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Averaging, symplectic reduction, and central extensions

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Abstract

We show that the averaged equation for a one-frequency fast-oscillating Hamiltonian system is the result of symplectic reduction of a certain natural system on the corresponding S^1-bundle with respect to the circle action. Furthermore, if the reduced configuration space happens to be a group, then under natural assumptions the averaged system turns out to be the Euler equation on a central extension of that group. This gives a new explanation of the drift, common in averaged system, as a similar shift is typically present in symplectic reductions and central extensions.

Contents

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1 Introduction

Dynamical systems with fast-oscillating conditions are ubiquitous in physics: they naturally arise in mechanics, astrophysics, fluid and air dynamics, and many other domains. They often exhibit surprising properties, a beautiful example of which is the inverted pendulum, which stabilizes via fast vibration of its pivot. The standard way of analyzing such equations includes a procedure of constructing an averaged system, whose solutions remain close to those of the original system for very long time (see e.g. [7, 10, 1, 2]).

In many examples of Hamiltonian one-frequency oscillating systems one obtains an additional term, a drift in the averaged equation. A similar drift (or shift) is observed in hydrodynamical-type systems, including the β-plane equation in meteorology (see e.g. [6]), infinite-conductivity equation for electron flows [8], and the Craik-Leibovich equation for an ideal fluid confined to a domain with oscillating boundary [5, 14]. In those hydrodynamical systems such a shift is often related to the consideration of a central extension of an appropriate Lie algebra [13].

Below we explain this phenomenon by building a general connection between the averaging method and symplectic reduction in appropriate, possibly nontrivial, S^1-bundles. Namely, one starts with symplectic reduction of the cotangent bundle over a circle action, which is one of the most studied objects in symplectic geometry. One observes two features for the reduction over a nonzero value of the momentum map: the appearance of a twisted symplectic structure (similar to how the curvature arises in the description of gyroscopes on surfaces [4]), where a new magnetic term supplements the canonical symplectic form of the reduced space, and the appearance of an amended potential function, see Section 2. It turns out that exactly these two phenomena occur in the averaging procedure. This can be summarized in the following statement (which is a combined version of Theorems 3.6 and 5.4):

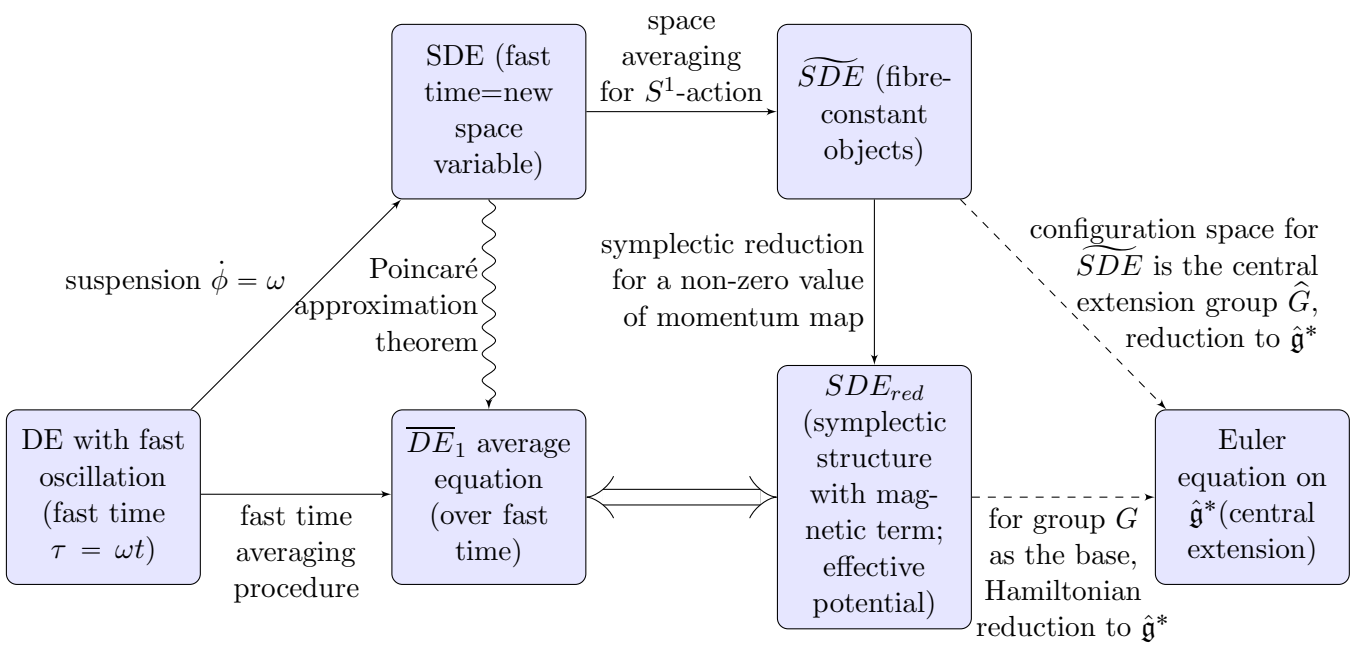

Figure 1: DIAGRAM

Theorem 1.1. *For a natural slow-fast Hamiltonian system the resulting slow (averaged) system coincides with the one obtained by space averaging over the fibers of an appropriate S^1-bundle and performing the symplectic reduction of the corresponding cotangent bundle over S^1-action at the momentum value related to the fast frequency. The averaged system turns out to be a natural Hamiltonian system with an amended potential function with respect to a twisted (magnetic) symplectic structure.*

Furthermore, central extensions appear whenever the base of the reduction turns out to be a group by itself, as discussed in Section 4. This can be regarded as a manifestation of the reduction by stages developed in [11]. The second main result of the paper is the following abbreviated version of Theorem 4.3:

Theorem 1.2. *If the slow manifold is a group G and the perturbed Hamiltonian system is invariant relative to the G-action, then the second reduction of such a fast oscillating system gives an Euler equation, Hamiltonian with respect to the Poisson-Lie bracket on a central extension $\widehat{\mathfrak{g}}$ of the corresponding Lie algebra $\mathfrak{g}$.*

The essence of the paper is described in the diagram in Figure 1: we show how to view the fast time averaging approximation on the left by going via the averaging on the top and reduction in the right column of the diagram. We describe this averaging-reduction procedure in Section 3, and compare its result with the one obtained by using the classical fast-time averaging method in Section 5.

In Section 6 we describe three examples by using the averaging-reduction procedure developed in this paper: the vibrating pendulum manifests the appearance of the amended potential, the Craik-Leibovich equation for oscillating boundary is related to the magnetic term in the symplectic structure and a central extension, while the motion of particles in rapidly oscillating potentials has both magnetic term and additional potential present upon averaging. (Note that, instead of the classical approach of applying ingenious canonical transformations [3], the present paper gives an alternative method of averaging natural Hamiltonian systems: one can average the metric, which contains all relevant information, and then obtain the averaged natural system directly from that metric.)

While the symplectic reduction part of this paper is also valid for high-dimensional torus action, i.e. for many-frequency case, the approximation theorem does not work in this generality, as for several frequencies resonances can appear unavoidably in such systems, as e.g. KAM theory manifests. Note also that in many examples the two contributions appearing in averaging, the magnetic term and the potential amendment, are of different order in the small parameter of perturbation. It would be interesting to see if it is always the case.

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2 Symplectic reduction of cotangent bundles

We start by recalling (following [11]) general results on the symplectic reduction. Consider an action of an abelian group $\mathbb{T} := S^1$ (or more generally, a torus $\mathbb{T} = T^k$) common in averaging, while the results with appropriate amendments hold for a reduction by any Lie group. Assume

that the group $\mathbb{T}$ acts on a configuration space Q (from the right) properly and freely, so that the quotient space $Q/\mathbb{T}$ is a manifold. Our first goal is to reduce Q by the $\mathbb{T}$-action and describe structures on the reduced phase space. The quotient projection $\pi : Q \to B := Q/\mathbb{T}$ defines a principal fiber bundle over the base B. It turns out that the curvature of this $\mathbb{T}$-bundle enters the symplectic structure of the reduced manifold. The gyroscope example below can be regarded as an illustration of the abstract reduction procedure.

Namely, the group $\mathbb{T}$ acts on T^*Q by cotangent lifts, and we denote the momentum map of this action by $J : T^*Q \to \mathfrak{t}^*$. The momentum map is a natural projection of T_q^*Q at any $q \in Q$ to the cotangent space to the fiber, $\mathfrak{t}^*$. For an arbitrary value $\mu \in \mathfrak{t}^*$ of the momentum map consider the reduced phase space[1] $(T^*Q)_\mu := J^{-1}(\mu)/\mathbb{T}$.

Theorem 2.1. (see e.g. [11]) *Let $\mathbb{T}$ be an abelian group acting on a manifold Q so that $\pi : Q \to Q/\mathbb{T} =: B$ is a principal fiber bundle, and fix $\mu \in \mathfrak{t}^*$. Let $\mathcal{A} : TQ \to \mathfrak{t}$ be a principle connection 1-form on this bundle. Then*

i) for $\mu = 0$ there is a symplectic diffeomorphism between $(T^*Q)_0$ and $T^*B = T^*(Q/\mathbb{T})$ equipped with the canonical symplectic form ω_{can};

ii) for $\mu \neq 0$ there is a symplectic diffeomorphism between $(T^*Q)_\mu$ and T^*B, where the latter is equipped with symplectic form $\omega_\mu := \omega_{can} - \beta_\mu$. Here the 2-form $\beta_\mu := \pi_P^*\sigma_\mu$ on T^*B is obtained by the pull-back via the cotangent bundle projection $\pi_P : T^*B \to B$ from the 2-form σ_μ on B. The latter 2-form is the μ-component of the curvature of the principal fiber bundle Q over B, namely $\pi^*\sigma_\mu = \mathbf{d}\langle \mu, \mathcal{A}\rangle$.

Proof outline. We just recall an explicit form of the isomorphism between $(T^*Q)_\mu$ and $T^*(Q/\mathbb{T})$, see Theorem 2.3.3 in [11] for more detail.

The isomorphism $\varphi_0 : (T^*Q)_0 \to T^*(Q/\mathbb{T})$ is defined by noting that

$$J^{-1}(0) = \{p_q \in T^*Q : \langle p_q, \xi_Q(q)\rangle = 0 \quad \text{for all } \xi \in \mathfrak{t}\},$$

where ξ_Q is the vector field on Q corresponding to the infinitesimal action ξ, i.e. vectors $\xi_Q(q)$ span the vertical subspace at q. Thus the map $\Phi : J^{-1}(0) \to T^*(Q/\mathbb{T})$ given by

$$\langle \Phi(p_q), \pi_*(v_q)\rangle = \langle p_q, v_q\rangle \tag{1}$$

is well defined. The map Φ is $\mathbb{T}$-invariant and surjective, and hence induces a quotient map $\varphi_0 : (T^*Q)_0 \to T^*(Q/\mathbb{T})$.

The isomorphism $\varphi_\mu : (T^*Q)_\mu \to T^*(Q/\mathbb{T})$ is the composition $\varphi_\mu = \varphi_0 \circ \mathrm{shift}_\mu$ of φ_0 with the isomorphism $\mathrm{shift}_\mu : (T^*Q)_\mu \to (T^*Q)_0$ defined as follows. Introduce a map $\mathrm{Shift}_\mu : J^{-1}(\mu) \to J^{-1}(0)$ by

$$\mathrm{Shift}_\mu(p_q) = p_q - \langle \mu, \mathcal{A}(q)\rangle$$

for any $p_q \in J^{-1}(\mu)$. It is $\mathbb{T}$-invariant, so it drops to a quotient map $\mathrm{shift}_\mu : (T^*Q)_\mu \to (T^*Q)_0$. The $\mathfrak{t}$-valued 2-form $\mathbf{d}\mathcal{A}$ is the curvature of the (abelian) connection $\mathcal{A}$, while to construct the 2-form σ_μ one uses its μ-component, cf. [11]. □

Remark 2.2. The isomorphism between $(T^*Q)_\mu$ and $T^*B = T^*(Q/\mathbb{T})$ is connection-dependent. The reduced symplectic form on T^*B is modified by the curvature 2-form σ_μ on B, which is traditionally called a *magnetic term*, since it also appears in the description of motion of a charged particle in a magnetic field on B.

[1] For an arbitrary Lie group the reduced space is defined as $J^{-1}(\mu)/\mathbb{T}_\mu$ where $\mathbb{T}_\mu$ is the stationary subgroup of μ. In this section we use the fact that $\mathbb{T}$ is abelian, and hence the stationary group $\mathbb{T}_\mu$ coincides with the full group: $\mathbb{T}_\mu = \mathbb{T}$.

Definition 2.3. Let the space Q be equipped with a $\mathbb{T}$-invariant metric. This metric defines an invariant distribution of horizontal spaces: at each point $q \in Q$ there is a subspace of T_qQ orthogonal to the fiber (i.e. the $\mathbb{T}$-orbit) at q. Hence the metric defines an invariant connection 1-form $\mathcal{A} : TQ \to \mathfrak{t}$ on this fiber bundle. This 1-form is called a *mechanical connection.*

Consider a natural system on T^*Q with Hamiltonian $H(q,p) = (1/2)(p,\, p)_q + U(q)$ invariant with respect to the $\mathbb{T}$-action. (Here and below $(.\,,.)_q$ stands for the metric on Q, i.e. the inner product on TQ, or the induced one on T^*_qQ, depending on the context. The Euclidean inner product in $\mathbb{R}^n$ is denoted by dot.) This system descends to a Hamiltonian system on the quotient $(T^*Q)_\mu$ with respect to the symplectic structure $\omega_\mu = \omega_{can} - \beta_\mu$. The new Hamiltonian H_μ is obtained from H by applying the map Shift_μ and the corresponding potential $U(q)$ acquires an additional term, as we discuss below.

Example 2.4. The following example of a spinning disk (a gyroscope) on a curved surface shed the light on the geometry behind the symplectic reduction above. Cox and Levi proved in [4] that the motion of the disk center coincides with the motion of a charged particle in a magnetic field which is normal to the surface and equal in magnitude to the Gaussian curvature of the surface.

To explain their result in the context of reduction theory, let $q = (q_1, q_2)$ be orthogonal local coordinates on a surface $B \subset \mathbb{R}^3$, so that metric on the surface is given by $ds^2 = a_{11}(q)dq_1^2 + a_{22}(q)dq_2^2$. When the disk is not spinning, its kinetic energy is a function of its position q and linear velocity $\dot{q}$, i.e. a function on TB. It is given by

$$E_0 = \frac{m}{2}(G\dot{q}, \dot{q}) + \frac{\mathbb{I}_d}{2}h(\dot{q}, \dot{q}),$$

where $G = diag(a_{11}, a_{22})$, h is the second fundamental form of B, and $\mathbb{I}_d$ is the moment of inertia of the disk along its diameter. Denote by $\mathbb{I}_a$ the moment of inertia about the disk axis.

Theorem 2.5. [4] *For a spinning disk on a surface B the angular momentum $\mu = \mathbb{I}_a\omega_a$ of the disk about its axis is constant and the disk's center satisfies the following equation*

$$\frac{d}{dt}\frac{\partial E_0}{\partial \dot{q}} - \frac{\partial E_0}{\partial q} = \sqrt{a_{11}a_{22}}\,\mu\, K(q) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \dot{q}, \tag{2}$$

where $K(q)$ is the Gaussian curvature of the surface B.

Remark 2.6. Before we provide a different proof of this result via symplectic reduction, note that the configuration space Q of this system is the circle bundle over the surface: the disk position is defined by the position of its center on the surface B and the angle of rotation. Globally Q may be a nontrivial $\mathbb{T}$-bundle over $B = Q/\mathbb{T}$. However, for the local consideration below it suffices to consider the case $Q = B \times \mathbb{T}$. The phase space is the corresponding tangent bundle TQ. The metric in $\mathbb{R}^3$ allows one to identify TQ and T^*Q, while the disk motion is a Hamiltonian system on the cotangent bundle T^*Q. The trajectory of the disk center can be obtained as the symplectic reduction of the system on T^*Q with respect to the $\mathbb{T}$-action, as we quotient out the disk rotation. Different angular velocities of the disk lead to different values μ of the momentum map, over which one takes the quotient. According to Theorem 2.1 the resulting system is a Hamiltonian system on T^*B, with two amendments. The corresponding symplectic structure after the reduction will be twisted by a magnetic term. In this setting it will be proportional to the curvature $K(q)$ of the surface, which one observes in Equation (2). Moreover, the corresponding Hamiltonian undergoes a shift by μ. However, in the gyroscope

case the shift reduces to adding a constant to the Hamiltonian and does not appear in the equations.[2]

Proof. Let us identify TB and T^*B by means of the metric. First note that Equation (2) is the Euler-Lagrange equation for a Lagrangian system, which can be rewritten as a Hamiltonian system with the Hamiltonian energy function E_0 on the cotangent bundle of the surface B (thanks to the metric identification) with a twist symplectic structure given in local coordinates by

$$\omega_\mu = \omega_{can} - \mu\sqrt{a_{11}a_{22}}\,K(q)\,dq_1 \wedge dq_2\,,$$

where ω_{can} is the canonical symplectic structure on T^*B.

Next, we show how to obtain this system via symplectic reduction. Denote by θ the angle between a fixed radius on the disk and the positive direction of the line $\{q_2 = const\}$. This gives us a principal $\mathbb{T}$-bundle Q with the curved surface B as the base.

The absolute angular velocity of a spinning disk is $\Omega_a = \dot\theta + A(q)\dot q$, where $A(q)\dot q$ is the transferred velocity, and $A(q) = (k_1\sqrt{a_{11}}, k_2\sqrt{a_{22}})$, where $k_1,\ k_2$ are the geodesic curvatures of coordinate lines $\{q_1 = const\}$ and $\{q_2 = const\}$.

So, in local coordinates, the metric on the principal $\mathbb{T}$-bundle Q is given by

$$\Big((\dot q,\dot\theta),(\dot q,\dot\theta)\Big)_{(q,\theta)} = \mathbb{I}_a(\dot\theta + A(q)\dot q)^2 + m(G\dot q,\dot q) + \mathbb{I}_d h(\dot q,\dot q).$$

Note that this metric is invariant under the $\mathbb{T}$-rotations.

Therefore, the momentum map $J : TQ \to \mathfrak{t}^* = \mathbb{R}$ is $J(q,\theta;\dot q,\dot\theta) = \mathbb{I}_a(\dot\theta + A(q)\dot q)$ and the mechanical connection $\mathcal{A} : TQ \to \mathfrak{t} = \mathbb{R}$ is $\mathcal{A} = d\theta + A(q)\,dq$ (here we again identify TQ and T^*Q). By Theorem 2.1, for a fixed value $\mu = \mathbb{I}_a\Omega_a$ of the momentum map J, the system can be reduced to the (co)tangent bundle of the surface B with the magnetic symplectic structure

$$\omega_\mu = \omega_{can} - \mu\, d(A(q)dq) = \omega_{can} - \mu\,\sqrt{a_{11}a_{22}}\,K(q)\,dq_1 \wedge dq_2,$$

where $K(q)$ is the Gaussian curvature of the surface B.

The energy Lagrangian $E = \frac{1}{2}\Big((\dot q,\dot\theta),(\dot q,\dot\theta)\Big)_{(q,\theta)}$ on Q defines the reduced Hamiltonian on B, which turns out to be $E_0 = 1/2(m(G\dot q,\dot q) + \mathbb{I}_d h(\dot q,\dot q))$. Here we omit the constant term $\mathbb{I}_a(\dot\theta + A(q)\dot q)^2 = \langle\Omega_a,\, \mathbb{I}_a\Omega_a\rangle = \langle\mu,\, \mathbb{I}_a^{-1}\mu\rangle$ in the energy expression, since the value μ of the momentum map (i.e. the angular momentum of the disk) is conserved.

This reduced Hamiltonian system with Hamiltonian function E_0 on the cotangent bundle (T^*B,ω_μ) of the surface describes the motion of the disk center. □

3 Averaging-Reduction procedure for a natural system

3.1 Averaging

Let $\pi : Q \to B$ be a principal $\mathbb{T}$-bundle. From now on we assume that $\mathbb{T} = S^1$ (and occasionally comment on $\mathbb{T} = T^n$). The cotangent lift of $\mathbb{T}$-action on Q induces $\mathbb{T}$-action on T^*Q. Denote by ρ, ρ^*, and ρ_* the $\mathbb{T}$-action on Q, T^*Q and TQ, respectively. Let $d\eta$ be the standard Euclidean measure on the group $\mathbb{T}$.

[2] Jumping ahead, in order to see this one can use an explicit formula of Theorem 3.6, which gives an additional term $\frac{1}{2}\langle\mu,\, \mathbb{I}(q)^{-1}\mu\rangle$. It is indeed constant, since in the gyroscope case the inertia operator $\mathbb{I}$ does not depend on q, while μ is a constant angular velocity.

Consider a natural Hamiltonian system on the cotangent bundle T^*Q:

$$H(q,p) = \frac{1}{2}(p,\, p)_q + U(q). \tag{3}$$

Here Q is the configuration space of the motion, we assume that this Hamiltonian system has slow motion on the base B and fast motion on the fibers isomorphic to $\mathbb{T}$. The Hamiltonian function $H(q,p)$ is not necessarily invariant under the $\mathbb{T}$-action on T^*Q. As the first step one passes to the space $\mathbb{T}$-average $\overline{H(q,p)}^{\mathbb{T}}$, the $\mathbb{T}$-invariant function on T^*Q defined by the following formula:

$$\overline{H(q,p)}^{\mathbb{T}} := \frac{1}{\eta(\mathbb{T})}\int_{g\in\mathbb{T}} H(\rho_g^*(q,p))\, d\eta(g).$$

For the natural system (3), one averages both the kinetic and potential parts of the energy:

$$\overline{H(q,p)}^{\mathbb{T}} = \frac{1}{2}\overline{(p,p)}_q^{\mathbb{T}} + \overline{U(q)}^{\mathbb{T}},$$

where $\overline{U(q)}^{\mathbb{T}} = \frac{1}{\eta(\mathbb{T})}\int_{\mathbb{T}} U(\rho_g(q))\, d\eta(g)$ and $\overline{(p,p)}_q^{\mathbb{T}} = \frac{1}{\eta(\mathbb{T})}\int_{\mathbb{T}}(\rho_g^*p, \rho_g^*p)_{\rho_{g^{-1}}(q)}\, d\eta(g)$ is defined via the following averaged metric on Q:

Definition 3.1. The *averaged metric* $\overline{(\cdot,\cdot)}^{\mathbb{T}}$ on the principal $\mathbb{T}$-bundle Q is given by

$$\overline{(v,v)}_q^{\mathbb{T}} := \frac{1}{\eta(\mathbb{T})}\int_{\mathbb{T}}(\rho_{g*}v, \rho_{g*}v)_{\rho_g(q)}\, d\eta(g),$$

for any $v \in T_qQ$. This defines a $\mathbb{T}$-invariant metric on Q.

Now define the connection on Q corresponding to the averaged metric:

Definition 3.2. The *averaged connection* $\bar{\mathcal{A}} \in \Omega^1(Q,\mathfrak{t})$ on the principal $\mathbb{T}$-bundle Q is the connection induced by the averaged metric $\overline{(\,,\,)}_q^{\mathbb{T}}$ by the invariant distribution of horizontal spaces: at each point $q \in Q$ there is a subspace of T_qQ orthogonal to the fiber (i.e. the $\mathbb{T}$-orbit) at q.

The connection induced by an invariant averaged metric on Q is the mechanical connection, according to Definition 2.3.

Remark 3.3. We would like to give a more explicit description of averaged metrics and connections. First note that a $\mathbb{T}$-invariant metric $(.\,,.)_q$ on Q can be defined by means of a metric operator $\mathbb{I}_Q(q) : T_qQ \to T_q^*Q$ for $q \in Q$, where $\mathbb{I}_Q(q) : v \mapsto v^\flat$, i.e. $(v,v)_q := \langle v, \mathbb{I}_Q(q)v\rangle$ for $v \in T_qQ$. This defines the "fiber inertia operator" $\mathbb{I}(q) : \mathfrak{t} \to \mathfrak{t}^*$ by restricting to $\mathfrak{t} = T_q\mathbb{T} \subset T_qQ$ the metric operator $\mathbb{I}_Q(q)$ for $q \in Q$. (Recall, that for $\mathbb{T} = S^1$, we have $\mathfrak{t} = \mathbb{R}$.) The $\mathbb{T}$-invariance of metric implies that the fiber inertia operator $\mathbb{I}$ is equivariant, $\mathbb{I}(g(q)) = Ad^*_{g^{-1}}\mathbb{I}(q)$, i.e. it depends on the base point $\pi(q) \in B = Q/\mathbb{T}$ only.

The invariant metric on TQ also induces the momentum map $J : T^*Q \to \mathfrak{t}^*$ for the action of the group $\mathbb{T}$. In these terms the averaged mechanical connection can be defined explicitly by

$$\bar{\mathcal{A}}(v_q) = \mathbb{I}(q)^{-1}J(p_q),$$

where v_q is a tangent vector in T_qQ, $\;p_q := \mathbb{I}_Q(q)v_q = v_q^\flat \in T_q^*Q$ is the corresponding metric-dual cotangent vector, and $\mathbb{I}$ is the inertia operator on $\mathfrak{t}$ in the fiber at q.

Remark 3.4. More specifically, in coordinates for a trivial bundle Q the general form for a $\mathbb{T}$-invariant metric on $Q = B \times \mathbb{T}$ is as follows:

$$((u,\gamma),(u,\gamma))_{(x,\tau)} = (u,u)_x + 2\gamma\, h(x)\langle A(x),u\rangle + h(x)\gamma^2, \tag{4}$$

where $(u,\gamma) \in T_{(x,\tau)}(B\times\mathbb{T}) = T_xB \times \mathfrak{t}$, $A(x) \in \Omega^1(B,\mathfrak{t}) = T^*_xB$, $h(x) \in \mathbb{R}^+$, and $\mathfrak{t} \simeq \mathbb{R}$. For a non-trivial Q this general form is valid locally on the base.

Proposition 3.5. *For a trivial bundle $Q = B\times\mathbb{T}$ the averaged connection $\bar{\mathcal{A}} \in \Omega^1(B\times\mathbb{T},\mathfrak{t}) = T^*_{(x,\tau)}(B\times\mathbb{T})$ corresponding to the averaged metric (4) is given by $\bar{\mathcal{A}}(x,\tau) = A(x) + d\tau$. The summands can be regarded as connections on the base $A(x) \in T^*_xB$ and in the fiber $d\tau$.*

Proof. For a trivial bundle Q the momentum map $J : T^*_{(x,\tau)}(B\times\mathbb{T}) \to \mathfrak{t}^*$ is given by

$$J_{(x,\tau)}(a,\eta) = h(x)\langle A(x),u\rangle + h(x)\langle d\tau,\xi\rangle,$$

where $(a,\eta) \in T^*_{(x,\tau)}(B\times\mathbb{T})$ and $(u,\xi) \in T_{(x,\tau)}(B\times\mathbb{T})$ is the image of (a,η) under the metric identification. Indeed, by definition of the momentum map, for any $\zeta \in \mathfrak{t}$,

$$\langle J_{(x,\tau)}(a,\eta),\zeta\rangle = \langle (a,\eta),(0,\zeta)\rangle = ((u,\xi),(0,\zeta))$$

$$= (u,0)_x + \zeta h(x)\langle A(x),u\rangle + \xi h(x)\langle A(x),0\rangle + \xi h(x)\zeta = \langle h(x)\langle A(x),u\rangle + h(x)\xi,\zeta\rangle.$$

Furthermore, the inertia operator $\mathbb{I}(x) : \mathfrak{t} \to \mathfrak{t}^*$ at $x \in B$ is given by $\mathbb{I}(x)\gamma = h(x)\gamma$ for any $\gamma \in \mathfrak{t}$, hence the average mechanical connection assumes the form $\bar{\mathcal{A}}(u,\xi) = \mathbb{I}(x)^{-1}J(a,\eta) = \langle A(x),u\rangle + \langle d\tau,\xi\rangle$, as required. □

3.2 Reduction

By considering the $\mathbb{T}$-invariant metric and Hamiltonian (obtained by $\mathbb{T}$-averaging) we are now in the framework of Section 2. The dynamics defined by the averaged Hamiltonian $\overline{H}^{\mathbb{T}}$ on T^*Q can be derived from the corresponding *averaged* or *slow motion*, i.e. the dynamics on T^*B of the base space $B = Q/\mathbb{T}$. However, unlike the standard averaging method discussed below in Section 5.1, now we obtain this slow motion via symplectic reduction.

Recall that, for a fixed value μ of the momentum map, the reduced space $J^{-1}(\mu)/\mathbb{T}$ is symplectomorphic to the cotangent bundle T^*B of the base space B with the twisted symplectic form $\omega_\mu = \omega_{can} - \beta_\mu$, where ω_{can} and β_μ are the canonical and magnetic 2-forms on T^*B (see Theorem 2.1). The averaged/slow system turns out to be a Hamiltonian system on the symplectic manifold (T^*B,ω_μ) with the following reduced Hamiltonian function $\bar{H}_\mu$.

Theorem 3.6. *For a natural system on a $\mathbb{T}$-bundle Q over the slow manifold B with Hamiltonian function $H(q,p) = (1/2)(p,\,p)_q + U(q)$ the result of the symplectic reduction with respect to the $\mathbb{T}$-action of the averaged system is a natural system with the Hamiltonian function $\bar{H}_\mu$,*

$$\bar{H}_\mu(q,p) = \frac{1}{2}(p,\,p)_B + U_\mu(q)\,, \tag{5}$$

on the symplectic manifold (T^*B,ω_μ). Here $(q,p) \in T^*B$, $(\cdot,\cdot)_B$ stands for the metric on the base $B = Q/\mathbb{T}$ obtained as a Riemannian submersion $Q \to B$ from the metric $\overline{(\cdot,\cdot)}^{\mathbb{T}}$ on Q, while $U_\mu(q) := \overline{U(q)}^{\mathbb{T}} + \frac{1}{2}\langle\mu,\,\mathbb{I}(q)^{-1}\mu\rangle$ is the effective potential.

Proof. We start by computing the result of averaging and consequent symplectic reduction on T^*Q with respect to the $\mathbb{T}$-action. Upon averaging along $\mathbb{T}$-orbits one can assume that the Hamiltonian $\bar{H}$ on Q is $\mathbb{T}$-invariant, $\bar{H}(q,p) = \overline{H(q,p)}^{\mathbb{T}}$. The reduced Hamiltonian system on the quotient $(T^*Q)_\mu$ is Hamiltonian with respect to the symplectic structure $\omega_\mu = \omega_{can} - \beta_\mu$. The new Hamiltonian is obtained from $\bar{H}$ by applying the map Shift_μ. Namely, abusing the notation, for $(q,p) \in T^*B$ and a connection $\bar{\mathcal{A}}$ in the $\mathbb{T}$-bundle Q one has

$$\bar{H}_\mu(q,p) = \bar{H}(q,\, p + \langle \mu, \bar{\mathcal{A}}(q)\rangle) = \frac{1}{2}\overline{(p + \langle \mu, \bar{\mathcal{A}}(q)\rangle,\, p + \langle \mu, \bar{\mathcal{A}}(q)\rangle)_q}^{\mathbb{T}} + \overline{U(q)}^{\mathbb{T}}$$

$$= \frac{1}{2}(p,\, p)_B + \overline{(p,\, \langle \mu, \bar{\mathcal{A}}(q)\rangle)_q}^{\mathbb{T}} + \frac{1}{2}\overline{(\langle \mu, \bar{\mathcal{A}}(q)\rangle,\, \langle \mu, \bar{\mathcal{A}}(q)\rangle)_q}^{\mathbb{T}} + \overline{U(q)}^{\mathbb{T}} = \frac{1}{2}(p,\, p)_B + \overline{U_\mu(q)}^{\mathbb{T}}$$

for $U_\mu(q) := \frac{1}{2}\overline{(\langle \mu, \bar{\mathcal{A}}(q)\rangle,\, \langle \mu, \bar{\mathcal{A}}(q)\rangle)_q}^{\mathbb{T}} + \overline{U(q)}^{\mathbb{T}}$. Here we use that $\bar{\mathcal{A}}$ is the mechanical connection corresponding to the averaged metric $\overline{(\cdot,\cdot)_q}^{\mathbb{T}}$, and hence we have $\overline{(p,\, \langle \mu, \bar{\mathcal{A}}(q)\rangle)_q}^{\mathbb{T}} = \langle \mu,\, \bar{\mathcal{A}}(q)(v)\rangle = \langle \mu,\, \mathbb{I}(q)^{-1}J(p)\rangle = 0$, since $J(p) = 0$, and where $(q,p) \in T^*B$ is identified with $(q,v) \in TB$ by means of the averaged metric. Thus on the reduced symplectic manifold T^*B with the twisted symplectic form $\omega_\mu = \omega_{can} - \beta_\mu$ the new reduced Hamiltonian is

$$\bar{H}_\mu(q,p) = \frac{1}{2}(p,\, p)_B + U_\mu(q)$$

for $q \in B$ and $p \in T^*_qB$. It is a natural system with a new effective potential

$$U_\mu(q) = \frac{1}{2}\overline{(\langle \mu, \bar{\mathcal{A}}(q)\rangle,\, \langle \mu, \bar{\mathcal{A}}(q)\rangle)_q}^{\mathbb{T}} + \overline{U(q)}^{\mathbb{T}} = \frac{1}{2}\langle \mu, \mathbb{I}(q)^{-1}\mu\rangle + \overline{U(q)}^{\mathbb{T}}.$$

□

In Section 5.2 below we will prove the following corollary of Theorem 3.6 for averaging one-frequency fast-oscillating systems: Under certain conditions, solutions of the averaged system and projections to slow manifold of solutions of the actual system with the same initial conditions remain ϵ-close to each other for $0 \leqslant t \leqslant 1/\epsilon$.

Remark 3.7. The two features of the averaged-reduced Hamiltonian system are the additional term in the effective potential U_μ and the magnetic term $-\beta_\mu$ in the symplectic structure ω_μ. Therefore this averaging-reduction procedure provides a geometrical explanation of these two phenomena, often observed in the averaging theory.

Remark 3.8. In the classical averaging of fast-oscillating systems (cf. Section 5.1 below) one starts by fixing the action variable J. This can be regarded as a manifestation of symplectic reduction in flat coordinates, as this means fixing a certain value of the corresponding momentum map. The bundle averaging-reduction procedure described here is also applicable in that case, but the metric in this bundle turns out to be flat. Namely, in the reduction to a submanifold $J = \mu$ one chooses a flat connection on the principal bundle which corresponds to the direct product of the base and fibres, and hence no twisted symplectic structure appears on the reduced manifold: for the momentum value $J = \mu$, the averaged Hamiltonian function is $\epsilon\,\bar{H}(Q,P,\mu)$ on the "flat" cotangent bundle $(T^*\mathbb{R}^\ell, dP \wedge dQ)$.

4 Central extensions in symplectic reduction

Above we described the reduced phase space $(T^*Q)_\mu$ for the right action by the group $\mathbb{T}$. In this case, the reduced phase space $(T^*Q)_\mu$ coincides with $T^*(Q/\mathbb{T})$, equipped with the magnetic symplectic structure ω_μ described before. Now assume in addition that the base space $Q/\mathbb{T}$ has the structure of another Lie group G, which acts on itself from the left and leaves the metric on $G = Q/\mathbb{T}$ invariant. As a result, G acts on $T^*G = T^*(Q/\mathbb{T})$ and, as one can check, this action leaves the symplectic structure $\omega_\mu = \omega_{can} - \beta_\mu$ invariant. (Recall that the magnetic 2-form $\beta_\mu := \pi_G^*\sigma_\mu$ on T^*G is the pullback of the left-invariant 2-form σ_μ on the group G.) Hence another reduction for this G-action ("the reduction by stages") would take this magnetic symplectic structure on T^*G to an appropriate structure on the dual Lie algebra $\mathfrak{g}^*$, as described below.

Theorem 4.1. (Theorem 7.2.1 in [11]) *The Poisson reduced space for the left action of G on $(T^*G, \omega_\mu = \omega_{can} - \beta_\mu)$ is the dual Lie algebra $\mathfrak{g}^*$ with the Poisson bracket given by*

$$\{f, g\}_\mu(\nu) = -\left\langle \nu, \left[\frac{\delta f}{\delta \nu}, \frac{\delta g}{\delta \nu}\right]\right\rangle - \sigma_\mu(e)\left(\frac{\delta f}{\delta \nu}, \frac{\delta g}{\delta \nu}\right) \tag{6}$$

for $f, g \in C^\infty(\mathfrak{g}^*)$ at any $\nu \in \mathfrak{g}^*$, where $\sigma_\mu(e)$ is the value of the left-invariant 2-form σ_μ at $e \in G$ on the pair of tangent vectors $\frac{\delta f}{\delta \nu}, \frac{\delta g}{\delta \nu} \in T_eG = \mathfrak{g}$, and $\beta_\mu := \pi_G^*\sigma_\mu$ is the pullback of σ_μ to T^*G.

Remark 4.2. The above Poisson bracket is the Lie-Poisson bracket of the dual $\widehat{\mathfrak{g}}^*$ of the central extension $\widehat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$ by means of the $\mathfrak{t}$-valued 2-cocycle σ, such that $\langle \sigma, \mu\rangle := \sigma_\mu(e)$. Namely, the Lie algebra $\widehat{\mathfrak{g}}$ is the direct sum $\mathfrak{g} \oplus \mathfrak{t}$, as a vector space, with the commutator

$$[(u, a), (v, b)]_{\widehat{\mathfrak{g}}} := ([u, v]_{\mathfrak{g}}, \sigma(u, v))$$

for $u, v \in \mathfrak{g}$ and $a, b \in \mathfrak{t}$. It turns out that under certain integrality conditions, the space Q gives a realization of the corresponding centrally extended group $\widehat{G}$.

For the right action of G the bracket changes sign.

Theorem 4.3. *Let G be a group equipped with a closed integral left-invariant 2-form $\sigma_\mu/2\pi$. Then the $\mathbb{T}$-bundle Q over the group G with the curvature form σ_μ can be canonically identified with the central extension $\widehat{G}$ of the group G by means of $\mathbb{T}$, where the Lie algebra 2-cocycle is $\sigma_\mu(e)$, i.e. its value on a pair of Lie algebra elements ξ and η is $\sigma_\mu(e)(\xi, \eta)$.*

Proof. The proof is based on a version of Proposition 4.4.2 of [12] adjusted to the setting at hand. In fact, one can explicitly construct $\widehat{G}$ and identify it with Q, the $\mathbb{T}$-bundle over G. Namely, first for any oriented loop ℓ in G one associates an element $C(\ell) = \exp(i\int_{\partial^{-1}\ell}\sigma_\mu)$, where $\partial^{-1}\ell$ is an oriented 2D surface in G bounded by ℓ. The value $C(\ell)$ is well-defined, since for two different surfaces with the same boundary the integrals of σ_μ for an integral 2-form $\sigma_\mu/2\pi$ differ by a multiple of 2π.

The map $\ell \mapsto C(\ell)$ is independent of parametrization of ℓ, additive, and G-invariant. It defines a central extension $\widehat{G}$ of the group G by $\mathbb{T}$ as a set of triples (g, u, p), where $g \in G$, $u \in \mathbb{T}$ and p is a path in G from e to g, modulo the following equivalence. Two triples (g, u, p) and (g', u', p') are equivalent if $g' = g$ and $u' = C(p' \cup p^{-1})u$. The composition is $(g_1, u_1, p_1) \circ (g_2, u_2, p_2) = (g_1g_2, u_1u_2, p_1 \cup g_1(p_2)))$.

Recall that the space Q with an invariant metric has a structure of a $\mathbb{T}$-bundle with mechanical connection. Then a triple (g, u, p) modulo equivalence can be interpreted as the following point in Q: it is the point in the $\mathbb{T}$-fiber over $g \in G$, obtained from the point (e, u) of the $\mathbb{T}$-fiber over $e \in G$ by a horizontally lifted path p from e to g. Then the equivalence of triples stands for their correspondence to the same point in Q, since the form σ_μ is the curvature of the mechanical connection, while the formula $u' = C(p' \cup p^{-1})u$ describes the holonomy of the connection over a closed loop. $\square$

Remark 4.4. Theorems 4.1 and 4.3 can be extended to the case of a torus $\mathbb{T}$-bundle Q over G, where $\mathbb{T} = T^k$. In Theorem 4.3 one realizes Q as a group central extension of G by $\mathbb{T}$ by applying the above consideration to the "coordinate 2-forms" $\sigma_\mu = \langle \sigma, \mu \rangle$ of the $\mathfrak{t}$-valued 2-form σ.

Remark 4.5. Return to the 2-cocycle β_μ on the Lie algebra $\mathfrak{g}$, which defines the central extension and the magnetic term. In many examples, this 2-cocycle is a 2-boundary, i.e. the 2-form σ_μ on the Lie algebra can be represented as a linear functional of the Lie algebra commutator, $\sigma_\mu(\xi, \eta) = L([\xi, \eta])$ for some element $L \in \mathfrak{g}^*$. In that case, the corresponding Poisson structure on $\mathfrak{g}^*$ is the linear Lie-Poisson structure on the dual space $\mathfrak{g}^*$ shifted to the point L. The associated Euler equation also manifests a certain shift, observed, e.g. as a Stokes drift velocity related to surface waves in the Craik-Leibovich equation, cf. Section 6.2.

Remark 4.6. When considering dynamics on the reduced space T^*G, in order to use the second reduction over the G-action one has to confine to the natural systems with effective potential independent of q, i.e. $U_\mu(q) = const$. The latter are geodesic flows for the invariant metric on G defined by the inertia operator $\mathbb{I}_G : \mathfrak{g} \to \mathfrak{g}^*$. The second reduction defines the Euler equations for the quadratic Hamiltonian $H(p) := \frac{1}{2}(p, p)_e = \frac{1}{2}\langle p, \mathbb{I}_G^{-1} p \rangle$ on the dual $\widehat{\mathfrak{g}}^*$ of centrally extended Lie algebra $\widehat{\mathfrak{g}}$.

5 Reminder on averaging and examples

5.1 Averaging in one-frequency Hamiltonian systems

Consider a Hamiltonian system with $\ell + 1$ degrees of freedom and Hamiltonian of the form $H(q, p, I, \phi) = H_0(I) + \epsilon H_1(q, p, I, \phi)$, where $\phi (\mathrm{mod}\, 2\pi) \in \mathbb{T}$, while H is 2π-periodic in ϕ, and $(q, p, I) \in D \subset \mathbb{R}^{2\ell+1}$. (Such perturbations of properly degenerate Hamiltonian systems are typical in celestial mechanics.) The corresponding Hamiltonian equations for the standard symplectic structure are as follows:

$$\begin{cases} \dot{q} = \epsilon\, \partial H_1/\partial p & \dot{I} = -\epsilon\, \partial H_1/\partial \phi, \\ \dot{p} = -\epsilon\, \partial H_1/\partial q, & \dot{\phi} = \partial H_0/\partial I + \epsilon\, \partial H_1/\partial I. \end{cases} \tag{7}$$

Definition 5.1. The *averaged system* for the above Hamiltonian $H = H_0 + \epsilon H_1$ is the system of $2\ell + 1$ equations:

$$\begin{cases} \dot{Q} = \epsilon\, \partial \bar{H}_1/\partial P, & \dot{J} = 0, \\ \dot{P} = -\epsilon\, \partial \bar{H}_1/\partial Q, \end{cases} \tag{8}$$

where $\bar{H}_1(Q, P, J) := \frac{1}{2\pi} \int_0^{2\pi} H_1(Q, P, J, \phi)\, d\phi$.

Since there is no evolution of J in the averaged system, one can fix it and regard J as a parameter for the Hamiltonian system with ℓ degrees of freedom, where $\bar{H}(Q,P) = \bar{H}_J(Q,P) = H_0(J) + \epsilon\bar{H}_1(Q,P,J)$.

Let (q,p,I) belong to a domain $D \subset \mathbb{R}^{2\ell+1}$, and $D_\delta \subset D$ stands for a subdomain whose δ-neighbourhood belongs to D. Assume that the Hamiltonian H is C^3-bounded for $(q,p,I,\phi) \in D \times \mathbb{T}$, as well as $\partial H_0(I)/\partial I > C > 0$ in D and $(Q(t),P(t),J(t)) \in D_\delta$ for all $0 \leqslant t \leqslant 1/\epsilon$.

Theorem 5.2. (cf. [1, 2]) *For sufficiently small positive ϵ (i.e. $0 < \epsilon < \epsilon_0$) solutions of the actual system* (7) *and averaged system* (8) *with the same initial conditions* $(q(0),p(0),I(0)) = (Q(0),P(0),J(0))$ *remain ϵ-close to each other for $0 \leqslant t \leqslant 1/\epsilon$: $|I(t) - I(0)| < C_0\epsilon$ and $|q(t) - Q(t)| + |p(t) - P(t)| < C_0\epsilon$, where C_0 does not depend on ϵ.*

Proof of this theorem consists of constructing a canonical transformation ϵ-close to the identity and mapping the original system to the averaged one modulo ϵ^2-terms. Then for the time $0 \leqslant t \leqslant 1/\epsilon$ solutions of the averaged system remain ϵ-close to those of the original one, see [1].

Remark 5.3. Consider now a fast-oscillating nonautonomous Hamiltonian system with "ℓ and a half" degrees of freedom, whose Hamiltonian is $H = H(p,q,\omega t)$ with high frequency $\omega = \mu/\epsilon$. The associated Hamiltonian equations are $\dot{q} = \partial H/\partial p$ and $\dot{p} = -\partial H/\partial q$. The fast variable $\phi = \omega t$ can be regarded as a new independent space variable by passing to the new autonomous Hamiltonian system for $\widetilde{H} = \omega I + H(p,q,\phi)$ with $\ell+1$ degrees of freedom, where variable I is conjugate to ϕ. Combined with the reparametrization $t \mapsto \tau = t/\epsilon$ this leads to the system of the above type (where now the upper dot stands for the derivative in the fast time τ):

$$\begin{cases} \dot{q} = \quad \epsilon\,\partial H/\partial p\,, & \dot{I} = -\epsilon\,\partial H/\partial \tau, \\ \dot{p} = -\epsilon\,\partial H/\partial q\,, & \dot{\phi} = \ \mu. \end{cases} \tag{9}$$

5.2 Averaging in natural systems

In this section we study averaging in natural systems on the cotangent bundle of a principle circle bundle $\pi : Q \to B$. The general form of a natural Hamiltonian function on T^*Q is

$$H(x,y) = \frac{1}{2}(y,\, y)_x + U(x)$$

for $(x,y) \in T^*_xQ$.

We start by considering natural systems on the cotangent bundle $T^*(\mathbb{R}^\ell \times \mathbb{T})$ of a direct product with Hamiltonians

$$H(q,\phi;p,\gamma) = \frac{1}{2}\left((p,\gamma),(p,\gamma)\right)_{(q,\phi)} + U(q,\phi), \tag{10}$$

where $(q,\phi) \in \mathbb{R}^\ell \times \mathbb{T}$ and $(p,\gamma) \in T^*_{(q,\phi)}(\mathbb{R}^\ell \times \mathbb{T})$. Assume that the function $H(q,\phi;p,\gamma)$ is 2π-periodic in ϕ, $U(q,\phi) = U_0(q) + \epsilon U_1(q,\phi)$, and the metric on $\mathbb{R}^\ell \times \mathbb{T}$ has the form

$$\left((p,\gamma),(p,\gamma)\right)_{(q,\phi)} = p \cdot p + 2\gamma a(q,\phi) \cdot p + h(q,\phi)\,\gamma^2, \tag{11}$$

where dot stands for the Euclidean inner product and where the corresponding coefficients a and h have the following expansions in ϵ as $\epsilon \to 0$:

$$a(q,\phi) = a_0(q) + \epsilon a_1(q,\phi), \quad h(q,\phi) = h_0(q) + \epsilon h_1(q,\phi),$$

with functions $a_1(q,\phi)$, $h_1(q,\phi)$ and $U_1(q,\phi)$ of zero mean with respect to ϕ.

The Hamiltonian equations for this Hamiltonian function $H(q,\phi;p,\gamma)$ and symplectic structure $\omega = (1/\epsilon)\, dq \wedge dp + d\phi \wedge d\gamma$ on $T^*(\mathbb{R}^\ell \times \mathbb{T})$ are

$$\begin{cases} \dot q = \ \ \epsilon\,(p + \gamma a(q,\phi)), \\ \dot\phi = a(q,\phi)\cdot p + h(q,\phi)\,\gamma, \\ \dot p = -\epsilon\,\partial(\gamma a(q,\phi)\cdot p + (1/2)\,h(q,\phi)\gamma^2 + U(q,\phi))/\partial q\,, \\ \dot\gamma = -\epsilon\,\partial(\gamma a_1(x,\phi)\cdot p + (1/2)\,h_1(q,\phi)\gamma^2 + U_1(q,\phi))/\partial\phi. \end{cases} \tag{12}$$

In these equations ϕ is the fast variable. Below we prove that the averaged Hamiltonian for (10) is

$$\bar H(Q,P,\mu) = \frac{1}{2}\,P\cdot P + \mu\, a_0(Q)\cdot P + \frac{1}{2}\,\mu^2 h_0(Q) + U_0(Q)\,, \tag{13}$$

where the part $(1/2)\,\mu^2 h_0(Q) + U_0(Q)$ is related to an effective potential, while the term $\mu\, a_0(Q)\cdot P$ linear in impulses is related to a magnetic-gyroscopic-like force. Namely, one has the following statement, which is an adaptation of Theorem 5.2 to the system (12).

Theorem 5.4. *For sufficiently small $\epsilon > 0$ solutions for the original Hamiltonian* (10) *and the averaged Hamiltonian* (13) *with the same initial conditions* $(q(0),p(0),\gamma(0)) = (Q(0),P(0),\mu(0))$ *remain ϵ-close to each other for $0 \leqslant t \leqslant 1/\epsilon$: $|\gamma(t)-\mu(t)| + |q(t)-Q(t)| + |p(t)-P(t)| < C_0\epsilon$, where C_0 does not depend on ϵ.*

Proof. The Hamiltonian equations (12) can be rewritten in the following form

$$\begin{cases} \dot q = \ \ \epsilon\,(p + \gamma a_0(q)) + \epsilon^2\gamma a_1(q,\phi)\,, \\ \dot\phi = a(q,\phi)\cdot p + h(q,\phi)\,\gamma, \\ \dot p = -\epsilon\,\partial(\gamma a_0(q)\cdot p + (1/2)\,h_0(q)\gamma^2 + U_0(q))/\partial q \\ \qquad\qquad\qquad\qquad - \epsilon^2\partial(\gamma a_1(q,\phi)\cdot p + (1/2)\,h_1(q,\phi)\gamma^2 + U_1(q,\phi))/\partial q\,, \\ \dot\gamma = -\epsilon\,\partial(\gamma a_1(x,\phi)\cdot p + (1/2)\,h_1(q,\phi)\gamma^2 + U_1(q,\phi))/\partial\phi. \end{cases} \tag{14}$$

These equations differ by ϵ^2-terms from those for the averaged Hamiltonian (13) in the symplectic structure $\Omega = dP \wedge dQ$:

$$\begin{cases} \dot Q = \epsilon\,(P + \gamma a_0(Q))\,, \\ \dot P = -\epsilon\,\partial(\gamma a_0(Q)\cdot P + (1/2)\,h_0(Q)\gamma^2 + U_0(q))/\partial Q\,, \\ \dot\mu = 0, \end{cases} \tag{15}$$

so according to Theorem 5.2, we obtain the required proximity of solutions for the original and averaged equations. □

Remark 5.5. In the shifted coordinates $(Q,P) \to (Q, P_1 := P + \mu\, a_0(Q))$, the averaged Hamiltonian (13) becomes

$$\bar H(Q,P_1,\mu) = \frac{1}{2}\,P_1\cdot P_1 + \frac{1}{2}\,\mu^2(h_0(Q) - a_0(Q)\cdot a_0(Q)) + U_0(Q), \tag{16}$$

while the symplectic structure becomes

$$\Omega = d(PdQ) = d((P_1 - \mu\, a_0(Q))\, dQ) = dP_1 \wedge dQ - \mu\, d(a_0(Q)\, dQ).$$

One notices that a new (effective) potential now includes an additional term,

$$\bar U_\mu(Q) := \frac{1}{2}\,\mu^2(h_0(Q) - a_0(Q)\cdot a_0(Q)) + U_0(Q),$$

while the new symplectic structure acquires the magnetic term $\mu\, d(a_0(Q)\, dQ)$.

Now we are ready to prove

Corollary 5.6. *The solutions of the reduced Hamiltonian system and projections to slow manifold of solutions of the actual system with the same initial conditions remain ϵ-close to each other for $0 \leqslant t \leqslant 1/\epsilon$.*

Proof. Comparing the result of Theorem 3.6 with Remark 5.5 we observe that the Hamiltonian (5) in the theorem coincides with the averaged Hamiltonian of the natural system obtained above.

As a matter of fact, the consideration of Remark 5.5 can be seen as a local (coordinate) version of the proof of Theorem 3.6. Indeed, the averaged metric in local coordinates can be expressed as

$$\overline{((p,\mu),\ (p,\mu))}^{\mathbb{T}}_{(q,g)} = p \cdot p + 2\mu\, a_0(q) \cdot p + \mu^2 h_0(q),$$

where $(q,g) \in B \times \mathbb{T}$ and $(p,\mu) \in T^*_{(q,g)}(B \times \mathbb{T})$, cf. (13). Therefore the corresponding mechanical connection $\bar{\mathcal{A}} \in \Omega^1(B,\mathbb{R})$ in local coordinates is $\bar{\mathcal{A}} = a_0(q)\ dq$. One can see that the symplectic structure and effective potential in Remark 5.5 coincide with those in Theorem 3.6.

Since one obtains the same averaged system both via the "local" proof of Section 5.2 and via the "global" proof of Theorem 3.6, then Theorem 5.4 guarantees the required closeness of averaged and original solutions. □

Remark 5.7. While Theorem 5.4 deals with the topologically trivial $\mathbb{T}$-bundle $\mathbb{R}^\ell \times \mathbb{T}$, the result on the existence of an averaged system holds for a topologically nontrivial bundle as well. In the general case of a nontrivial $\mathbb{T}$-bundle $\pi : Q \to B$ we assume that the Hamiltonian system has fast motions along fibers and slow motions on the base B. Under the assumption that the oscillatory parts of the Hamiltonian are of order ϵ, i.e.

$$U(q) - \overline{U(q)}^{\mathbb{T}} \sim\ O(\epsilon) \ \text{ and } \ (p,p)_q - \overline{(p,p)}^{\mathbb{T}}_q \sim\ O(\epsilon), \tag{17}$$

one can introduce the averaged kinetic energy $(1/2)\overline{(p,p)}^{\mathbb{T}}_q$ and averaged potential energy $\overline{U(q)}^{\mathbb{T}}$ by averaging the system along the fibers, see Section 3. Then one proves the averaging theorem by considering only the local picture of the principal bundle $\pi : Q \to B$, since any bundle is locally trivial.

5.3 Examples of averaged systems

Example 5.8 *(A pendulum with rapidly oscillating suspension point)***.** Consider the motion of a pendulum with vertically vibrating suspension point. Set θ to be the angle of deviation of the pendulum from the vertical, a and ω are the amplitude and frequency of the oscillation of the suspension point, l is the length of the pendulum and g is the acceleration of gravity. We assume that the amplitude ϵa is of order ϵ and the frequency $\omega = \mu/\epsilon$ is of order $1/\epsilon$, i.e. the suspension point oscillates with high frequency and small amplitude. The corresponding potential is $U(\theta) = -gl\cos\theta$ and the Hamiltonian function is

$$H(\theta,p,t) = \frac{1}{2}\left(\frac{p}{l} - a\mu \sin\omega t \sin\theta\right)^2 - gl\cos\theta\,.$$

This system differs from a natural one because of the shift in p, and the standard reasoning goes as follows, see e.g. [2]. Let $\phi = \omega t$ be the fast variable. In order to get rid of the ϕ-dependence in the Hamiltonian of vibrating pendulum, we seek for a canonical transformation

$(\theta, p) \mapsto (\theta_1, p_1)$ with a generating function $p_1\theta + \epsilon S_1(\theta, p_1, \phi)$, where the function S_1 is 2π periodic in ϕ. Then the new Hamiltonian becomes

$$\mathscr{H}(\theta_1, p_1, \phi) = \mu \frac{\partial S_1}{\partial \phi} + H(\theta_1, p_1, \phi) + O(\epsilon),$$

as $\epsilon \to 0$. By taking $S_1(\theta, p_1, \phi) = -(p_1/l)\, a \cos\phi \sin\theta + (1/8)\, a^2 \mu \sin^2\theta \sin 2\phi$, we obtain the following Hamiltonian averaged to the first order in ϵ:

$$\mathscr{H}(\theta_1, p_1, \phi) = \frac{p_1^2}{2l^2} - gl\cos\theta_1 + \frac{1}{4} a^2 \mu^2 \sin^2\theta_1 + O(\epsilon).$$

Notice the appearance of an additional positive definite quadratic term in the effective potential

$$U_\mu(\theta_1) = -gl\cos\theta_1 + (1/4) a^2 \mu^2 \sin^2\theta_1 \,.$$

It causes such an interesting dynamical phenomenon as the stability of the upper position of the pendulum.

Example 5.9 *(A particle in a rapidly oscillating potential)***.** Consider the motion of a particle in a rapidly oscillating potential, following [3]. The corresponding Hamiltonian function is $H(q, p, t) = (1/2)\, p \cdot p + U(q, t/\epsilon)$, where the potential function $U(q, \tau)$ is 2π-periodic with respect to τ. To obtain an averaged Hamiltonian modulo the third order in ϵ one needs to iteratively apply canonical transformations $(q, p) \to (Q, P)$ four times (see [3] for details). The results is

$$\bar{H}(Q, P) = \frac{1}{2} P \cdot P + \bar{U}(Q) + \frac{\epsilon^2}{2} \overline{V' \cdot V'} - \epsilon^3 \overline{S'' V'} P,$$

where $\bar{U}(q) = \frac{1}{2\pi} \int_0^{2\pi} U(q, \tau)\, d\tau$ is the time average of U over one temporal period, functions V and S stand for temporal antiderivatives $V(q, \tau) := \int^\tau [U(q, \theta) - \bar{U}(q)]\, d\theta$, $S(q, \tau) := \int^\tau V(q, \theta)\, d\theta$, with the constants of integration chosen such that $\bar{V} = \bar{S} = 0$, and the prime denotes the derivative with respect to the new space variable Q.

Example 5.10 *(Foucault pendulum)***.** By considering small and rapid oscillations of an ideal pendulum on a sphere rotating with angular velocity Ω, one can observe that the plane of oscillation will be rotating with the angular velocity $-\Omega \sin\lambda$, where λ is the pendulum latitude. This is an example of the Foucault pendulum, see [1] for a detailed description. More generally, one can consider a curved surface and a pendulum slowly transported along a path on the surface. In this case the plane of oscillation turns out to be parallel transported along the path on the surface, see the discussion in [9], and it can be regarded as a physical interpretation of the Levi-Civita connection.

One can use the averaging-reduction theory to interpret this phenomenon. First, notice that this system has the same $\mathbb{T}$-bundle structure as the one in the gyroscope Example 2.4. Hence by shifting the momentum similarly to that in Theorem 2.1, one obtains a Hamiltonian system on (the cotangent bundle of) the tangent bundle of the surface with a twisted symplectic structure, where the twist is given by an additional curvature term related to the Levi-Civita connection. Then the averaging-reduction procedure allows one to descend this system to (the cotangent bundle of) the unit tangent bundle of the surface, on which the parallel transport is observed. We plan to discuss this example in detail elsewhere.

6 Applications

6.1 Pendulum with a vibrating suspension point

In Section 5.3, we obtained the averaged Hamiltonian for a pendulum with a vibrating suspension point using the classical averaging method, and observed the appearance of an additional quadratic term in the effective potential. In this section, using the averaging-reduction procedure of Section 3, we show that this additional term is the result of symplectic reduction.

We start with the following Hamiltonian function describing a natural mechanical system with a rapidly oscillating potential:

$$H(x,p,t) = \frac{1}{2}(p,\, p) + U(x) + \epsilon\,\omega^2\, \widetilde{U}(x,\omega t), \tag{18}$$

where the frequency $\omega = \mu/\epsilon$ is of order $1/\epsilon$, and the oscillating part of the potential $\widetilde{U}(x,\phi)$ is 2π-periodic and has zero mean with respect to ϕ.

Introduce the fast time $\tau = t/\epsilon$ and fast variable $\phi = \omega t = \mu\tau$. We split any (vector-) function $f = f(t,\phi)$ depending on two times t and τ (and 2π-periodic in $\phi = \mu\tau$) into the mean and oscillatory parts:

$$f(t,\phi) = \overline{f}(t) + \widetilde{f}(t,\phi),$$

where $\overline{f} = (1/2\pi)\int_0^{2\pi} f(t,\phi)\,d\phi$. Now regard the fast variable ϕ as a new coordinate. Note that when the fast time τ changes by 1, the slow time t changes only by ϵ. So for a motion $x(t,\phi) = \overline{x}(t) + \widetilde{x}(t,\phi)$ described by the Hamiltonian function (18), one can fix $\overline{x}(t)$ and regard $\widetilde{x}(t,\phi) = \widetilde{x}(\overline{x},\phi)$ as a function of $\overline{x}$ and ϕ modulo $O(\epsilon)$ as $\epsilon \to 0$. In other words, one can consider a map from a suspension over M, a manifold $M \times \mathbb{T}$, to M itself, where a point $(\overline{x},\phi) \in M \times \mathbb{T}$ is mapped to $x = \overline{x}(t) + \widetilde{x}(\overline{x},\phi) \in M$. Here $\widetilde{x}(\overline{x},\phi)$ can be obtained by solving the Hamiltonian system corresponding to the above Hamiltonian function H in variables $(\widetilde{x},\widetilde{p})$:

$$H(\widetilde{x},\widetilde{p},t) = \frac{1}{2}(\widetilde{p},\widetilde{p}) + U(\overline{x}) + \epsilon\,\omega^2\, \widetilde{U}(\overline{x}+\widetilde{x},\omega t), \tag{19}$$

with the initial conditions such that the solution $(\widetilde{x}(\bar{x},\phi),\widetilde{p}(\bar{x},\phi))$ has zero mean value with respect to ϕ.

In order to compute the metric on the suspension manifold $M \times \mathbb{T}$ we set $\mu = 1$ (i.e. $\omega = 1/\epsilon$ and $\phi = \tau$). Thinking of position $\widetilde{x}$ and momentum $\widetilde{p}$ as depending on the average position $\overline{x}$ and fast time τ, we denote by $(\widetilde{x}(\overline{x},\tau),\widetilde{p}(\overline{x},\tau))$ the solution of the corresponding Hamiltonian system with initial conditions corresponding to zero mean value relative to τ.

According to the above consideration, the configuration space is a principal $\mathbb{T}$-bundle $\pi : M \times \mathbb{T} \to M$, where the $\mathbb{T}$-action on $Q = M \times \mathbb{T}$ is the shift in fibres $\phi \circ (\overline{x}, e^{i\phi'}) = (\overline{x}, e^{i(\phi+\phi')})$. Applying the averaging-reduction theory of Section 3, we obtain the following statement.

Theorem 6.1. *The Hamiltonian system (18) averaged using the standard averaging method in Example 5.8 coincides with the system obtained on the reduced symplectic manifold (T^*M, ω_{can}) with the canonical symplectic structure, the Hamiltonian function*

$$H_{slow}(\overline{x},\overline{p}) = \frac{1}{2}(\overline{p},\overline{p}) + U_\mu(\overline{x}), \tag{20}$$

for $(\overline{x},\overline{p}) \in T^*M$, and with the effective potential

$$U_\mu(\overline{x}) = U(\overline{x}) + \frac{\epsilon^2\omega^2}{4\pi}\int_0^{2\pi} (\widetilde{v}(\overline{x},\tau),\widetilde{v}(\overline{x},\tau))\,d\tau\,.$$

Proof. Now the averaged metric is given by

$$((v,\gamma),(v,\gamma))_{(\overline{x},\phi)} = (v,v) + 2\pi\,\gamma^2 \left(\int_0^{2\pi} (\widetilde{v}(\overline{x},\tau),\widetilde{v}(\overline{x},\tau))\,d\tau\right)^{-1},$$

where $(v,\gamma) \in T_{\overline{x}}M \times T_\phi\mathbb{T}$. Also, note that the value of $(1/2\pi)\int_0^{2\pi}(\widetilde{v}(\overline{x},\tau),\widetilde{v}(\overline{x},\tau))\,d\tau$ depends on $\overline{x}$ only.

The corresponding fiber inertia operator $\mathbb{I}(\overline{x}) : \mathfrak{t} = \mathbb{R} \to \mathfrak{t}^* = \mathbb{R}$ at $\overline{x} \in M$ is given by $\mathbb{I}(\overline{x})\gamma = 2\pi\gamma\left(\int_0^{2\pi}(\widetilde{v}(\overline{x},\tau),\widetilde{v}(\overline{x},\tau))\,d\tau\right)^{-1}$, while the momentum map $J : T^*(M\times\mathbb{T}) \to \mathfrak{t}^* = \mathbb{R}$ is given by $J(\bar{x},\phi,p,\eta) = \eta$. The averaged connection $\bar{\mathcal{A}} \in \Omega^1(M\times\mathbb{T},\mathbb{R})$ on the principal trivial $\mathbb{T}$-bundle $M\times\mathbb{T}$ is given by $\bar{\mathcal{A}}(\bar{x},\phi,v,\gamma) = \gamma$. This connection is flat, $\bar{\mathcal{A}} = d\phi$.

Recall that the magnetic term is proportional to the curvature of the bundle $\pi : Q \to M$. The flatness of $\bar{\mathcal{A}}$ implies that the reduced symplectic structure on the manifold $T^*M = J^{-1}(\mu)/\mathbb{T}$ has no magnetic term, i.e. it coincides with the canonical symplectic structure ω_{can}. By taking $\mu = d\phi/d\tau = \epsilon\omega$ in Theorem 3.6, the effective potential U_μ in the Hamiltonian function (20) is

$$U_\mu(\overline{x}) = U(\overline{x}) + 1/2\,\langle\mu,\mathbb{I}(\overline{x})^{-1}\mu\rangle = U(\overline{x}) + (\mu^2/4\pi)\int_0^{2\pi}(\widetilde{v}(\overline{x},\tau),\widetilde{v}(\overline{x},\tau))\,d\tau$$

$$= U(\overline{x}) + \epsilon^2(\omega^2/4\pi)\int_0^{2\pi}(\widetilde{v}(\overline{x},\tau),\widetilde{v}(\overline{x},\tau))\,d\tau.$$

□

Now return to Example 5.8 in Section 5.3. The Hamiltonian of a pendulum with a vibrating suspension point can be rewritten in the following form:

$$H(\theta,p) = \frac{1}{2}\left(\frac{p}{l}\right)^2 - (g - \epsilon\,a\,\omega^2\sin\tau)\,l\cos\theta. \tag{21}$$

Recall our assumption that the amplitude ϵa is of order ϵ and the frequency $\omega = \mu/\epsilon$ is of order $1/\epsilon$, which means that the above Hamiltonian has the form (18). Hence we obtain the following result on its slow motion.

Theorem 6.2. *The averaged Hamiltonian function reduces to the following Hamiltonian function describing the slow motion:*

$$H_{slow}(\overline{\theta},\overline{p}) = \frac{1}{2}\left(\frac{\overline{p}}{l}\right)^2 + U_\mu(\overline{\theta}), \tag{22}$$

where the effective potential is $U_\mu(\overline{\theta}) = (1/4)\,\mu^2a^2\sin^2\overline{\theta} - gl\cos\overline{\theta}$.

Proof. Let $\tau = t/\epsilon$ and $(\widetilde{\theta}(\overline{\theta},\tau),\widetilde{p}(\overline{\theta},\tau))$ be the solution of Hamiltonian system corresponding to the function

$$\widetilde{H}(\widetilde{\theta},\widetilde{p}) = \frac{1}{2}\left(\frac{\widetilde{p}}{l}\right)^2 - (g - \frac{a}{\epsilon}\sin\tau)\,l\cos(\overline{\theta}+\widetilde{\theta}),$$

with the initial values determined by the zero mean conditions of $(\widetilde{\theta}(\overline{\theta},\tau),\widetilde{p}(\overline{\theta},\tau))$ with respect to τ.

Omitting terms of order ϵ^2 we obtain the following Newton equation for $\widetilde{\theta}$:

$$d^2\widetilde{\theta}/dt^2 = -(a/(\epsilon l))\sin\tau\sin\overline{\theta}\,.$$

By rewriting it for the fast time $\tau = t/\epsilon$ and integrating we obtain $d\widetilde{\theta}/d\tau = (\epsilon a/l)\cos\tau\sin\overline{\theta}$ and $\widetilde{\theta} = (\epsilon a/l)\sin\tau\sin\overline{\theta}$. (In this integration one regards the right-hand side as a function of τ modulo higher order terms in ϵ, and uses the zero mean condition on $d\widetilde{\theta}/d\tau$ and $\widetilde{\theta}$.)

Furthermore, the momentum map $J : T^*(\mathbb{T}_{\bar{\theta}} \times \mathbb{T}_{\phi}) \to \mathbb{R}$, corresponding to the $\mathbb{T}$-action on $\mathbb{T}_{\bar{\theta}} \times \mathbb{T}_{\phi}$ equipped with the averaged metric is $J(\bar{\theta}, \phi, p, \mu) = \mu$, and $(l^2/2\pi)\int_0^{2\pi}(d\widetilde{\theta}/d\tau, d\widetilde{\theta}/d\tau)\,d\tau = \epsilon^2(a^2/2)\sin^2\overline{\theta}$.

Therefore, by Theorem 6.1, the reduced (or slow) symplectic manifold is $(T^*\mathbb{T}_{\bar{\theta}}, \omega)$ with symplectic structure $\omega = d\bar{\theta} \wedge d\bar{p}$ and the Hamiltonian of the slow motion is

$$H_{slow}(\overline{\theta}, \overline{p}) = \frac{1}{2}\left(\frac{\overline{p}}{l}\right)^2 + U_{\mu}(\overline{\theta}),$$

where the effective potential is

$$U_{\mu}(\overline{\theta}) = \frac{\omega^2 l^2}{4\pi}\int_0^{2\pi}(d\widetilde{\theta}/d\tau, d\widetilde{\theta}/d\tau)\,d\tau + U(\bar{\theta})$$

$$= \epsilon^2\frac{\omega^2 a^2}{4}\sin^2\overline{\theta} - gl\cos\overline{\theta} = \frac{1}{4}\mu^2 a^2\sin^2\overline{\theta} - gl\cos\overline{\theta}\,.$$

□

6.2 Craik-Leibovich equation

Consider the motion of an ideal fluid confined to a three-dimensional domain D with fast oscillating boundary ∂D. The averaged fluid motion is described by the following Craik-Leibovich (CL) equation on the fluid velocity field v:

$$\begin{cases} \frac{\partial v}{\partial t} + (v, \nabla)v + \operatorname{curl} v \times V_s = -\nabla p, \\ (v + V_s) \,||\, \partial D, \\ \operatorname{div} v = 0, \end{cases} \tag{23}$$

where V_s is a (time-dependent) prescribed Stokes drift velocity related to the average of surface waves. We refer to [14] for a derivation of the CL equations via perturbation theory. In [15, 16] the Hamiltonian structure of the CL equation was studied, along with a generalization of the perturbation theory to a principal $\mathbb{T}$-bundle over any group G and derivation of the Euler equation associated with a certain central extension of G.

In a more general setting, let $\mathrm{SDiff}(D)$ be the group of all volume-preserving diffeomorphisms of an n-dimensional Riemannian manifold D with boundary ∂D. Its Lie algebra $\mathfrak{g} = \mathrm{SVect}(D)$ consists of all the divergence-free vector fields in D tangent to the boundary ∂D, while the regular dual space $\mathfrak{g}^* = \Omega^1(D)/d\Omega^0(D)$ of this Lie algebra is the space of cosets of 1-forms on D modulo exact 1-forms.

Theorem 6.3. (see [16]) *The n-dimensional Craik-Leibovich (CL) equation on the space $\Omega^1(D)/d\Omega^0(D)$ has the form*

$$\frac{d}{dt}\,[u] = -\mathcal{L}_{v+V_s}\,[u], \tag{24}$$

where $v + V_s \in \mathrm{SVect}(D)$, and $[u] = [v^\flat] \in \Omega^1(D)/d\Omega^0(D)$ is the coset of the 1-form $v^\flat$ metric-related to the vector field v on D.

Remark 6.4. The requirement $v + V_s \in \mathrm{SVect}(D)$ provides the boundary condition of tangency of the vector field $v+V_s$ to the boundary ∂D. Although the divergence-free vector fields v and V_s are not necessarily tangent to the boundary, the 1-forms $v^\flat$ and $V_s^\flat$, considered up to the differential of a function, are well-defined elements in the dual space $\Omega^1(D)/d\Omega^0(D)$ of Lie algebra $\mathrm{SVect}(D)$.

Upon shifting the origin in $\mathfrak{g}^* = \Omega^1(D)/d\Omega^0(D)$ by $[V_s^\flat]$, equation (24) becomes

$$\frac{d}{dt}\,[w] = -\mathcal{L}_{\mathbb{I}^{-1}[w]}\,\Big([w] - [V_s^\flat]\Big) \tag{25}$$

for $[w] := [u + V_s^\flat]$.

Theorem 6.5. (see [16]) *The equation (25) is the Euler equation on the central extension $\widehat{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g} = \mathrm{SVect}(D)$ by means of the 2-cocycle*

$$\sigma_{V_s}(X,Y) := -\Big\langle \mathcal{L}_X\, V_s^\flat, Y\Big\rangle$$

associated with the vector field V_s.

Remark 6.6. More generally, for a "shift 2-cocycle"

$$\sigma_{V_s}(X,Y) := -\langle \mathrm{ad}_X^*\mathbb{I}(V_s), Y\rangle = -\langle \mathbb{I}(V_s), [X,Y]\rangle$$

on an arbitrary Lie algebra $\mathfrak{g}$, consider the corresponding (possibly trivial) central extension $\widehat{\mathfrak{g}}$ of Lie algebra $\mathfrak{g}$ by means of this 2-cocycle. Then the Euler equation on the centrally extended Lie algebra $\widehat{\mathfrak{g}}$ is

$$\frac{d}{dt}\,\xi = -\mathrm{ad}^*_{\mathbb{I}^{-1}\xi}\left(\xi - \mathbb{I}(V_s)\right), \tag{26}$$

where $\xi \in \mathfrak{g}^*$. Applying this to the Lie algebra $\mathrm{SVect}(D)$, where $\mathrm{ad}_X^*\xi = \mathcal{L}_X\xi$, we obtain the above theorem.

The averaging-reduction procedure gives us an explanation of the structure of central extension for the CL equation, which appeared as the result of averaging, as well as the origin of the vector field V_s. More specifically, by applying the symplectic reduction procedure to a natural system on the principal $\mathbb{T}$-bundle $G\times\mathbb{T} \to G$ for a group G (in the case of an oscillating flow, G is the group $\mathrm{SDiff}(D)$ of volume-preserving diffeomorphisms of D) one obtains the averaged system on the cotangent bundle T^*G of the base G with the symplectic structure endowed with a magnetic term. Furthermore, Theorem 4.1 provides the corresponding central extension. First we prove the averaging-reduction theorem for a trivial bundle $G \times \mathbb{T} \to G$, and then explain the necessary changes in the general case.

Theorem 6.7. *The averaging of a natural G-invariant Hamiltonian system on $T^*(G \times \mathbb{T})$ reduces to a slow Hamiltonian system on the reduced symplectic manifold (T^*G, ω_μ) with the Hamiltonian function $H_{slow}(\bar g, \bar r) = \frac{1}{2}(\bar r,\, \bar r)$, where $(\bar g, \bar r) \in T^*G$, and the symplectic structure ω_μ given by*

$$\omega_\mu = \omega_{can} - \mu\,\pi_G^*\,d\widetilde{\alpha},$$

for the canonical symplectic form ω_{can} on T^*G. Here $\pi_G : T^*G \to G$ is the cotangent bundle projection and $\widetilde{\alpha}$ is a certain 1-form on the base G depending on the averaged metric.

Proof. For a point $q = (g, x) \in G \times \mathbb{T}$ let $p = (\nu, y)$ be a covector at that point. A natural G-invariant Hamiltonian system on $T^*(G \times \mathbb{T})$ with Hamiltonian $H(q, p) = \frac{1}{2}(p,\, p) + U(x)$ for a G-invariant metric on $G \times \mathbb{T}$ has the form:

$$H(g, x; \nu, y) = \frac{1}{2}((\nu, y),\ (\nu, y))_{(g,x)} + U(x).$$

To write it more explicitly,[3] let $\mathcal{B}^*(x) : \mathfrak{g}^* \to T_x^*\mathbb{T}$ be the linear map associated with the G-invariant metric,[4] and identify covector components ν with elements of $\mathfrak{g}^*$ by right translations. Then the Hamiltonian can be rewritten as

$$H(g, x; \nu, y) = \frac{1}{2}(y,\ y)_x + (y,\ \mathcal{B}^*\nu)_x + \frac{1}{2}(\nu,\ \nu)_x + U(x). \tag{27}$$

Now fix a position g of the slow motion and consider the fast motion (i.e. x-dependence) of the system. In this setting we have $\nu = 0$, since the fast motion has zero mean, omitting higher order terms in ϵ. Then the Hamiltonian for the fast motion becomes

$$\widetilde{H}(\widetilde{x}, \widetilde{y}) = \frac{1}{2}(\widetilde{y}, \widetilde{y})_{\widetilde{x}} + U(\widetilde{x})\,. \tag{28}$$

Note that the Hamiltonian of fast motion does not depend on the group element g.

Suppose that the period of this motion is $2\pi/\omega$ and set the fast variable to be $\phi = \omega t$, where the frequency $\omega = \mu/\epsilon$. Let $(\widetilde{x}(\phi), \widetilde{y}(\phi))$ be any Hamiltonian trajectory for Hamiltonian function (28) satisfying the following two conditions: it is 2π-periodic with respect to the fast variable ϕ and its initial condition is chosen in a way to provide zero mean for $(\widetilde{x}(\phi), \widetilde{y}(\phi))$. Such a vanishing condition generically defines a 1-parameter family of solutions parametrized by the energy level (e.g. such solutions differ by scaling for a quadratic potential U). We use the fast variable ϕ to define the $\mathbb{T}$-action on $G \times \mathbb{T}$, which is given by $\phi \circ (g, e^{i\phi'}) = (g, e^{i(\phi+\phi')})$.

Since the Hamiltonian of fast motion does not depend on the group element g, the averaged kinetic energy $(1/2\pi)\int_0^{2\pi}(\widetilde{v}(\phi), \widetilde{v}(\phi))\, d\phi$ does not depend on g as well, where $\widetilde{v}$ is related to $\widetilde{y}$ by means of the metric: $\widetilde{v}^\flat = \widetilde{y}$. Take the solution $(\widetilde{x}(\phi), \widetilde{y}(\phi))$ for which the averaged kinetic energy $(1/2\pi)\int_0^{2\pi}(\widetilde{v}(\phi), \widetilde{v}(\phi))\, d\phi$ is $1/\epsilon^2$.

Then the averaged metric $\overline{(\cdot\,,\cdot)}^{\mathbb{T}}$ on $G \times \mathbb{T}$ (cf. Remark 3.4) is given by

$$\begin{aligned} \overline{((v,\gamma),(v,\gamma))}^{\mathbb{T}} &= (v, v) + 2\pi(2\gamma\,\langle\widetilde{\alpha}, v\rangle + \gamma^2)\left(\int_0^{2\pi}(\widetilde{v}(\phi), \widetilde{v}(\phi))\, d\phi\right)^{-1} \\ &= (v, v) + \epsilon^2\,(2\gamma\,\langle\widetilde{\alpha}, v\rangle + \gamma^2)\,, \end{aligned} \tag{29}$$

where $(v, \gamma) \in T_gG \times T_\phi\mathbb{T} \simeq T_gG \times \mathbb{R}$, while the 1-form $\widetilde{\alpha} \in T^*G$ depends on $\widetilde{y}$. This 1-form is right-invariant, i.e. it satisfies $\widetilde{\alpha}(hg) = R^*_{g^{-1}}\widetilde{\alpha}(h)$. Indeed, the metric on $G \times \mathbb{T}$ is right-invariant under the G-action, hence its average with respect to the $\mathbb{T}$-action is right-invariant under the G-action as well.

The averaged inertia operator $\mathbb{I}(g) : \mathfrak{t} = \mathbb{R} \to \mathfrak{t}^* = \mathbb{R}$ is $\mathbb{I}(g)\gamma = \epsilon^2\gamma$, while the averaged momentum map $J : T(G \times \mathbb{T}) \to \mathfrak{t}^* = \mathbb{R}$ is given by $J(g, \phi, v, \gamma) = \langle\epsilon^2\widetilde{\alpha}, v\rangle + \epsilon^2\gamma$, cf. Proposition 3.5. Finally, the averaged connection $\bar{\mathcal{A}} \in \Omega^1(G \times \mathbb{T}, \mathbb{R})$ on the principal $\mathbb{T}$-bundle $G \times \mathbb{T}$ is given by $\bar{\mathcal{A}}(g, \phi) = \widetilde{\alpha} + d\phi$.

According to Theorem 3.6 the magnetic term in the symplectic structure is $\beta_\mu = \mu\,\pi_G^* d\bar{\mathcal{A}} = \mu\,\pi_G^* d\widetilde{\alpha}$ for the corresponding $\mu \in \mathbb{R}$. Therefore, the symplectic structure on the reduced manifold $T^*G \cong J^{-1}(\mu)/\mathbb{T}$ is the 2-form $\omega_\mu = \omega_{can} - \mu\,\pi_G^* d\widetilde{\alpha}$. The correction in the effective potential U_μ of Hamiltonian function (5) is constant, and hence can be omitted. □

[3] This is based on the same consideration as Remark 3.4.

[4] As a matter of fact, $\mathcal{B}^*$ will turn out to be dual of a flat connection $\mathcal{B}$ in the bundle $G \times \mathbb{T} \to \mathbb{T}$, see Remark 6.8.

Remark 6.8. For the general case of a topologically nontrivial G-bundle $\pi : M \to N$, consider an open subset $\mathcal{O} \subset N$. Then locally in the base trivialize $M|_{\pi^{-1}(\mathcal{O})} \cong \mathcal{O} \times G$ and the cotangent bundle $T^*M|_{\pi^{-1}(\mathcal{O})} \cong T^*\mathcal{O} \times G \times \mathfrak{g}^*$. This decomposition of the tangent/cotangent spaces gives us a flat connection $\mathcal{B} \in \Omega^1(\mathcal{O}, \mathfrak{g})$ on $\mathcal{O} \times G$. Therefore, for the local representation $(x, g; y, \nu)$ of (q, p), we have $q = (x, g)$ and $p = (y + \mathcal{B}^*\nu, \nu)$, and then the Hamiltonian $H(q, p) = \frac{1}{2}(p, p) + U(\pi(q))$ in T^*M becomes

$$\begin{aligned} H(x, g; y, \nu) &= \tfrac{1}{2}(y + \mathcal{B}^*\nu, y + \mathcal{B}^*\nu)_x + \tfrac{1}{2}(\nu, \nu)_x + U(x) \\ &= \tfrac{1}{2}(y, y)_x + (\mathcal{B}^*\nu, y)_x + \tfrac{1}{2}(\mathcal{B}^*\nu, \mathcal{B}^*\nu)_x + \tfrac{1}{2}(\nu, \nu)_x + U(x). \end{aligned}$$

By combining the third and forth terms in the above expression of the general Hamiltonian and by switching the order of coordinates, we obtain the Hamiltonian (27) in the proof. Finally, by confining ourselves to x-periodic solutions in the base lying inside $\mathcal{O}$ one can reduce the setting to a Hamiltonian on $T^*(G \times \mathbb{T})$.

Remark 6.9. For the oscillating flow we consider the principal G-bundle $\pi : M \to N$ with $G = \mathrm{SDiff}(D)$ to be the group of volume-preserving diffeomorphisms. The infinite-dimensional manifold M is the space of all volume-preserving embeddings of the reference manifold D to $\mathbb{R}^n$, while N is the manifold of all boundaries of such embeddings, i.e. hypersurfaces diffeomorphic to ∂D and bounding diffeomorphic manifolds with the same volumes.

For the averaged metric, the 1-form $\widetilde{\alpha}$ on G is metric-related to a vector field V_s on G via $\widetilde{\alpha} = V_s^\flat$. This defines the 2-cocycle $\sigma_{V_s}(X, Y) := -\left\langle \mathcal{L}_X\, V_s^\flat, Y \right\rangle$ on the Lie algebra $\mathrm{SVect}(D)$ in Theorem 6.5.

6.3 Particles in rapidly oscillating potentials

It turns out that the motion of a particle in a rapidly oscillating potential field [3] can also be viewed in the context of the symplectic averaging-reduction setting. Moreover, in this example one observes both phenomena: additional terms in the effective potential and magnetic correction to the symplectic structure.

Namely, consider the Hamiltonian function

$$H(x, p, \omega t) = \frac{1}{2}\, p \cdot p + \epsilon^2 \mu^2\, U(x, \omega t), \tag{30}$$

where $x \in \mathbb{R}^n$, the potential function $U(x, \phi)$ is 2π-periodic with respect to (the fast variable) ϕ and the frequency $\omega = \mu/\epsilon$ is of order $1/\epsilon$. By ingenious repeated application of canonical transformations (see [3] for details, where $\mu = 1$) one obtains an averaged Hamiltonian up to the third order in ϵ:

$$\overline{H}(\overline{x}, \overline{p}) = \frac{1}{2}\overline{p} \cdot \overline{p} + \overline{U} + \frac{\epsilon^2 \mu^2}{2}\overline{V' \cdot V'} - \epsilon^3 \mu\, \overline{S'' V'}\, \overline{p}, \tag{31}$$

where $\overline{U}(x) = \frac{1}{2\pi}\int_0^{2\pi} U(x, \tau)\, d\tau$, $V(x, \tau) = \int^\tau U(x, \theta) - \overline{U}(x)\, d\theta$, and $S(x, \tau) = \int^\tau V(x, \theta)\, d\theta$ with the constant of integration chosen so that $\overline{V} = \overline{S} = 0$, and where prime denotes the derivative with respect to the space variable x.

On the other hand, the symplectic averaging-reduction procedure applied to the Hamiltonian (30) is as follows. Regard the fast variable $\phi \in \mathbb{T}$ as a new (periodic) coordinate, while the quotient along ϕ-fibers is the slow manifold with coordinates $\overline{x} \in \mathbb{R}^n$, mean values of the solutions. The $\mathbb{T}$-action on the principal $\mathbb{T}$-bundle $\pi : \mathbb{R}^n \times \mathbb{T} \to \mathbb{R}^n$ is given by $\phi \circ (\overline{x}, e^{i\phi'}) = (\overline{x}, e^{i(\phi + \phi')})$.

Theorem 6.10. *The averaged Hamiltonian system* (31) *for the natural system* (30) *is equivalent to the result of the symplectic averaging-reduction procedure, i.e. the Hamiltonian system on the reduced symplectic manifold* $(T^*\mathbb{R}^n, \omega_\mu)$ *with the Hamiltonian function:*

$$H_{slow}(\overline{x}, \overline{p}) = \frac{1}{2}\,\overline{p}\cdot\overline{p} + U_\mu(\overline{x}), \tag{32}$$

where $(\overline{x}, \overline{p}) \in T^*\mathbb{R}^n$*, and the effective potential is* $U_\mu(\overline{x}) = \overline{U}(\overline{x}) + \frac{\epsilon^2\mu^2}{2}\overline{V'\cdot V'}$*. The reduced symplectic structure is given by*

$$\omega_\mu = d\overline{x}\wedge d\overline{p} - \epsilon^3\mu\; d\{\overline{S''V'}^\flat\} \tag{33}$$

with the above notations for $\overline{U}, V, S$ *and the prime as above, and* $\flat$ *stands for the lifting indices operator* $\mathrm{Vect}(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ *corresponding to the Euclidean metric on* $\mathbb{R}^n$*.*

Proof. For the fast time $\tau = t/\epsilon$ let $(\widetilde{x}(\overline{x}, \tau, \widetilde{p}(\overline{x}, \tau)))$ be a solution of the Hamiltonian system corresponding to the function

$$\widetilde{H}(\widetilde{x}, \widetilde{p}) = \frac{1}{2}\,\widetilde{p}\cdot\widetilde{p} + \widetilde{U}(\overline{x} + \widetilde{x}, \tau)$$

and satisfying the periodicity and zero mean requirement in τ. (Recall that one splits any solution $x = \overline{x} + \widetilde{x}$ into the mean and periodic parts.) Upon discarding higher order terms in ϵ, the Newton's equation on $\widetilde{x}$ becomes

$$d^2\widetilde{x}/dt^2 = -\widetilde{U}'(\overline{x} + \widetilde{x}, \tau),$$

where the prime stands for the derivative with respect to the space variable. We consecutively obtain

$$d\widetilde{x}/dt = -\epsilon\int^\tau U'(\overline{x} + \widetilde{x}, \tau)\; d\tau = -\epsilon V' \qquad \text{and} \qquad \widetilde{x} = -\epsilon\int^\tau \epsilon\, V'(\overline{x} + \widetilde{x}, \tau)\; d\tau = -\epsilon^2 S'\,,$$

where one integrates by using the zero mean condition on the constants of integration.

Then the averaged metric is given by

$$((v,\gamma),(v,\gamma))_{(\overline{x},\phi)} = v\cdot v + 2\,\epsilon\gamma\,\langle\overline{S''V'}, v\rangle\,(\overline{V'\cdot V'})^{-1} + \gamma^2(\epsilon^2\overline{V'\cdot V'})^{-1},$$

where $(v,\gamma) \in T_{\overline{x}}\mathbb{R}^n \times T_\phi\mathbb{T} \simeq T_{\overline{x}}\mathbb{R}^n \times \mathbb{R}$. Note that the above metric has the form (4) discussed in Remark 3.4.

The corresponding fiber inertia operator $\mathbb{I}(\overline{x}) : \mathfrak{t} = \mathbb{R} \to \mathfrak{t}^* = \mathbb{R}$, $\overline{x} \in M$ is given by $\mathbb{I}(\overline{x})\gamma = \gamma(\epsilon^2\overline{V'\cdot V'})^{-1}$ for $\gamma \in \mathbb{R}$. Then, as follows from Proposition 3.5, the momentum map $J : T(\mathbb{R}^n \times \mathbb{T}) \to \mathfrak{t}^* = \mathbb{R}$ for $Q = \mathbb{R}^n \times \mathbb{T}$ is given by $J(\bar{x}, \phi, v, \gamma) = \gamma(\epsilon^2\overline{V'\cdot V'})^{-1} + \epsilon\langle\overline{S''V'}, v\rangle\,(\overline{V'\cdot V'})^{-1}$, where v is the image of p under the metric identification.

Finally, the averaged connection $\bar{\mathcal{A}} \in \Omega^1(\mathbb{R}^n \times \mathbb{T}, \mathbb{R})$ on the principal $\mathbb{T}$-bundle $\mathbb{R}^n \times \mathbb{T}$ is given by $\bar{\mathcal{A}}(\bar{x}, v, \phi, \gamma) = d\phi + \epsilon^3\overline{S''V'}^\flat$.

We choose μ to be the value of the momentum map. By Theorem 3.6, the reduced symplectic structure on the reduced manifold $T^*\mathbb{R}^n = J^{-1}(\mu)/\mathbb{T}$ has a magnetic term

$$\omega_\mu = d\overline{x}\wedge d\overline{p} - \epsilon^3\mu\, d\{\overline{S''V'}d\overline{x}\}.$$

The reduced Hamiltonian function on $(T^*\mathbb{R}^n, \omega_\mu)$ turns out to be

$$H_{slow}(\overline{x}, \overline{p}) = \frac{1}{2}\,\overline{p}\cdot\overline{p} + U_\mu(\overline{x})$$

at any $(\overline{x}, \overline{p}) \in T^*M$, where the effective potential is $U_\mu(\overline{x}) = \overline{U}(\overline{x}) + \frac{\epsilon^2\mu^2}{2}\overline{V'\cdot V'}$. □

Thus the system obtained by applying the symplectic averaging-reduction procedure is equivalent to the one obtained via the classical averaging method in [3].

Remark 6.11. It remains an open question to describe the magnetic term related to the curvature of an appropriate $\mathbb{T}$-bundle in purely geometric terms, similar to the gyroscope description in [4].

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