Dynamics of symplectic fluids and point vortices

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Abstract

We present the Hamiltonian formalism for the Euler equation of symplectic fluids, introduce symplectic vorticity, and study related invariants. In particular, this allows one to extend D. Ebin's long-time existence result for geodesics on the symplectomorphism group to metrics not necessarily compatible with the symplectic structure. We also study the dynamics of symplectic point vortices, describe their symmetry groups and integrability.

In 1966 V. Arnold showed how the Euler equation describing dynamics of an ideal incompressible fluid on a Riemannian manifold can be viewed as a geodesic equation on the group of volume-preserving diffeomorphisms of this manifold [1]. Consider a similar problem for a symplectic fluid.

Let (M^{2m}, ω) be a closed symplectic manifold equipped with a Riemannian metric. A symplectic fluid filling M is an ideal fluid whose motions preserve not only the volume element, but also the symplectic structure ω. (In 2D symplectic and ideal incompressible fluids coincide.) Such motions are governed by the corresponding Euler-Arnold equation, i.e. the equation describing geodesics on the infinite-dimensional group $\mathrm{Symp}_\omega(M)$ of symplectomorphisms of M with respect to the right-invariant L^2-metric. The corresponding problem of studying this dynamics was posed in [2] (see Section IV.8).

Recently D. Ebin [4] considered the corresponding Euler equation of the symplectic fluid and proved the existence of solutions for all times for compatible metrics and symplectic structures. His proof uses the existence of a pointwise invariant transported by the flow, similar to the vorticity function in 2D. This symplectic vorticity allows one to proceed with the existence proof in the symplectic case similarly to the 2D setting.

The purpose of this note is three-fold. First, we describe the Hamiltonian formalism of the Euler-Arnold equation for symplectic fluids, the corresponding dual spaces, inertia operators, and Casimir invariants. This formalism manifests a curious duality to the incompressible case: its natural setting is a quotient space of $(2m-1)$-forms for symplectic fluids vs. that of 1-forms for incompressible ones. Second, we show the geometric origin of the symplectic vorticity arising from the general approach to ideal fluids. In particular we prove that this quantity is a pointwise invariant for any metric, not only for a metric compatible with the symplectic structure, which allows one to extend the corresponding long-time existence theorem for solutions of the symplectic Euler equation, see [4]. We also present a variational description of the Hamiltonian stationary

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flows. Finally, we present and study the problems of symplectic filaments and symplectic point vortices. The Hamiltonian of the latter problem has more similarities with the N-body problem in celestial mechanics than that of the classical problem of N point vortices on the plane. We prove that this problem of symplectic vortices is completely integrable for $N = 2$ point vortices in any dimension. We conjecture that it is not completely integrable for $N \geq 3$ in the Arnold-Liouville sense, similarly to the N-body problem in celestial mechanics.

To make the paper relatively self-contained we recall the setting of an ideal fluid and classical point vortices and compare them to the new symplectic framework. While the analytical side of the problem was explored in [4], in this paper we are concerned with the geometric and Hamiltonian aspects of the problem.

1 Arnold's framework for the Euler equation of an ideal incompressible fluid

We start with a brief reminder of the main setting of an ideal hydrodynamics. Consider the Euler equation for an inviscid incompressible fluid filling some domain M in $\mathbb{R}^n$. The fluid motion is described as an evolution of the fluid velocity field $v(t, x)$ in M which is governed by the *classical Euler equation*:

$$\partial_t v + v \cdot \nabla v = -\nabla p\,. \tag{1.1}$$

Here the field v is assumed to be divergence-free ($\operatorname{div} v = 0$) and tangent to the boundary of M. The pressure function p is defined uniquely modulo an additive constant by these restrictions on the velocity v.

The same equation describes the motion of an ideal incompressible fluid filling an arbitrary Riemannian manifold M equipped with a volume form μ [1, 5]. In the latter case v is a divergence-free vector field on M, while $v \cdot \nabla v$ stands for the Riemannian covariant derivative $\nabla_v v$ of the field v in the direction of itself, while the divergence $\operatorname{div} v$ is taken with respect to the volume form μ.

The Euler equation as a geodesic flow

Equation (1.1) has a natural interpretation as a geodesic equation. Indeed, the flow $(t, x) \to g(t, x)$ describing the motion of fluid particles is defined by its velocity field $v(t, x)$:

$$\partial_t g(t, x) = v(t, g(t, x)),\ \ g(0, x) = x.$$

The chain rule immediately gives $\partial^2_{tt} g(t, x) = (\partial_t v + v \cdot \nabla v)(t, g(t, x))$, and hence the Euler equation is equivalent to

$$\partial^2_{tt} g(t, x) = -(\nabla p)(t, g(t, x)),$$

while the incompressibility condition is $\det(\partial_x g(t, x)) = 1$. The latter form of the Euler equation (for a smooth flow $g(t, x)$) means that it describes a geodesic on the set $\operatorname{Diff}_\mu(M)$ of volume-preserving diffeomorphisms of the manifold M. Indeed, the acceleration of the flow, $\partial^2_{tt} g$, being given by a gradient, $-\nabla p$, is L^2-orthogonal to all divergence-free fields, which constitute the tangent space to this set $\operatorname{Diff}_\mu(M)$.

In the case of any Riemannian manifold M the Euler equation defines the geodesic flow on the group of volume-preserving diffeomorphisms of M with respect to the right-invariant L^2-metric on $\operatorname{Diff}_\mu(M)$. (A proper analysis framework deals with the Sobolev H^s spaces of vector fields with $s > \frac{n}{2} + 1$ and it is described in [5].)

In the Euler equation for a symplectic fluid the same derivation replaces the gradient term ∇p in (1.1) by a vector field q which is L^2-orthogonal to the space of all symplectic vector fields and can be found in terms of the Hodge decomposition.

Arnold's framework for the Euler-type equations

In [1] V. Arnold suggested the following general framework for the Euler equation on an arbitrary group describing a geodesic flow with respect to a suitable one-sided invariant Riemannian metric on this group.

Consider a (possibly infinite-dimensional) Lie group G, which can be thought of as the configuration space of some physical system. The tangent space at the identity of the Lie group G is the corresponding Lie algebra $\mathfrak{g}$. Fix some (positive definite) quadratic form, the "energy," $E(v) = \frac{1}{2}\langle v, Av\rangle$ on $\mathfrak{g}$ and consider right translations of this quadratic form to the tangent space at any point of the group (the "translational symmetry" of the energy). This way the energy defines a right-invariant Riemannian metric on the group G. The geodesic flow on G with respect to this energy metric represents the extremals of the least action principle, i.e., the actual motions of our physical system.

To describe a geodesic on the Lie group with an initial velocity $v(0) = \xi$, we transport its velocity vector at any moment t to the identity of the group (by using the right translation). This way we obtain the evolution law for $v(t)$, given by a (non-linear) dynamical system $dv/dt = F(v)$ on the Lie algebra $\mathfrak{g}$. The system on the Lie algebra $\mathfrak{g}$, describing the evolution of the velocity vector along a geodesic in a right-invariant metric on the Lie group G, is called the *Euler* (or *Euler-Arnold*) *equation* corresponding to this metric on G.

The Euler equation on the Lie algebra $\mathfrak{g}$ has a more explicit Hamiltonian reformulation on the dual space $\mathfrak{g}^*$ of the Lie algebra $\mathfrak{g}$. Identify the Lie algebra and its dual with the help of the above quadratic form $E(v) = \frac{1}{2}\langle v, Av\rangle$. This identification $A : \mathfrak{g} \to \mathfrak{g}^*$ (called the *inertia operator*) allows one to rewrite the Euler equation on the dual space $\mathfrak{g}^*$.

It turns out that the Euler equation on $\mathfrak{g}^*$ is Hamiltonian with respect to the natural Lie–Poisson structure on the dual space [1]. Moreover, the corresponding Hamiltonian function is the energy quadratic form lifted from the Lie algebra to its dual space by the same identification: $H(m) = \frac{1}{2}\langle A^{-1}m, m\rangle$, where $m = Av$. Here we are going to take it as the *definition* of the Euler equation on the dual space $\mathfrak{g}^*$.

Definition 1.1. (see, e.g., [2]) *The* Euler equation on $\mathfrak{g}^*$*, corresponding to the right-invariant metric* $E(m) = \frac{1}{2}\langle Av, v\rangle$ *on the group, is given by the following explicit formula:*

$$\frac{dm}{dt} = -\mathrm{ad}^*_{A^{-1}m} m, \tag{1.2}$$

as an evolution of a point $m \in \mathfrak{g}^*$.

Here ad^* is the operator of the coadjoint representation of the Lie algebra $\mathfrak{g}$ on its dual $\mathfrak{g}^*$: $\langle \mathrm{ad}^*_v u, w\rangle := \langle u, [v, w]\rangle$ for any elements $v, w \in \mathfrak{g}$ and $u \in \mathfrak{g}^*$.

2 Hamiltonian approach to incompressible fluids

In this section we recall the Hamiltonian framework for the classical Euler hydrodynamics of an incompressible fluid, which we are going to generalize to symplectic fluids in the next section.

Let M be an n-dimensional Riemannian manifold with a volume form μ and filled with an ideal incompressible fluid. The corresponding Lie group $G = \mathrm{Diff}_\mu(M)$ is the group of volume-preserving diffeomorphisms of M. The corresponding Lie algebra $\mathfrak{g} = \mathrm{Vect}_\mu(M)$ consists of divergence-free vector field in M: $\mathrm{Vect}_\mu(M) = \{v \in \mathrm{Vect}(M) \mid L_v\mu = 0\}$.

Equip the group G with the right-invariant metric by using the L^2-product on divergence-free vector fields on M. The formalism of the hydrodynamical Euler equation can be summarized in the following theorem:

Theorem 2.1. (see [2]) *a) The (regular) dual space $\mathfrak{g}^* = \Omega^1(M)/\Omega^0(M)$ is the space of cosets of 1-forms on M modulo exact 1-forms. The group coadjoint action is the change of coordinates in the 1-form, while the corresponding Lie algebra coadjoint action is the Lie derivative along a vector field: $\mathrm{ad}^*_v = L_v$. Its action on cosets $[u] \in \Omega^1/d\Omega^0$ is well-defined.*

b) The inertia operator is lifting the indices: $A : v \mapsto [v^\flat]$, where one considers the coset of the 1-form $v^\flat$ on M. More precisely, for a manifold M equipped with a Riemannian metric $(.,.)$ one defines the 1-form $v^\flat$ as the pointwise inner product with vectors of the velocity field v: $v^\flat(\eta) := (v, \eta)$ for all $\eta \in T_xM$.

c) The Euler equation (1.2) on the dual space has the form

$$\partial_t[u] = -L_v[u],$$

where $[u] \in \Omega^1(M)/\Omega^0(M)$ stands for a coset of 1-forms and the vector field v is related with the 1-form u by means of the Riemannian metric on M: $u = v^\flat$.

The idea of the proof is that the map $v \mapsto i_v\mu$ provides an isomorphism of the space of divergence-free vector fields and the space of closed $(n-1)$-forms on M: $\mathrm{Vect}_\mu(M) \cong Z^{n-1}(M)$, since $d(i_v\mu) = L_v\mu = 0$. Then the dual space to the Lie algebra $\mathfrak{g} = Z^{n-1}(M)$ is $\mathfrak{g}^* = \Omega^1(M)/d\Omega^0(M)$, and the pairing is

$$\langle v, [u] \rangle := \int_M (i_v u)\, \mu\,.$$

The coadjoint action is the change of coordinates in differential forms. The substitution of the inertia and coadjoint operators to the formula (1.2) yields the Euler equation.

Remark 2.2. The Euler equation for a coset $[u]$ can be rewritten as an equation for a representative 1-form modulo a function differential dp:

$$\partial_t u + L_v u = -dp,$$

where one can recognize the elements of the Euler equation (1.1) for an ideal fluid.

Note that each coset $[u]$ has the unique 1-form $\bar{u} \in [u]$ related to a *divergence-free* vector field by means of the metric. This is a coclosed 1-form: $\delta\bar{u} = 0$ on M. Such a choice of the coclosed representative $\bar{u} \in [u]$ determines the pressure p uniquely (modulo a constant), since this condition prescribes $\Delta p := \delta d\, p$ for each time t.

Remark 2.3. Define the vorticity 2-form $\xi = du$. It is well defined for a coset $[u]$. The vorticity form of the Euler equation is

$$\partial_t\xi = -L_v\xi,$$

which means that the vorticity 2-form ξ is transported by the flow. The frozenness of the vorticity form allows one to define various invariants of the hydrodynamical Euler equation.

Remark 2.4. Recall, that the Euler equation of an ideal fluid (1.1) filling a three-dimensional simply connected manifold M has the helicity (or asymptotic Hopf) invariant. Topologically helicity in 3D describes the mutual linking of the trajectories of the vorticity field $\operatorname{curl} v$, and has the form $J(v) = \int_{M^3} (\operatorname{curl} v,\ v)\ \mu$.

For an ideal 2D fluid one has an infinite number of conserved quantities, so called *enstrophies*:

$$J_k(v) = \int_{M^2} (\operatorname{curl}\ v)^k\ \mu \qquad \text{for } k = 1, 2, \dots,$$

where $\operatorname{curl}\ v := du/\mu = \partial v_1/\partial x_2 - \partial v_2/\partial x_1$ is a *vorticity function* of a 2D flow.

It turns out that enstrophy-type integrals exist for all even-dimensional flows, and so do helicity-type integrals for all odd-dimensional ideal fluid flows, see e.g. [11, 2]. The invariance of the helicity and enstrophies follows, in fact, from their coordinate-free definition: they are invariant with respect to volume-preserving coordinate changes, and hence, are first integrals of the corresponding Euler equations.

3 Hamiltonian approach to symplectic fluids

Let (M^{2m}, ω) be a closed symplectic manifold of dimension $n = 2m$ which is also equipped with a Riemannian metric. Consider the dynamics of a fluid in M preserving the symplectic 2-form ω.

The configuration space of a symplectic fluid on M is the Lie group $G = \mathrm{Symp}_\omega(M)$ and it is equipped with the right-invariant L^2-metric. It is a subgroup of the group of volume-preserving diffeomorphisms: $G = \mathrm{Symp}_\omega(M) \subset \mathrm{Diff}_\mu(M)$, where $\mu = \omega^m/m$ is the symplectic volume. We will also assume that the metric volume coincides with the symplectic one, but do not assume that the symplectic structure and metric are compatible.

The corresponding Lie algebra is $\mathfrak{g} = \mathrm{symp}_\omega(M) = \{v \in \mathrm{Vect}(M) \mid L_v\omega = 0\}$.

Theorem 3.1. *a) The (regular) dual space to* $\mathfrak{g} = \mathrm{symp}_\omega(M)$ *is* $\mathfrak{g}^* = \mathrm{symp}^*_\omega(M) \cong \Omega^{n-1}(M)/d\Omega^{n-2}(M)$. *The pairing between* $v \in \mathfrak{g} = \mathrm{symp}_\omega(M)$ *and* $\alpha \in \Omega^{n-1}(M)$ *is given by the formula*

$$\langle v, \alpha \rangle := \int_M \alpha \wedge i_v\omega\,, \tag{3.3}$$

and it is well-defined on cosets $[\alpha] \in \Omega^{n-1}(M)/d\Omega^{n-2}(M)$.

The algebra coadjoint action is the Lie derivative: $\mathrm{ad}^*_v = L_v$ *and it is well-defined on cosets.*

b) The inertia operator A is lifting the indices and wedging with ω^{m-1}, i.e. $A : v \mapsto [v^\flat \wedge \omega^{m-1}]$, *where $v^\flat$ is the 1-form on M and one takes the corresponding coset of the form* $\alpha = v^\flat \wedge \omega^{m-1} \in \Omega^{2m-1}(M)$.

c) The Euler equation (1.2) on the dual space has the form

$$\partial_t[\alpha] = -L_v[\alpha],$$

where the vector field v is related to the $(n-1)$-form α by means of the inertia operator: $\alpha = v^\flat \wedge \omega^{m-1}$ *and* $[\alpha] \in \Omega^{n-1}/d\Omega^{n-2}$ *stands for its coset.*

Proof. The Lie algebra $\mathrm{symp}_\omega(M)$ is naturally isomorphic to the space of closed 1-forms $Z^1(M)$. Indeed, the requirement for a field v to be symplectic, $0 = L_v\omega = i_v d\omega + d i_v\omega = d i_v\omega$, is equivalent to closedness of the 1-form $i_v\omega$ since $d\omega = 0$, while each closed 1-form on M can be obtained as $i_v\omega$ due to the nondegeneracy of the symplectic form ω.

The isomorphism $\mathrm{symp}_\omega(M) \cong Z^1(M)$ implies that $\mathrm{symp}^*_\omega(M) \cong \Omega^{n-1}(M)/d\Omega^{n-2}(M)$ for the (regular) dual spaces with pairing given by wedging of differential forms.

The explicit expression for the inertia operator follows from the following transformations:

$$E(v) = \frac{1}{2}\int_M (v,v)\,\frac{\omega^m}{m} = \frac{1}{2m}\int_M (i_v v^\flat)\,\omega^m = \frac{1}{2m}\int_M v^\flat \wedge i_v\omega^m = \frac{1}{2}\int_M v^\flat \wedge \omega^{m-1} \wedge i_v\omega\,.$$

Hence the energy $E(v) = \frac{1}{2}\langle v, Av\rangle$ is given by the inertia operator $Av = [v^\flat \wedge \omega^{m-1}]$, due to the pairing (3.3) between symplectic fields and cosets of $(n-1)$-forms.

Note that for $m = 1$, the 2D case, the inertia operator $A : v \mapsto [v^\flat]$ coincides with that for an ideal incompressible 2D fluid. □

Remark 3.2. In terms of representatives in the cosets one obtains the equation on $(n-1)$-forms modulo exact forms:

$$\partial_t \alpha + L_v \alpha = -d\beta, \tag{3.4}$$

where $\beta \in \Omega^{n-2}$, and by applying the inverse inertia operator one arrives at the symplectic Euler equation. One can show that each coset $[\alpha]$ has a unique 1-form $\bar{\alpha} \in [\alpha]$ related to a symplectic vector field by means of the metric.

Symplectic vorticity

Definition 3.3. *Define the* symplectic vorticity *for a symplectic vector field v to be the n-form $\xi := d\alpha \in \Omega^n(M)$, where $n = 2m = \dim M$, while the field v and the $(n-1)$-form α are related by the inertia operator: $[\alpha] = [v^\flat \wedge \omega^{m-1}]$.*

The symplectic vorticity function *ν is the ratio $\nu := d\alpha/\omega^m$ of the symplectic vorticity form and the symplectic volume.*

Proposition 3.4. *Both the symplectic vorticity form ξ and symplectic vorticity function ν are transported by the symplectic flow.*

Indeed, by taking the differential of both sides of the Euler equation (3.4) we obtain the vorticity form of the symplectic Euler equation

$$\partial_t \xi = -L_v \xi\,,$$

which expresses the fact that the n-form $\xi = d\alpha$ is transported by the symplectic flow. The symplectic vorticity function ν is also transported by the flow, just like in the ideal 2D case:

$$\partial_t \nu = -L_v \nu\,,$$

since the symplectic volume ω^m is invariant under the flow.

This geometric observation allows one to extend Ebin's theorem to symplectic manifolds with metrics not necessarily compatible with symplectic structures.

Corollary 3.5. (cf. [4]) *The solutions of the symplectic Euler equation* (3.4) *on a closed Riemannian symplectic manifold (M^{2m}, ω) in spaces H^s with $s > \frac{n}{2} + 1$ exist for all t for any metric whose volume element coincides with the symplectic volume.*

Proof. The corresponding theorem is proved by D. Ebin in [4] for metrics g compatible with the symplectic structure ω, i.e. for which there is an almost complex structure $\mathbf{J}$ on the manifold M, so that $\omega(v, w) = g(\mathbf{J}v, w)$, see e.g. [7].

The proof is based on the existence of an invariant quantity, similar to the vorticity of an ideal incompressible 2D fluid, which allowed one to reduce the existence questions in the symplectic case to similar questions in the 2D case and to adapt the corresponding 2D long-time existence proof to the symplectic setting. The compatibility is used in the proof of the invariance of this quantity.

On the other hand, the symplectic vorticity ν is exactly the invariant quantity defined in the paper by D. Ebin, and the above geometric point of view proves its frozenness into symplectic fluid without the requirement of compatibility. □

Remark 3.6. The transported symplectic vorticity also allows one to obtain infinitely many conserved quantities (Casimirs):

$$I_k(\alpha) = \int_M \nu^k \, \omega^m \qquad \text{for any} \quad k = 1, 2, 3, ...,$$

which are invariants of the symplectic Euler equation for any metric and symplectic form on M.

Remark 3.7. Since symplectic vector fields form a Lie subalgebra in divergence-free ones, $\mathrm{symp}_\omega(M) \subset \mathrm{Vect}_\mu(M)$ for $\mu = \omega^m/m$, there is the natural projection of the dual spaces $\mathrm{Vect}^*_\mu \to \mathrm{symp}^*_\omega$, i.e. the projection $\Omega^1/d\Omega^0 \to \Omega^{n-1}/d\Omega^{n-2}$ given by $[u] \mapsto [u \wedge \omega^{m-1}]$. This projection respects the coadjoint action, while the vorticity 2-form of an ideal incompressible fluid under the projection becomes the symplectic vorticity. This is yet one more way to check that the symplectic vorticity is frozen into a symplectic flow for any metric on M.

Remark 3.8. We also mention the necessary changes to describe Hamiltonian fluids. Now for a closed symplectic manifold (M^{2m}, ω) consider the group of Hamiltonian diffeomorphisms $G = \mathrm{Ham}_\omega(M)$, i.e. those diffeomorphisms that are attainable from the identity by Hamiltonian vector fields.

Its Lie algebra is $\mathfrak{g} = \mathrm{ham}_\omega(M) = \{v \in \mathrm{Vect}(M) \mid i_v\omega = dH\}$. By definition, this Lie algebra is naturally isomorphic to the space of exact 1-forms, $\mathrm{ham}_\omega(M) \cong d\Omega^0(M) \cong C^\infty(M)/\{\text{constants}\}$.

Then its regular dual space is $\mathfrak{g}^* = \mathrm{ham}_\omega(M)^* = \Omega^{n-1}(M)/Z^{n-1}(M)$, i.e. the space of all $(n-1)$-forms modulo closed ones. The coadjoint action is again the Lie derivative $\mathrm{ad}^*_v = L_v$. The energy and inertia operators are the same as for symplectic fluids.

Note that the dual space $\mathfrak{g}^* = \Omega^{n-1}(M)/Z^{n-1}(M)$ is naturally isomorphic to exact n-forms $d\Omega^{n-1}(M)$ on M, since for $\alpha \in \Omega^{n-1}(M)$ and cosets defined this way the map $[\alpha] \mapsto d\alpha$ is an isomorphism. Hence the Euler equation $\partial_t[\alpha] = -L_v[\alpha]$ is now *equivalent* to its vorticity formulation:

$$\partial_t \nu = -L_v \nu \,,$$

for the symplectic vorticity function $\nu = d\alpha/\omega^m$.

Finally, note that for a Hamiltonian field v one can rewrite the above equation with the help of the Poisson bracket on the symplectic manifold (M, ω) as follows

$$\partial_t \nu = \{\psi, \nu\} \,,$$

where ψ is the Hamiltonian function for the velocity field $v = \ \mathrm{sgrad}\, \psi$, while ν is its symplectic vorticity function and they are related via $\Delta\psi = \nu$. Indeed, the latter relation between ψ and ν is equivalent to the relation furnished by the inertia operator: $dv^\flat \wedge \omega^{m-1} = \nu \cdot \omega^m$. This shows that the above Poisson-bracket-formulation of the 2D Euler equation is valid for the Hamiltonian fluid in any dimension.

Hamiltonian steady flows

Steady solutions ($\partial_t \nu = 0$) to the Euler equation for Hamiltonian fluids are given by those vector fields on M whose Hamiltonians ψ Poisson commute with their Laplacians $\nu = \Delta\psi$:

$$\{\psi, \nu\} = 0\,.$$

For generic Hamiltonians in 2D this means that they are functionally dependent with their Laplacians, cf. [1], the fact used by V.Arnold in the 60s to obtain stability conditions in ideal hydrodynamics. In higher dimensions this merely means that that the two functions ψ and $\nu = \Delta\psi$ are in involution with respect to the natural Poisson bracket on M. For generic Hamiltonians in 4D this implies that the corresponding steady flows represent integrable systems with two degrees of freedom, similarly to the case of an incompressible 4D fluid studied in [6].

For a Riemannian symplectic manifold M consider the Dirichlet functional

$$D(\psi) := \int_M (\text{sgrad }\psi, \text{sgrad }\psi)\,\omega^m$$

on Hamiltonian functions on M obtained from a given function ψ_0 by the action of Hamiltonian diffeomorphisms.

Proposition 3.9. *Smooth extremals (in particular, smooth minimizers) of the Dirichlet functional with respect to the action of the Hamiltonian diffeomorphisms on functions are given by Hamiltonians of steady vector fields, i.e. Hamiltonian functions ψ satisfying*

$$\{\psi, \Delta\psi\} = 0\,.$$

Proof. Indeed, the Dirichlet functional is (up to a factor) the energy functional $E = \frac{1}{2m}\int_M (v, v)\,\omega^m$ on Hamiltonian fields. The variational problem described above can be reformulated as funding extrema of E on the group *adjoint* orbit containing sgrad ψ_0 in the Lie algebra $\text{ham}_\omega(M)$. These extrema are in 1-1 correspondence with extrema of the energy functional on the group *coadjoint* orbits, see the general theorem in [2], section II.2.C. In turn, extrema for the kinetic energy on coadjoint orbits in $\text{ham}^*_\omega(M)$ are given by stationary Hamiltonian fields. □

Note that for metrics compatible with the symplectic structure the above functional becomes the genuine Dirichlet functional on functions: $D(\psi) := \int_M (\nabla\psi, \nabla\psi)\,\omega^m$. One can also give similar variational and direct descriptions for steady symplectic fields, i.e. for the group $\text{Symp}_\omega(M)$.

4 Classical and symplectic point vortices

In this section we recall several facts about the classical problem of point vortices in the 2D plane and consider the symplectic analog of point vortices for higher-dimensional symplectic spaces. For an ideal incompressible 2D fluid the systems of 2 and 3 point vortices are known to be completely integrable, while systems of ≥ 4 point vortices are not. It turns out that the corresponding evolution of symplectic vortices in higher dimension $2m > 2$ is integrable for $N = 2$ and presumably it is non-integrable for $N \geq 3$. This generalized system of symplectic vortices, in a sense, looks more like a many-body problem in space, which is non-integrable already in the three-body case.

Consider the 2D Euler equation in the vorticity form:

$$\dot{\nu} = \{\psi, \nu\}\,,$$

where ν is the vorticity function and the stream function (or Hamiltonian) ψ of the flow satisfies $\Delta\psi = \nu$. The same equation governs the evolution of the symplectic vorticity ν of a Hamiltonian fluid on any symplectic manifold (M^{2m}, ω), where the symplectic vorticity ν and the Hamiltonian function ψ of the flow are related in the same way: $\Delta\psi = \nu$ and Δ is the Laplace-Beltrami operator, see the preceding section.

Now we consider the symplectic space $\mathbb{R}^{2m}$ with the standard symplectic structure $\omega = \sum_{\alpha=1}^{m} dx_\alpha \wedge dy_\alpha = d\tilde{x} \wedge d\tilde{y}$. Let the symplectic vorticity ν be supported on N point vortices: $\nu = \sum_{j=1}^{N} \Gamma_j \, \delta(\tilde{z} - \tilde{z}_j)$, where $\tilde{z}_j = (\tilde{x}_j, \tilde{y}_j)$ are coordinates of the jth point vortex in the space $\mathbb{R}^{2m} = \mathbb{C}^m$ with $m \geq 1$.

Theorem 4.1. *The evolution of vortices according to the Euler equation is described by the system*

$$\Gamma_j \dot{x}_{j,\alpha} = \frac{\partial \mathcal{H}}{\partial y_{j,\alpha}}, \qquad \Gamma_j \dot{y}_{j,\alpha} = -\frac{\partial \mathcal{H}}{\partial x_{j,\alpha}}, \qquad 1 \leq j \leq N, \quad 1 \leq \alpha \leq m\,.$$

This is a Hamiltonian system on $(\mathbb{R}^{2m})^N$ *with the Hamiltonian function*

$$\mathcal{H} = 2 \cdot C(2m) \sum_{j<k}^{N} \Gamma_j \Gamma_k \, |\tilde{z}_j - \tilde{z}_k|^{2-2m} \qquad \textit{for } m > 1 \qquad \textit{or}$$

$$\mathcal{H} = -\frac{1}{4\pi} \sum_{j<k}^{N} \Gamma_j \Gamma_k \, \ln |\tilde{z}_j - \tilde{z}_k|^2 \qquad \textit{for } m = 1\,.$$

Here the distance $|\tilde{z}_j - \tilde{z}_k|$ is defined in $(\mathbb{R}^{2m})^N$, the constant $C(2m)$ is the constant of the Laplace fundamental solution in $\mathbb{R}^{2m}$, and the Poisson structure is given by the bracket

$$\{f, g\} = \sum_{j=1}^{N} \frac{1}{\Gamma_j} \left(\frac{\partial f}{\partial \tilde{x}_j} \frac{\partial g}{\partial \tilde{y}_j} - \frac{\partial f}{\partial \tilde{y}_j} \frac{\partial g}{\partial \tilde{x}_j} \right) = \sum_{j=1}^{N} \frac{1}{\Gamma_j} \sum_{\alpha=1}^{m} \left(\frac{\partial f}{\partial x_{j,\alpha}} \frac{\partial g}{\partial y_{j,\alpha}} - \frac{\partial f}{\partial y_{j,\alpha}} \frac{\partial g}{\partial x_{j,\alpha}} \right).$$

The case $m = 1$ goes back to Kirchhoff. The case $m > 1$ apparently did not appear in the literature before.

Proof. Any (non-autonomous) Hamiltonian equation $\dot{z} = \operatorname{sgrad} H(t, z)$ in a symplectic manifold M has an alternative (Liouville) version: $\dot{\rho} = \{H, \rho\}$ for any smooth function ρ on M. The reduction of the Liouville version to the Hamiltonian one is obtained by taking the limit as ρ tends to a delta-function supported at a given point $z \in M$.

In particular, the vorticity equation $\dot{\nu} = \{\psi, \nu\}$ describes a Hamiltonian equation on $\mathbb{R}^{2m}$ with instantaneous Hamiltonian function ψ. By assuming that ν is of the form $\nu = \sum_{j=1}^{N} \Gamma_j \, \delta(\tilde{z} - \tilde{z}_j), \quad \tilde{z} \in \mathbb{R}^{2m}$, one obtains the instantaneous Hamiltonian

$$\psi = \Delta^{-1}\nu = C(2m) \sum_{j=1}^{N} \Gamma_j \, |\tilde{z} - \tilde{z}_j|^{2-2m}\,.$$

Thus the corresponding Hamiltonian form of the vorticity equation for point vortices is $\dot{\tilde{z}}_j = \operatorname{sgrad} \psi|_{\tilde{z}=\tilde{z}_j}$. Now the straightforward differentiation of ψ at $\tilde{z} = \tilde{z}_j$ (in which one discards the singular term at $\tilde{z}_j$ itself) shows that the equation for $\dot{\tilde{z}}_j$ coincides with the Hamiltonian vector field for the Hamiltonian $\mathcal{H}$ on $(\mathbb{R}^{2m})^N$.

Indeed, for the standard complex structure $\mathbf{J}$ in $\mathbb{R}^{2m} = \mathbb{C}^m$, $m > 1$ one has

$$\text{sgrad}\,\psi|_{\tilde{z}=\tilde{z}_j} = \mathbf{J}\,\text{grad}\,|_{\tilde{z}=\tilde{z}_j}\left(C(2m)\cdot\sum_{k=1,k\neq j}^{N}\Gamma_k\,|\tilde{z}-\tilde{z}_k|^{2-2m}\right) = \frac{1}{\Gamma_j}\,\mathbf{J}\,\frac{\partial\mathcal{H}}{\partial\tilde{z}_j}\,.$$

as required. □

Theorem 4.2. *The above dynamical system of symplectic vortices is invariant with respect to the group $E(2m) := U(m) \ltimes \mathbb{R}^{2m}$ of unitary motions of $\mathbb{R}^{2m} = \mathbb{C}^m$. The corresponding Noether $m^2 + 2m$ conserved quantities, which commute with $\mathcal{H}$, are:*

$$Q_\alpha = \sum_{j=1}^{N}\Gamma_j x_{j,\alpha}\,, \qquad P_\alpha = \sum_{j=1}^{N}\Gamma_j y_{j,\alpha} \qquad \textit{for } 1 \le \alpha \le m\,,$$

$$F^+_{\alpha\beta} = \sum_{j=1}^{N}\Gamma_j(x_{j,\alpha}x_{j,\beta} + y_{j,\alpha}y_{j,\beta}) \qquad \textit{for } 1 \le \alpha \le \beta \le m\,, \quad \textit{and}$$

$$F^-_{\alpha\beta} = \sum_{j=1}^{N}\Gamma_j(x_{j,\alpha}y_{j,\beta} - x_{j,\beta}y_{j,\alpha}) \qquad \textit{for } 1 \le \alpha < \beta \le m\,.$$

Proof. The standard complex structure $\mathbf{J}$ in $\mathbb{R}^{2m} = \mathbb{C}^m$ is compatible with the symplectic and Euclidean structures on the space. Since unitary motions, i.e. group elements of $E(2m) := U(m) \ltimes \mathbb{R}^{2m}$, preserve the Euclidean and complex structures, they also preserve the symplectic one and hence preserve the equation, which is defined in terms of these structures. The above quantities Q_α and P_α are generators of translations in the plane (x_α, y_α), while $F^\pm_{\alpha\beta}$ generate unitary rotations in the space $(x_\alpha, y_\alpha, x_\beta, y_\beta)$.

Indeed, rewrite the Poisson structure and integrals in the $(z, \bar{z})$-variables for $z = x + iy$. Namely, for the Poisson structure $2\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = i\frac{\partial}{\partial \bar{z}} \wedge \frac{\partial}{\partial z}$ consider, respectively, the quadratic and linear functions

$$F_{\alpha\beta} := -iz_\alpha\bar{z}_\beta \qquad \text{and} \qquad R_\alpha := z_\alpha$$

for $1 \le \alpha, \beta \le m$. Note that

$$\overline{F}_{\beta\alpha} = -F_{\alpha\beta} \qquad \text{and} \qquad \{F_{\alpha\beta}, F_{\beta\gamma}\} = F_{\alpha\gamma}\,,$$

that is the Hamiltonians $F_{\alpha\beta}$ generate the unitary Lie algebra $u(m)$ with respect to the Poisson bracket. The corresponding Hamiltonian fields for R_α generate translations in the $\mathbb{C}$-lines (z_α), while $F_{\alpha\beta}$ generate unitary rotations in the $\mathbb{C}$-planes (z_α, z_β) preserving the norm $\langle z, z\rangle := \sum_\alpha z_\alpha\bar{z}_\alpha$. Assuming the summation in $j = 1, ..., N$ with weights Γ_j we see that Q_α and P_α are real and imaginary parts for R_α, while $F^\pm_{\alpha\beta}$ so are for the complex functionals $F_{\alpha\beta}$.

Since the corresponding Hamiltonian flows yield unitary motions of the space $\mathbb{C}^m$, the corresponding transformations commute with the equation of symplectic point vortices. □

Remark 4.3. As we mentioned, the quadratic functionals $F_{\alpha\beta}$ form the Lie algebra $u(m)$ with respect to the Poisson bracket, while Q_α and P_α form $\mathbb{R}^{2m}$. Furthermore, since $\{Q_\alpha, P_\alpha\} = \sum_{j=1}^{N}\Gamma_j$, together the functionals $F_{\alpha\beta}, Q_\alpha$, and P_α form the central extension of the Lie algebra for the semi-direct product group $E(2m) = U(m) \ltimes \mathbb{R}^{2m}$ of unitary motions. In the case $m = 1$ this extension of $E(2)$ was described in [9].

One can use the above functionals to construct involutive integrals.

Corollary 4.4. *Functionals $\mathcal{H}$, $F^+_{\alpha\alpha}$ and $Q_\alpha^2 + P_\alpha^2$ for $1 \le \alpha \le m$ provide $2m+1$ integrals in involution on $(\mathbb{R}^{2m})^N$.*

This follows from the commutation relations $\{P_\alpha, F^+_{\alpha\alpha}\} = -2Q_\alpha$ and $\{Q_\alpha, F^+_{\alpha\alpha}\} = 2P_\alpha$.

Corollary 4.5. *(a) In 2D (i.e. for $m=1$) the problems of $N=2$ and $N=3$ point vortices are completely integrable.*

(b) The $N=2$ symplectic vortex problem in $\mathbb{R}^{2m}$ is integrable for any dimension $2m$.

Indeed, for $m=1$ the phase space of $N=3$ vortices is 6 dimensional, while one has $2m+1=3$ independent involutive integrals.

For any dimension m and N vortices the phase space is $(\mathbb{R}^{2m})^N$. For integrability one needs $m\cdot N$ independent integrals in involution. Thus $2m+1$ integrals $\mathcal{H}$, $F^+_{\alpha\alpha}$ and $Q_\alpha^2+P_\alpha^2$ are sufficient for integrability of $N=2$ symplectic vortices for any m.

Remark 4.6. Note that the evolution of $N \ge 4$ point vortices in 2D is non-integrable [13]. When $m>1$ already for $N=3$ one does not have enough integrals for the Arnold-Liouville integrability: for $m=2$ one has 5 integrals, but integrability requires 6.

Conjecture 4.7. *The system of $N=3$ symplectic point vortices is not completely integrable on $(\mathbb{R}^{2m})^N$ for $m>1$ in the Arnold-Liouville sense.*

Note that for any two point vortices in $\mathbb{R}^{2m}$, their velocities, given by $\operatorname{sgrad}\psi(\tilde z)$ at $\tilde z = \tilde z_j$ for $j=1,2$ lie in a fixed two-dimensional plane depending on the initial positions $\tilde z_1, \tilde z_2$. Thus for $N=2$ the dynamics reduces to the 2D case. (Alternatively one notice that the $N=2$ dynamics can be written in terms of the single quantity $\tilde z_{12} := \tilde z_1 - \tilde z_2$.) The motivation for the above conjecture is that for $N=3$ the vectors $\operatorname{sgrad}\psi(\tilde z_j)$ at $\tilde z_j$ do not necessarily lie in one and the same plane passing through the vortices $\tilde z_1, \tilde z_2, \tilde z_3$ once $m>1$, i.e. the problem becomes indeed higher-dimensional. In a sense, the systems of symplectic point vortices in higher dimensions are somewhat similar to the N-body problem in mechanics, cf. [10].

Another interesting question is to describe the cases when the symplectic vortex problem is weakly integrable on $\mathbb{R}^{2m}$ in the sense of [3] thanks to a large number of conserved quantities that are not in involution.

Remark 4.8. The evolution of point vortices on the sphere S^2 or the hyperbolic plane $\mathbb{H}^2$ is invariant for the groups $SO(3)$ and $SO(2,1)$ respectively. The corresponding problems of $N \le 3$ vortices are integrable [8]. Furthermore, one can make the first integrals in these cases deform to each other by tracing the change of curvature for the corresponding symplectic manifolds (see the case of $SU(2)$, $E(2)$ and $SU(1,1)$ for $m=1$ in [9]).

Similarly, one can consider the evolution of symplectic point vortices on the projective space $\mathbb{CP}^m$ or other homogeneous symplectic spaces with invariance with respect to the groups $SU(m)$ or $SU(k,l)$.

Note that for a compact symplectic manifold M^{2m} one needs to normalize the vorticity supported on N point vortices by subtracting an appropriate constant:

$$\nu = \sum_{j=1}^N \Gamma_j\, \delta(\tilde z - \tilde z_j) - C\,, \qquad \text{where} \qquad C = \frac{1}{\operatorname{Vol}(M)} \sum_{j=1}^N \Gamma_j\,.$$

Indeed, for the existence of ψ satisfying the equation $\Delta\psi = \nu$ on a compact M, the function ν has to have zero mean.

Symplectic filament equation

Recall that in the ideal incompressible hydrodynamics the vorticity is geometrically a 2-form. While above we considered the 2D case in which vorticity was supported at N point vortices, in higher dimensions singular vorticity can be supported on submanifolds of codimension ≤ 2 (for instance, it describes an evolution of curves in $\mathbb{R}^3$). The corresponding Euler dynamics of the vorticity 2-form is nonlocal, since it requires finding curl^{-1}. In $\mathbb{R}^3$ the localized induction approximation of vorticity motion describes the filament (or binormal) equation $\partial_t\gamma = \gamma' \times \gamma''$, where $\gamma(t,\cdot) \subset \mathbb{R}^3$ is a time-dependent arc-parametrized space curve. In any parametrization the equation is

$$\partial_t\gamma = \kappa \cdot \mathbf{b}\,,$$

where κ and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ are, respectively, the curvature and binormal unit vector at the corresponding point of the curve γ. This equation is known to be Hamiltonian relative to the Marsden-Weinstein symplectic structure on non-parametrized space curves with the Hamiltonian functional being the length of the curve: $H(\gamma) = \mathrm{length}(\gamma)$, see e.g. [2]. Furthermore, this is equation turns out to be integrable and equivalent to the NLS equation by means of the Hasimoto transformation.

Remark 4.9. For symplectic fluids on (M^{2m}, ω) the symplectic vorticity is a $2m$-form. Isotropic submanifolds Γ of any codimension $k \le m$ with volume k-forms ρ on them are examples of singular symplectic vorticities: here $\xi_{\mathrm{sing}} := (\Gamma, \rho) = \rho \wedge \delta_\Gamma$ can be viewed as a δ-type $2m$-form supported on Γ. The space of such pairs (Γ, ρ) with diffeomorphic embedded $\Gamma \subset M$ and positive normalized forms ρ carry a natural symplectic structure, see [12]. Such pairs form coadjoint orbits in (a completion of) the Hamiltonian Lie algebra dual $\mathrm{ham}_\omega(M)^*$. For a pair of variations U, V of (Γ, ρ), understood as vector fields attached at Γ, the corresponding symplectic structure is "a pointwise integration," $\Omega(U, V) := \int_\Gamma \omega(U, V)\,\rho$.

As an example consider a symplectic fluid in the standard symplectic space $\mathbb{R}^{2m}$ and a symplectic filament, i.e. a curve Γ with a 1-form ρ on it. The pair (Γ, ρ) can be regarded as a parametrized curve $\gamma : S^1 \to \mathbb{R}^{2m}$, since the form ρ fixes the parameter θ on S^1 modulo rotations: $d\theta = \gamma^*\rho$. Let H be the Hamiltonian functional defined as the length of this curve: $H(\gamma) = \mathrm{length}(\gamma) = \int_\gamma |\gamma'(\theta)|\, d\theta$.

Proposition 4.10. *The symplectic filament dynamics for the Hamiltonian length functional $H(\gamma)$ and the natural symplectic structure on parametrized curves is*

$$\partial_t\gamma = -\kappa \cdot \mathbf{J}\,\mathbf{n},$$

where κ is the curvature and $\mathbf{J}\,\mathbf{n}$ is the operator $\mathbf{J}$ of complex structure in $\mathbb{R}^{2m}$ applied to the unit normal $\mathbf{n}$ to the curve γ.

Indeed, the variational derivative, i.e. the "gradient," of the length functional H is $\delta H/\delta\gamma = -\mathbf{t}' = -\kappa \cdot \mathbf{n}$, where $\mathbf{t}$ and $\mathbf{n}$ are, respectively, the unit tangent and normal fields to the curve γ. The dynamics is given by the corresponding skew-gradient, which is obtained from $\delta H/\delta\gamma$ by applying $\mathbf{J}$ for the above symplectic structure.

Note that the filament dynamics in 2D is trivial: $\mathbf{J}\,\mathbf{n} = \mathbf{t}$, i.e. the Hamiltonian field $\partial_t\gamma$ is everywhere directed along the curve tangents and its value is equal to the curve curvature. Hence this is an autonomous system with no change in the curve shape, while the curve inflections (where $\kappa = 0$) are stationary points of the dynamics. In higher dimensions this Hamiltonian vector field does change the shape of the symplectic filament. It would be interesting to find integrable cases for this dynamics.

In 2D the vorticity forms supported on curves can be interpreted as vorticity sheets and the above consideration shows that such sheets have a natural symplectic structure. Apparently, unlike the $\mathbb{R}^3$ case, the symplectic filament dynamics in $\mathbb{R}^{2m}$ has no relation to the localized induction approximation for symplectic fluid.

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