Gamma + (\varepsilon^3)$

Then $L_\varepsilon \Gamma_\varepsilon = 0 \xRightarrow{d/d\varepsilon} \dot L \Gamma + L \dot \Gamma = 0$

$\dot L = [Q_+^2, L]$ consistent with $'' L Q_+^2 \Gamma$, since

$(Q_2 \overset{0}{L} - L Q_2)\Gamma + L Q_2 \Gamma = 0$ $\boxed{}$

Diff oper of order $< m$.