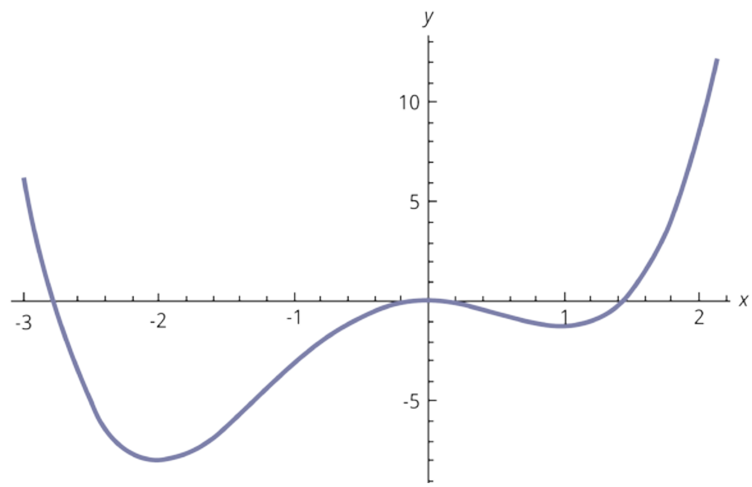
Puzzle Time: Solution

In Volume XVII of SCGP News the puzzle titled **Quartic Oscillators** was contributed by **Boris Khesin** (Professor of Mathematics, University of Toronto, Canada). Revealed here is the solution:

Previous puzzle:

Consider a particle of unit mass moving on the line according to Newton's equation $\ddot{x} = -\partial U/\partial x$ in a potential, which is a 4th degree polynomial $U(x)$ with two wells, a deep and a shallow one. Find explicitly the period of oscillations of the particle in the deeper well at the energy level corresponding to the bottom of the shallow one.



For instance, for Newton's equation $\ddot{x} = -3x(x+2)(x-1)$, corresponding to the quartic potential $U(x) = 3x^4/4 + x^3 - 3x^2$, with minima at $x = -2$ and $x = 1$, find this period of oscillations in the left well around the point $x = -2$ at the energy level $E = U(1) = -5/4$, see Figure 1.
(Hint: Compare the periods of oscillations in two wells at the same energy levels.)

SOLUTION: Back in 1991 Vladimir Arnold published *A Mathematical Trivium*, the list of 100 problems which, in his opinion, any math graduate should be able to solve [1]. The list was very elaborate, containing problems from different subjects and of various difficulty. The problems were mostly rather hard if one was trying to solve them by brute force, but often admitted an ingenious solution.

Here is a sample, **Problem #75:** *On account of the annual fluctuation of temperature the ground at a given town freezes to a depth of 2 metres. To what depth would it freeze on account of the daily fluctuation of the same amplitude?* (Spoiler: however puzzling the formulation is, it seems that the application of the invariance of the 1D heat equation – for the space variable directed downwards – with respect to the rescaling group $(x,t) \mapsto (cx, c^2t)$ solves the problem in one line.)

Almost all of Arnold's students (at least in the 80's), including the author, when they were sophomores, underwent a 'placement test', which was a take-home exam with ODE-related problems. That placement test had seemingly served as a prototype for the whole *Mathematical Trivium* by Arnold (see more details on that in [2]).

The problem on quartic oscillators arose as a variation on the following problem from Arnold's *Trivium* list:

Problem #54: *Let $\ddot{x} = 3x - x^3 - 1$. In which of the potential wells is the period of oscillation greater (in the more shallow one or in the deeper one) for equal values of the total energy?*

Recall that for a harmonic oscillator $\ddot{x} = -kx$ its potential energy is $U = kx^2/2$, the period of oscillations is equal to $T = 2\pi/\sqrt{k}$ and it does not depend on the amplitude. If the potential U in the Newton equation $\ddot{x} = -\partial U/\partial x$ is not quadratic, but has a nondegenerate minimum at a point x_0, the periods of oscillations near x_0 tend to $T = 2\pi/\sqrt{U''(x_0)}$ as the amplitude vanishes.

For Arnold's problem, $U(x) = x^4/4 - 3x^2/2 + x$, which is a quartic polynomial with two wells. Given the energy E consider the (compactified) Riemann surface $y^2/2 + U(x) = E$, which is the E-level of the Hamiltonian corresponding to the Newton equation. The periods are the integrals of the "time" 1-form $dt := (dx)/y$, where $y = \sqrt{2(E - U(x))}$ over the closed paths on this Riemann surface.

For a typical polynomial $U(x)$ of degree 4 (and in particular, for the above potential), the corresponding Riemann surface is an elliptic curve $\mathcal{E}$, while the 1-form $dt = (dx)/y$ is a holomorphic form on it. The oscillations in the two different wells correspond to homologically equivalent cycles on this elliptic curve $\mathcal{E}$. This implies that the oscillation periods are equal.

Returning to the original puzzle, we see that the required period in the deep well around x_{min} is equal to the period of small oscillations near the bottom x_{shal} in the shallow one, while the latter is given by $T = 2\pi/\sqrt{U''(x_{\text{shal}})}$. This explicit formula gives the answer to the puzzle.

(For quartic potential $U(x) = 3x^4/4 + x^3 - 3x^2$ there are two minima, at $x_{\text{min}} = -2$ and $x_{\text{shal}} = 1$. At the shallow minimum $U''(1) = 9 + 6 - 6 = 9$, and hence both the period of small oscillations near $x_{\text{shal}} = 1$ and oscillations at $E = U(1) = -5/4$ are equal to $T = 2\pi/\sqrt{9} = 2\pi/3$.)

References

[1] V. I. Arnold, *A Mathematical Trivium*, Russian Math. Surveys, 46:1 (1991), 271-278.

[2] B. Khesin and S. Tabachnikov, *Comments on "A Mathematical Trivium" by V.Arnold*, in *ARNOLD: Swimming Against the Tide*, Amer. Math. Soc. (2014), 57-66.