

THE PENTAGRAM MAP IN HIGHER DIMENSIONS AND KdV FLOWS

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(Communicated by Leonid Polterovich)

ABSTRACT. We extend the definition of the pentagram map from 2D to higher dimensions and describe its integrability properties for both closed and twisted polygons by presenting its Lax form. The corresponding continuous limit of the pentagram map in dimension d is shown to be the $(2, d + 1)$ -equation of the KdV hierarchy, generalizing the Boussinesq equation in 2D.

INTRODUCTION

The pentagram map was originally defined in [8] as a map on plane convex polygons considered up to their projective equivalence, where a new polygon is spanned by the shortest diagonals of the initial one, see Fig. 1. It was shown to exhibit quasi-periodic behavior under iterations. This map was extended to the case of twisted polygons and its integrability in 2D was proved in [6]. The integrability of the pentagram map in the case of closed polygons was proved in [9, 7].

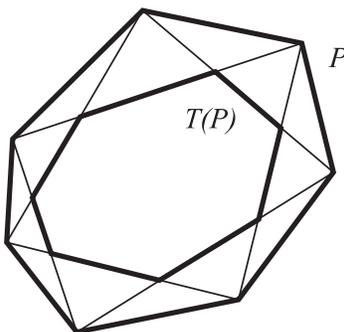


FIGURE 1. The image $T(P)$ of a heptagon P ($n = 7$; $M = \text{Id}$) under the 2D pentagram map.

Received by the editors May 16, 2012.

2010 *Mathematics Subject Classification*. Primary: 37J35; Secondary: 37K10, 37K25, 53A20.

Key words and phrases. pentagram map, completely integrable system, Lax equation, KdV hierarchy, spectral curve.

We are grateful to M. Gekhtman and S. Tabachnikov for useful discussions and to Olga Solovieva for help with drawing the figures. B.K. was partially supported by the Simonyi Fund and an NSERC research grant.

In this paper we extend this definition of the 2D pentagram map to any dimension and describe the 3D case in more detail. We prove algebro-geometric integrability of these general pentagram maps by presenting them as a discrete zero-curvature equation (a Lax-type equation, which implies Arnold–Liouville integrability), and study the continuous limit of the maps. We refer to [2] for more results, proofs, and details of the constructions described in this announcement.

Note that a higher dimensional generalization for the class of corrugated polygons, which have the property that their consecutive diagonals are not skew but do intersect, was treated in [1]. A variety of possible higher-dimensional generalizations with an integrable continuous limit was considered in [5]. In this paper we propose a definition of “diagonal hyperplanes” for *generic* higher dimensional polygons, which leads to integrable systems not only in the continuous limit but to genuine discrete integrable systems and can be regarded as natural integrable discretizations of the KdV-type equations. We start with a treatment of the 3D case and describe the d -dimensional case later. We outline the geometry of the pentagram map in the real setting, but complexify the spaces and maps to describe the algebro-geometric integrability.

1. INTEGRABILITY OF THE 3D PENTAGRAM MAP

We start by extending the set of space polygons to include so-called twisted ones, whose ends are related by a monodromy operator:

Definition 1.1. A *twisted n -gon* in a projective space \mathbb{RP}^3 with a monodromy $M \in SL_4$ is a map $\phi : \mathbb{Z} \rightarrow \mathbb{RP}^3$, such that $\phi(k+n) = M \circ \phi(k)$ for each $k \in \mathbb{Z}$ and where M acts naturally on \mathbb{RP}^3 . Two twisted n -gons are *equivalent* if there is a transformation $g \in SL_4$ such that $g \circ \phi_1 = \phi_2$.

We assume that the vertices $\phi(k)$, $k \in \mathbb{Z}$, are in general position, i.e., in particular, no 4 consecutive vertices of an n -gon belong to one and the same 2-dimensional plane in \mathbb{RP}^3 . The following pentagram map T is generically defined on the space \mathcal{P}_n of twisted n -gons considered up to the above equivalence:

Definition 1.2. Given an n -gon ϕ in \mathbb{RP}^3 , for each $k \in \mathbb{Z}$ consider the two-dimensional *short-diagonal plane* $P_k := (\phi(k-2), \phi(k), \phi(k+2))$ passing through 3 vertices $\phi(k-2), \phi(k), \phi(k+2)$. Take the intersection point of the three consecutive planes P_{k-1}, P_k, P_{k+1} and call it the image of the vertex $\phi(k)$ under the *space pentagram map* T . (We assume general position, so that every three consecutive planes P_k for the given n -gon intersect at a point.)

For simplicity, in 3D we discuss only the case of odd n in this research announcement. (Even values of n require more delicate treatment, for which we refer the reader to [2].) The coordinates on the space \mathcal{P}_n are introduced in the following way.

It turns out that for odd n there exists a unique lift of the vertices $\phi(k) \in \mathbb{RP}^3$, $k \in \mathbb{Z}$, of a given n -gon to the vectors $V_k \in \mathbb{R}^4$, $k \in \mathbb{Z}$, satisfying the conditions of nondegeneracy $\det(V_j, V_{j+1}, V_{j+2}, V_{j+3}) = 1$ and quasi-periodicity $V_{j+n} = MV_j$ for all $j \in \mathbb{Z}$, where $M \in SL_4$ is the monodromy matrix. These vectors satisfy the difference equations

$$V_{j+4} = a_j V_{j+3} + b_j V_{j+2} + c_j V_{j+1} - V_j, \quad j \in \mathbb{Z},$$

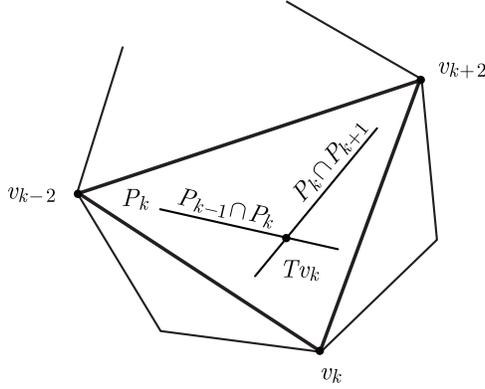


FIGURE 2. The image Tv_k of the vertex $v_k = \phi(k)$ in \mathbb{P}^3 .

where the sequences a_j, b_j, c_j are n -periodic. The numbers (a_j, b_j, c_j) , $0 \leq j \leq n-1$, form a system of $3n$ coordinates on the space of twisted n -gons \mathcal{P}_n . One can also introduce “local” coordinates on \mathcal{P}_n based on cross-ratios and somewhat similar to the so-called (x, y) -coordinates in the 2D case, see [2].

1.1. A discrete zero-curvature equation. Algebro-geometric integrability of the pentagram map is based on a zero-curvature equation with a spectral parameter. In the discrete case, it is an equation of the form

$$(1) \quad L_{i,t+1}(\lambda) = P_{i+1,t}(\lambda)L_{i,t}(\lambda)P_{i,t}^{-1}(\lambda),$$

which represents a dynamical system. This equation may be regarded as a compatibility condition of an over-determined system of equations:

$$\begin{cases} L_{i,t}(\lambda)\Psi_{i,t}(\lambda) = \Psi_{i+1,t}(\lambda) \\ P_{i,t}(\lambda)\Psi_{i,t}(\lambda) = \Psi_{i,t+1}(\lambda), \end{cases}$$

for an auxiliary function $\Psi_{i,t}(\lambda)$. To describe this equation we complexify the pentagram map and consider it over \mathbb{C} .

The above has an analogue in the continuous case: a zero-curvature equation $\partial_t L - \partial_x P = [P, L]$ is a compatibility condition which provides the existence of an auxiliary function $\psi = \psi(t, x)$ satisfying a system of differential equations $\partial_x \psi = L\psi$ and $\partial_t \psi = P\psi$.

Theorem 1.3. *The 3D pentagram map on twisted n -gons with odd n admits a zero-curvature representation with the Lax function $L_{j,t}(\lambda)$ given by*

$$L_{j,t}(\lambda) = \begin{pmatrix} c_j/\lambda & 1/\lambda & 0 & 0 \\ b_j & 0 & 1 & 0 \\ a_j/\lambda & 0 & 0 & 1/\lambda \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ \lambda & 0 & 0 & c_j \\ 0 & 1 & 0 & b_j \\ 0 & 0 & \lambda & a_j \end{pmatrix}^{-1}$$

in the coordinates a_j, b_j, c_j , $0 \leq j \leq n-1$, and an appropriate matrix function $P_{j,t}(\lambda)$ satisfying (1), where $\lambda \in \mathbb{C}$ is the spectral parameter.

In a sense, equation (1) implies all our integrability results in the discrete case, while the explicit form of $P_{i,t}$ is not important for the present exposition.

The pivotal property responsible for integrability of the pentagram map is its property of scaling invariance. In the 3D case in the coordinates (a_j, b_j, c_j) , $0 \leq j \leq n-1$, this means that the pentagram map T is invariant with respect to the transformations $a_j \rightarrow a_j s$, $b_j \rightarrow b_j$, $c_j \rightarrow c_j s$, $0 \leq j \leq n-1$. The invariance follows from the explicit formulas of the map. We prove that the existence of equation (1) and the corresponding formula for $L_{j,t}(\lambda)$ follow from such a scaling invariance. The spectral parameter λ is related to the scaling parameter s via $\lambda = 1/s^2$.

1.2. Spectral curve. In the discrete case, there exist analogues of monodromy and Floquet–Bloch solutions:

Definition 1.4. *The monodromy operators $M_{i,t}$, $i = 0, \dots, n-1$, are defined as the following ordered products of the corresponding Lax functions:*

$$M_{i,t} := L_{i+n-1,t} L_{i+n-2,t} \dots L_{i+1,t} L_{i,t},$$

where the (integer) index t represents the time variable, and the first index of L is n -periodic: $L_{j+n,t} = L_{j,t}$.

Definition 1.5. For an odd n define the *spectral function* $R(\lambda, k)$ as

$$R(\lambda, k) := \det(M_{0,0}(\lambda) - kI).$$

The *spectral curve* Γ is the normalization of the compactification of the curve $R(\lambda, k) = 0$.

Note that one could define the spectral function with help of any of the monodromies: $R(\lambda, k) := \det(M_{i,t}(\lambda) - kI)$. Indeed, by definition, the monodromy operators for different i are conjugate to each other, while equation (1) implies that the monodromy operators for different t are also conjugate (i.e., satisfy a discrete Lax equation):

$$M_{i,t+1}(\lambda) = P_{i,t}(\lambda) M_{i,t}(\lambda) P_{i,t}^{-1}(\lambda).$$

This implies that the spectral function $R(\lambda, k)$ is an invariant of the pentagram map. After a multiplication by a suitable power of λ , the equation $R(\lambda, k) = 0$ becomes a polynomial relation between λ and k , whereas its coefficients are integrals of motion for the pentagram map. Namely, we define the *integrals of motion* I_j, J_j, G_j , $0 \leq j \leq q = \lfloor n/2 \rfloor$ (where $\lfloor x \rfloor$ is the greatest integer less than or equal to x), as the coefficients of the expansion

$$R(\lambda, k) = k^4 - k^3 \left(\sum_{j=0}^q G_j \lambda^{j-n} \right) + k^2 \left(\sum_{j=0}^q J_j \lambda^{j-q-n} \right) - k \left(\sum_{j=0}^q I_j \lambda^{j-2n} \right) + \lambda^{-2n}.$$

The set $R(\lambda, k) = 0$ is an algebraic curve in \mathbb{C}^2 . A standard procedure (adding the infinite points and normalization with a few blow-ups) makes it into a compact Riemann surface, the corresponding spectral curve Γ . The integrals of motion are independent polynomials in the (a, b, c) -coordinates.

Theorem 1.6. *For generic twisted n -gons with odd n , the genus g of the spectral curve Γ is $g = 3q$, where $q = \lfloor n/2 \rfloor$.*

The spectral curve and its Jacobian, a natural torus associated with it, are starting points for the algebraic-geometric integrability. It turns out that one can recover Lax functions from the spectral curve and a point on the Jacobian, and vice versa: this correspondence is locally one-to-one.

1.3. Pentagonam dynamics in spectral data.

Definition 1.7. A *Floquet–Bloch solution* $\psi_{i,t}$ of a difference equation $\psi_{i+1,t} = L_{i,t}\psi_{i,t}$ is an eigenvector of the monodromy operator:

$$M_{i,t}\psi_{i,t} = k\psi_{i,t}.$$

To make them uniquely defined, we normalize the vectors $\psi_{i,t}$, $t \geq 0$ so that the sum of their components is equal to 1 and denote the normalized vectors by $\bar{\psi}_{i,t}$, i.e., $\bar{\psi}_{i,t} := \psi_{i,t} / \left(\sum_{j=1}^4 \psi_{i,t,j} \right)$.

Theorem 1.8. *Generically, a Floquet–Bloch solution $\bar{\psi}_{i,t}$ is a meromorphic vector function on Γ . Its pole divisor $D_{i,t}$ has degree $g + 3$.*

Definition 1.9. Let $J(\Gamma)$ be the Jacobian of the spectral curve Γ , and $[D_{i,t}]$ be the equivalence class of the divisor $D_{i,t}$ under the Abel map.

Theorem 1.10. *For an odd n , the spectral map $S : \mathcal{P}_n = (a_i, b_i, c_i, 0 \leq i \leq n-1) \rightarrow (\Gamma, [D_{i,t}])$ is non-degenerate at a generic point and an isomorphism of Zariski open subsets.*

The pentagram dynamics corresponds to the motion of the pole divisor $[D_{i,t}]$ along the corresponding Jacobian $J(\Gamma)$.

The equivalence class $[D_{i,t}] \in J(\Gamma)$ has the following linear time evolution along the Jacobian:

$$[D_{i,t}] = [D_{0,0} - t(O_1 + O_3) + i(O_2 + O_3) + (t-i)(W_1 + W_2)] \in J(\Gamma),$$

where the points $W_1, W_2 \in \Gamma$ correspond to $\lambda = \infty$, and $O_1, O_2, O_3 \in \Gamma$ are the points above $\lambda = 0$ (the point O_1 corresponds to a finite k , whereas O_2 and O_3 correspond to infinite k , and O_2 is a branch point).

This theorem implies that for an odd n the time evolution in $J(\Gamma)$ happens to be the same constant shift at each step. This way the above theorem describes the time evolution of the pentagram map and proves its algebro-geometric integrability.

For an even n (which we deal with in [2]) the dynamics turns out to be more complicated: the evolution goes along a “staircase”, i.e., its square is a constant shift. This dichotomy is similar to the 2D case, see [9].

1.4. Closed polygons. Closed polygons in \mathbb{P}^3 correspond to the monodromy $M = \pm \text{Id} \in SL_4$ and form a subspace of codimension $15 = \dim SL_4$ in the space of all twisted polygons \mathcal{P}_n of dimension $3n$. Such a monodromy corresponds to the spectral curves with either $(\lambda, k) = (1, 1)$ or $(\lambda, k) = (1, -1)$ being a quadruple point. More precisely, we have:

Theorem 1.11. *Closed polygons in \mathbb{P}^3 are singled out by the condition that either $(\lambda, k) = (1, 1)$ or $(\lambda, k) = (1, -1)$ is a quadruple point of Γ . Generically, the genus of Γ drops to $g = 3q - 6$ for $q = \lfloor n/2 \rfloor$ and odd n . The dimension of the Jacobian $J(\Gamma)$ drops by 6 for closed polygons. Theorem 1.8 holds with this genus adjustment, and Theorem 1.10 holds verbatim for closed polygons.*

Remark 1.12. The algebraic conditions implying that $(1, \pm 1)$ is a quadruple point are:

- $R(1, \pm 1) = 0$,
- $\partial_k R(1, \pm 1) = \partial_\lambda R(1, \pm 1) = 0$,

- $\partial_k^2 R(1, \pm 1) = \partial_\lambda^2 R(1, \pm 1) = \partial_{k\lambda}^2 R(1, \pm 1) = 0,$
- $\partial_k^3 R(1, \pm 1) = \partial_\lambda^3 R(1, \pm 1) = \partial_{kk\lambda}^3 R(1, \pm 1) = \partial_{k\lambda\lambda}^3 R(1, \pm 1) = 0.$

One can show that among these 10 linear equations on the integrals of motion there are only 9 independent ones due to the following relation:

$$R(1, \pm 1) = \pm \partial_k R(1, \pm 1) - \frac{1}{2} \partial_k^2 R(1, \pm 1) \pm \frac{1}{6} \partial_k^3 R(1, \pm 1).$$

At the same time, the dimension of the Jacobian of the spectral curve drops by 6. Thus the subspace \mathcal{C}_n has codimension 15, which matches the above calculation of dimensions.

1.5. An invariant symplectic structure. To describe an invariant symplectic structure on leaves of the space of twisted polygons \mathcal{P}_n in 3D we employ the Krichever–Phong universal formula [3, 4]. In the 2D case such a symplectic structure was shown to coincide with that induced on these leaves by the invariant Poisson structure found in [6], see [9]. The description below is somewhat implicit and, in a sense, universal: it is applicable in the higher-dimensional cases of \mathbb{P}^d with $d > 3$ as well. Finding an explicit expression, e.g., in the coordinates (a_i, b_i, c_i) , of the symplectic structure or of the corresponding Poisson structure is still an open problem.

Definition 1.13 ([3, 4]). Krichever–Phong’s universal formula defines a *pre-symplectic form* on the space of Lax operators, i.e., on the space \mathcal{P}_n . It is given by the expression:

$$\omega = -\frac{1}{2} \sum_{\lambda=0, \infty} \text{res Tr} (\Psi_0^{-1} M_0^{-1} \delta M_0 \wedge \delta \Psi_0) \frac{d\lambda}{\lambda}.$$

The matrix $\Psi_0(\lambda)$ is composed of the normalized eigenvectors $\bar{\psi}_{0,0}$ on different sheets of Γ over the λ -plane, and it diagonalizes the matrix $M_0 = M_{0,0}$. (In this definition we drop the second index, since all variables correspond to the same moment t .)

The *leaves* of the 2-form ω are defined as submanifolds of \mathcal{P}_n , where the expression $\delta \ln k (d\lambda/\lambda)$ is holomorphic. The latter expression is considered as a 1-form on the spectral curve Γ .

Theorem 1.14. *For an odd n the leaves of the 2-form ω in \mathcal{P}_n are singled out by the 3 conditions*

$$\delta G_0 = \delta I_0 = \delta J_q = 0,$$

where I_0, G_0, J_q are the integrals of motion defined above. The restriction of ω to these leaves is well-defined (i.e., independent of the normalization of the Floquet–Bloch solutions Ψ_0) and non-degenerate, and hence symplectic. This symplectic form is invariant with respect to the pentagram map, that is the evolution given by the Lax equation.

The rank of the invariant 2-form ω restricted to these leaves is equal to $2g$, where g is the genus of the spectral curve Γ , i.e., $g = 3\lfloor n/2 \rfloor$.

Recall that the dimension of the space \mathcal{P}_n is $6q + 3$, where $q = \lfloor n/2 \rfloor$. Since the codimension of the leaves is 3, their dimension matches the doubled dimension of the tori: $2g = 6q$. We also note that the Arnold–Liouville integrability in the complex case implies integrability in the real one, since the formula for the symplectic structure is algebraic.

2. PENTAGRAM MAPS IN HIGHER DIMENSIONS

2.1. Definition for any dimension. First we extend the notion of a twisted n -gon to an arbitrary dimension d .

Definition 2.1. A *twisted n -gon* in a projective space \mathbb{RP}^d with a monodromy $M \in SL_{d+1}$ is a map $\phi : \mathbb{Z} \rightarrow \mathbb{RP}^d$, such that $\phi(k+n) = M \circ \phi(k)$ for each $k \in \mathbb{Z}$ and where M acts naturally on \mathbb{RP}^d . Two twisted n -gons are *equivalent* if there is a transformation $g \in SL_{d+1}$ such that $g \circ \phi_1 = \phi_2$.

Similarly to the 3D case, we assume that the vertices $v_k := \phi(k)$, $k \in \mathbb{Z}$, are in general position (i.e., no $d+1$ consecutive vertices lie in the same hyperplane in \mathbb{RP}^d), and denote by \mathcal{P}_n the space of twisted n -gons considered up to the above equivalence.

For a generic twisted n -gon in \mathbb{RP}^d define the “short-diagonal” hyperplane P_k passing through d vertices of the n -gon by taking d -tuple consisting every other vertex and centered at a given vertex v_k . Namely, for odd dimension $d = 2\kappa + 1$ we consider the *short-diagonal hyperplane* P_k through the d vertices

$$P_k := (v_{k-2\kappa}, v_{k-2\kappa+2}, \dots, v_k, \dots, v_{k+2\kappa}),$$

while for even dimension $d = 2\kappa$ we take P_k passing through the d vertices

$$P_k := (v_{k-2\kappa+1}, v_{k-2\kappa+3}, \dots, v_{k-1}, v_{k+1}, \dots, v_{k+2\kappa-1}).$$

The following pentagram map T is generically defined on the space \mathcal{P}_n of twisted n -gons:

Definition 2.2. The *higher pentagram map* T takes a vertex v_k of a generic twisted n -gon in \mathbb{P}^d to the intersection point of the d consecutive short-diagonal planes P_i around v_k . Namely, for odd $d = 2\kappa + 1$ one takes the intersection of the planes

$$Tv_k = P_{k-\kappa} \cap P_{k-\kappa+1} \cap \dots \cap P_k \cap \dots \cap P_{k+\kappa},$$

while for even $d = 2\kappa$ one takes the intersection of the planes

$$Tv_k = P_{k-\kappa+1} \cap P_{k-\kappa+2} \cap \dots \cap P_k \cap \dots \cap P_{k+\kappa}.$$

As usual, we assume that the vertices are in “general position,” and every d consecutive hyperplanes P_i intersect at one point in \mathbb{P}^d . The map T is well defined on the equivalence classes of generic n -gons in \mathbb{P}^d .

Remark 2.3. One can show that there exists a unique lift of the vertices $v_k = \phi(k) \in \mathbb{RP}^d$ to the vectors $V_k \in \mathbb{R}^{d+1}$ satisfying $\det |V_j, V_{j+1}, \dots, V_{j+d}| = 1$ and $V_{j+n} = MV_j$, $j \in \mathbb{Z}$, where $M \in SL_{d+1}$, if and only if the condition $\gcd(n, d+1) = 1$ holds. The corresponding difference equations have the form

$$(2) \quad V_{j+d+1} = a_{j,d}V_{j+d} + a_{j,d-1}V_{j+d-1} + \dots + a_{j,1}V_{j+1} + (-1)^d V_j, \quad j \in \mathbb{Z},$$

with n -periodic coefficients in the index j . It allows one to introduce coordinates $\{a_{j,k}, 0 \leq j \leq n-1, 1 \leq k \leq d\}$ on the space of twisted n -gons in \mathbb{RP}^d .

2.2. Complete integrability. It turns out that the pentagram map defined this way has a special scaling invariance, which implies the existence of a Lax representation, and allows one to prove its integrability. First, we complexify the spaces of n -gons and the map.

Proposition-conjecture 2.4. (The scaling invariance) *The pentagram map on twisted n -gons in \mathbb{CP}^d is invariant with respect to the following scaling transformations:*

- for odd $d = 2\kappa + 1$ the transformations are

$$a_{j,1} \rightarrow sa_{j,1}, a_{j,3} \rightarrow sa_{j,3}, a_{j,5} \rightarrow sa_{j,5}, \dots, a_{j,d} \rightarrow sa_{j,d},$$

while other coefficients $a_{j,2l}$ with $l = 1, \dots, \kappa$ do not change;

- for even $d = 2\kappa$ the transformations are

$$a_{j,1} \rightarrow s^{-1}a_{j,1}, a_{j,2} \rightarrow s^{-2}a_{j,2}, \dots, a_{j,\kappa} \rightarrow s^{-\kappa}a_{j,\kappa},$$

$a_{j,\kappa+1} \rightarrow s^\kappa a_{j,\kappa+1}, a_{j,\kappa+2} \rightarrow s^{\kappa-1}a_{j,\kappa+2}, \dots, a_{j,d-1} \rightarrow s^2a_{j,d-1}, a_{j,d} \rightarrow sa_{j,d}$
for all $s \in \mathbb{C}$.

We proved this proposition up to dimension $d \leq 6$ by studying explicit formulas for the pentagram map, but have no general proof for $d > 6$. It would be very interesting to find it.

Problem 2.5. *Find a general proof of the scaling invariance of the pentagram map in any dimension d .*

For the following theorem we assume this conjecture on scaling invariance.

Theorem 2.6. *The scale-invariant pentagram map on twisted n -gons in any dimension d is a completely integrable system. It is described by the Lax matrix*

$$L_{j,t}(\lambda) = \left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & (-1)^d \\ \hline & & & & a_{j,1} \\ & & & & a_{j,2} \\ & & & & \cdots \\ & & & & a_{j,d} \end{array} \right)^{-1},$$

where $D(\lambda)$ is the following diagonal $(d \times d)$ -matrix:

- for odd $d = 2\kappa + 1$, one has $D(\lambda) = \text{diag}(\lambda, 1, \lambda, 1, \dots, \lambda)$;
- for even $d = 2\kappa$, one has

$$D_{ii}(\lambda) = \begin{cases} 1, & \text{if } i \neq \kappa + 1, \\ \lambda, & \text{if } i = \kappa + 1. \end{cases}$$

Sketch of proof. Rewrite the difference equation (2) in the matrix form. It is described by the transformation matrix

$$N_j := \left(\begin{array}{ccc|c} 0 & \cdots & 0 & (-1)^d \\ \hline & & & a_{j,1} \\ & & & \cdots \\ & & & a_{j,d} \end{array} \right),$$

where Id is the identity $(d \times d)$ -matrix. Then the monodromy M for twisted n -gons is the product $M = N_0 N_1 \dots N_{n-1}$. Note that the pentagram map defined on classes of projective equivalence preserves the conjugacy class of M . E.g., assume that d is odd. Then using the scaling invariance, replace $a_{j,2k+1}$ by $sa_{j,2k+1}$ for all k in the right column to obtain new matrices $N_j(s)$. The pentagram map preserves the conjugacy class of the new monodromy $M(s) := N_0(s) \dots N_{n-1}(s)$ for any s . Although one could already take $N_j(s)$ as a Lax matrix $L_{j,t}(s)$, for computations

it is convenient to modify it slightly and set $L_{j,t}^{-1}(\lambda) := (g^{-1}N_j(s)g)/s$ for $g = \text{diag}(1, s, 1, s, \dots, 1, s)$ and $\lambda := s^{-2}$. \square

2.3. Continuous limit of the pentagram map. Consider the continuous limit of polygons and the pentagram map on them. In the limit $n \rightarrow \infty$ a generic twisted n -gon becomes a smooth non-degenerate quasi-periodic curve $\gamma(x)$ in $\mathbb{R}\mathbb{P}^d$. Its lift $G(x)$ to \mathbb{R}^{d+1} is defined by the conditions that the components of the vector function $G(x) = (G_1, \dots, G_{d+1})(x)$ provide the homogeneous coordinates for $\gamma(x) = (G_1 : \dots : G_{d+1})(x)$ in $\mathbb{R}\mathbb{P}^d$ and $\det |G(x), G'(x), \dots, G^{(d)}(x)| = 1$ for all $x \in \mathbb{R}$. Furthermore, $G(x+2\pi) = MG(x)$ for a given $M \in SL_{d+1}$. Then $G(x)$ satisfies the linear differential equation of order $d+1$:

$$G^{(d+1)} + u_{d-1}(x)G^{(d-1)} + \dots + u_1(x)G' + u_0(x)G = 0$$

with periodic coefficients $u_i(x)$, which is a continuous limit of difference equation (2). (Here ' stands for d/dx .)

Fix a small $\epsilon > 0$ and consider the case of odd $d = 2\kappa + 1$. A continuous analog of the hyperplane P_k is the hyperplane $P_\epsilon(x)$ passing through d points $\gamma(x - \kappa\epsilon), \dots, \gamma(x), \dots, \gamma(x + \kappa\epsilon)$ of the curve γ .

Let $\ell_\epsilon(x)$ be the envelope curve for the family of hyperplanes $P_\epsilon(x)$ for a fixed ϵ . The envelope condition means that $P_\epsilon(x)$ are the osculating hyperplanes of the curve $\ell_\epsilon(x)$, that is the point $\ell_\epsilon(x)$ belongs to the plane $P_\epsilon(x)$, while the vector-derivatives $\ell'_\epsilon(x), \dots, \ell_\epsilon^{(d-1)}(x)$ span this plane for each x . It means that the lift of $\ell_\epsilon(x)$ to $L_\epsilon(x)$ in \mathbb{R}^{d+1} satisfies the system of $d = 2\kappa + 1$ equations (see Fig. 3 for $d = 3$):

$$\det |G(x - \kappa\epsilon), G(x - (\kappa - 1)\epsilon), \dots, G(x), \dots, G(x + \kappa\epsilon), L_\epsilon^{(j)}(x)| = 0, \quad j = 0, \dots, d-1.$$

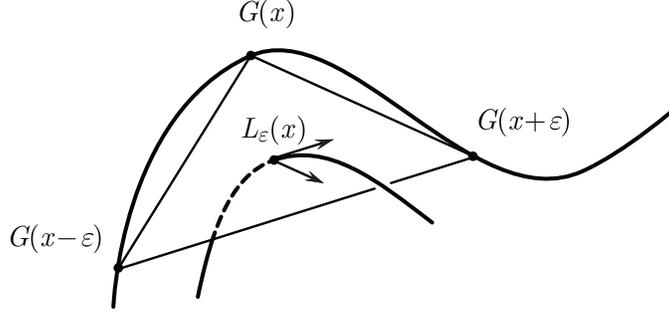


FIGURE 3. The envelope $L_\epsilon(x)$ in 3D. The point $L_\epsilon(x)$ and the vectors $L'_\epsilon(x)$ and $L''_\epsilon(x)$ belong to the plane $(G(x), G(x + \epsilon), G(x - \epsilon))$.

Similarly, for even $d = 2\kappa$ the lift $L_\epsilon(x)$ satisfies the system of d equations:

$$\det |G(x - (2\kappa - 1)\epsilon), G(x - (2\kappa - 3)\epsilon), \dots, G(x - \epsilon), G(x + \epsilon), \dots, G(x + (2\kappa - 1)\epsilon), L_\epsilon^{(j)}(x)| = 0, \quad j = 0, \dots, d-1.$$

A continuous limit of the pentagram map T is defined as the evolution of the curve γ in the direction of the envelope ℓ_ϵ , as ϵ changes. Namely, the expansion of $L_\epsilon(x)$ has the form

$$L_\epsilon(x) = G(x) + \epsilon^2 B(x) + \mathcal{O}(\epsilon^4)$$

and satisfies the family of differential equations:

$$L_\epsilon^{(d+1)} + u_{d-1,\epsilon}(x)L_\epsilon^{(d-1)} + \dots + u_{1,\epsilon}(x)L'_\epsilon + u_{0,\epsilon}(x)L_\epsilon = 0, \text{ where } u_{j,0}(x) = u_j(x).$$

Then the corresponding expansion of the coefficients $u_{j,\epsilon}(x)$ as $u_{j,\epsilon}(x) = u_j(x) + \epsilon^2 w_j(x) + \mathcal{O}(\epsilon^4)$, defines the continuous limit of the pentagram map as the system of evolution differential equations $du_j(x)/dt = w_j(x)$ for $j = 0, \dots, d-1$.

Theorem 2.7. *The continuous limit of the pentagram map T in dimension d defined by the system $du_j(x)/dt = w_j(x)$, $j = 0, \dots, d-1$ for $x \in S^1$ is the $(2, d+1)$ -KdV flow of the Adler-Gelfand-Dickey hierarchy on the circle.*

Recall that the $(m, d+1)$ -KdV flow is defined on linear differential operators $L = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + u_{d-2}(x)\partial^{d-2} + \dots + u_1(x)\partial + u_0(x)$ of order $d+1$ with periodic coefficients $u_j(x)$, where ∂^k stands for d^k/dx^k . One can define its fractional power $L^{m/d+1}$ as a pseudo-differential operator for any positive integer m and take its pure differential part $Q_m := (L^{m/d+1})_+$. In particular, for $m = 2$ one has $Q_2 = \partial^2 + \frac{2}{d+1}u_{d-1}(x)$. Then the $(m, d+1)$ -KdV equation is the evolution equation on (the coefficients of) L given by $dL/dt = [Q_m, L]$.

Remark 2.8. For $d = 2$ the $(2,3)$ -KdV equation is the classical Boussinesq equation, found in [6]. Apparently, the $(2, d+1)$ -KdV equation is a very robust continuous limit. One obtains it not only for the pentagram map defined by taking every other vertex, but also for a non-symmetric choice of vertices for the plane P_k . Also, the same limit was obtained in [5] for a map defined by taking intersections of various planes, rather than the envelopes.

The above scaling has a clear meaning in the continuous limit (in 2D this was proved in [6]):

Proposition 2.9. *The continuous limit of the scaling transformations corresponds to the spectral shift $L \rightarrow L + \lambda$ of the differential operator L .*

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