Pentagram maps and refactorization in Poisson-Lie groups

Anton Izosimov*

Abstract

The pentagram map was introduced by R. Schwartz in 1992 and is now one of the most renowned discrete integrable systems. In the present paper we prove that this map, as well as its all known integrable multidimensional generalizations, can be seen as refactorization-type mappings in the Poisson-Lie group of pseudo-difference operators. This brings the pentagram map into the rich framework of Poisson-Lie groups, both describing new structures and simplifying and revealing the origin of its known properties. In particular, the Poisson-Lie group setting provides invariant Poisson structures in all dimensions and a new Lax form with spectral parameter for multidimensional pentagram maps.

Furthermore, in the appendix written with B. Khesin we introduce and prove integrability of long-diagonal pentagram maps, encompassing all known integrable cases, as well as describe their continuous limit as the Boussinesq hierarchy.

Contents

1 Introduction and outline of main results 2
2 Pentagram-type maps associated with pairs of arithmetic progressions 5
3 Difference and pseudo-difference operators 7

3.1 Generalities on difference operators .................................................. 7
3.2 Difference operators and $J$-corrugated polygons .................................. 8
3.3 The Poisson-Lie group of pseudo-difference operators ........................... 9
3.4 Generalities on Poisson-Lie groups ..................................................... 11
3.5 Existence and properties of the Poisson structure ................................ 12
3.6 Relation with the $GL_n$ bracket ....................................................... 14
3.7 The subgroup of sparse operators ....................................................... 16

4 General refactorization maps associated with pairs of arithmetic progressions 17

4.1 The main theorem ................................................................. 17
4.2 Scaling invariance ............................................................... 21
4.3 Poisson brackets for the short-diagonal map in 3D ............................... 23
4.4 Refactorization and Y-meshes .................................................... 24

5 Open problems 26

6 Appendix: Refactorization as long-diagonal pentagram maps and its continuous limit (by Anton Izosimov and Boris Khesin) 27

6.1 Long-diagonal pentagram maps .................................................... 27
6.2 Continuous limit of refactorization pentagram maps ............................. 30

*Department of Mathematics, University of Arizona, e-mail: izosimov@math.arizona.edu
1 Introduction and outline of main results

The pentagram map, introduced by R. Schwartz in [30], is a discrete integrable system on the space of projective equivalence classes of planar polygons. The definition of this map is illustrated in Figure 1: the image of the polygon $P$ under the pentagram map is the polygon $P'$ whose vertices are the intersection points of consecutive shortest diagonals of $P$ (i.e., diagonals connecting second-nearest vertices). The pentagram map has been an especially popular subject in the last decade, mainly due to a combination of an elegant geometric definition and connections to such topics as cluster algebras, dimer models etc.

![Figure 1: The pentagram map.](image)

Integrability of the pentagram map was established, in different contexts, in [27, 28, 32]. Furthermore, it was shown that the pentagram map can be viewed as a particular case of several general constructions of integrable systems. In particular, it has an interpretation in terms of cluster algebras [11], networks of surfaces [10], T-systems [15], and Poisson-Lie groups [9]. In the present paper we suggest an alternative to [9] Poisson-Lie approach to the pentagram map. Namely, we show that the pentagram map can be seen as a refactorization in the Poisson-Lie group of pseudo-difference operators. The main advantage of our approach is that it is based on the geometric definition of the map and the explicit formulas are obtained as its corollaries. We thereby obtain all the ingredients needed to establish integrability, namely an invariant Poisson structure, Lax representation, and first integrals, directly from geometry. This can be compared with other frameworks, in particular, the ones based on cluster algebras [10] and Poisson-Lie groups [9], which lead to integrable maps shown to coincide with the pentagram map at the level of formulas.

By virtue of the geometric nature of our approach, it almost immediately generalizes to pentagram-type maps in higher dimensions and enables us to treat all these maps on an equal footing. It turns out that our scheme covers all previously known higher-dimensional integrable cases, and also gives rise to a large number of new ones. Furthermore, for many of the previously known integrable maps our approach provides certain missing ingredients, in particular invariant Poisson structures for short-diagonal and dented maps of [16] [18]. Construction of such structures has been an open problem since the introduction of these maps. Furthermore, for these maps we get new Lax representations which are, in a sense, dual to the ones given in [16] [18].

Recall that a refactorization is a mapping of the form $AB \mapsto BA$, where $A$ and $B$ are elements of a non-Abelian group, e.g. matrices. The relation between such mappings and integrability was pointed out in [36] [25] and put in the context of Poisson-Lie groups in [5]. Nowadays, refactorization in Poisson-Lie groups is viewed as one of the most universal mechanisms of integrability for discrete dynamical systems. In this paper we suggest such an interpretation for the pentagram map and its generalizations. Below we briefly describe the construction for the case of the classical pentagram map.

Let $\{v_i \in \mathbb{R}^2\}$ be a planar $n$-gon, and let $\{V_i \in \mathbb{R}^3\}$ be its arbitrary lift to $\mathbb{R}^3$ (here and in what follows we assume that the ground field is real numbers, although all the same constructions work over $\mathbb{C}$). The sequence $V_i$ can be encoded by writing down the relations between quadruples of consecutive vectors:

$$a_i V_i + b_i V_{i+1} + c_i V_{i+2} + d_i V_{i+3} = 0,$$
where \( i \in \mathbb{Z} \), and \( a_i, b_i, c_i, d_i \in \mathbb{R} \) are \( n \)-periodic sequences. This can be equivalently written as \( \mathcal{D} V = 0 \), where \( V \) is a bi-infinite sequence whose entries are the vectors \( V_i \), and \( \mathcal{D} \) is an \( n \)-periodic difference operator

\[
\mathcal{D} := a + bT + cT^2 + dT^3.
\]

Here \( T \) is the left shift operator on bi-infinite sequences, and \( a, b, c, d \) are scalar operators, i.e. operators of the form \( (aV)_i = a V_i \). Thus, one can encode planar polygons by third order difference operators. There is, however, more than one operator corresponding to a given polygon in \( \mathbb{R}P^2 \). Namely, one can multiply \( \mathcal{D} \) by scalar operators from the left or right without changing the corresponding polygon. This means that, for any mapping of the space of polygons to itself, its lift to difference operators is not a map, but a correspondence (a multivalued map). To explicitly describe this correspondence for the case of the pentagram map, we split the difference operator \( \mathcal{D} = a + bT + cT^2 + dT^3 \) into two parts:

\[
\mathcal{D}_+ := a + cT^2, \quad \mathcal{D}_- := bT + dT^3.
\]

**Theorem 1.1.** The pentagram map, written in terms of difference operators, is a multivalued map

\[
\mathcal{D} = \mathcal{D}_+ + \mathcal{D}_- \quad \mapsto \quad \tilde{\mathcal{D}} = \tilde{\mathcal{D}}_+ + \tilde{\mathcal{D}}_-
\]

determined by the relation

\[
\tilde{\mathcal{D}}_+ \mathcal{D}_- V = \tilde{\mathcal{D}}_- \mathcal{D}_+ V, \tag{1}
\]

**Proof sketch.** Equation (1) can be viewed as a homogeneous linear system on \( 4n \) unknown coefficients of the \( n \)-periodic operator \( \mathcal{D} \). Both sides of (1) are linear combinations of \( T \), \( T^3 \), and \( T^5 \) with \( n \)-periodic coefficients, so the number of equations is \( 3n \), which is less than the number of unknowns. Therefore, there always exists a non-trivial solution \( \tilde{\mathcal{D}} \) depending on \( \mathcal{D} \), and (1) indeed defines a multivalued map \( \mathcal{D} \mapsto \tilde{\mathcal{D}} \). To identify the latter with the pentagram map, we need to rewrite it in terms of bi-infinite sequences \( V, \tilde{V} \) annihilated by the operators \( \mathcal{D} \) and \( \tilde{\mathcal{D}} \) respectively. Applying both sides of (1) to \( V \), we get

\[
\tilde{\mathcal{D}}_+ \mathcal{D}_- V = \tilde{\mathcal{D}}_- \mathcal{D}_+ V,
\]

which, using that \( \mathcal{D} V = 0 \) and thus \( \mathcal{D}_- V = -\mathcal{D}_+ V \), can be rewritten as

\[
\tilde{\mathcal{D}} \mathcal{D}_+ V = 0.
\]

But the latter means that \( \tilde{V} = \mathcal{D}_+ V \), which is exactly the definition of the pentagram map. Indeed, by definition of \( \mathcal{D}_+ \), the vector \( (\mathcal{D}_+ V)_i \) belongs to the span \( \langle V_i, V_{i+2} \rangle \) of \( V_i, V_{i+2} \). At the same time, we have \( \mathcal{D}_+ V = -\mathcal{D}_- V \), so

\[
(\mathcal{D}_+ V)_i = -(\mathcal{D}_- V)_i \in \langle V_{i+1}, V_{i+3} \rangle.
\]

Therefore, we have

\[
(\mathcal{D}_+ V)_i \in \langle V_i, V_{i+2} \rangle \cap \langle V_{i+1}, V_{i+3} \rangle,
\]

which means that the corresponding point in \( \mathbb{R}P^2 \) is the intersection of consecutive shortest diagonals \( \langle v_i, v_{i+2} \rangle \) and \( \langle v_{i+1}, v_{i+3} \rangle \), as desired.

\[\square\]

**Corollary 1.2.** The pentagram map, written in terms of difference operators, is a refactorization relation.

**Proof.** Relation (1) can be rewritten as

\[
\tilde{\mathcal{D}}_+^{-1} \tilde{\mathcal{D}}_- = \mathcal{D}_- \mathcal{D}_+^{-1}, \tag{2}
\]

where the inverses of difference operators are understood as pseudo-difference operators. To see that this formula defines a refactorization mapping, consider the operator \( \mathcal{L} := \mathcal{D}_-^{-1} \mathcal{D}_+ \). Then (2) means that the dynamics of \( \mathcal{L} \) under the pentagram map is given by \( \mathcal{L} \mapsto \tilde{\mathcal{L}} \), where \( \tilde{\mathcal{L}} := \mathcal{D}_+ \mathcal{D}_-^{-1} \). Therefore, the pentagram map in terms of \( \mathcal{L} \) is a refactorization map

\[
\mathcal{L}^{-1} \mathcal{D}_+ \mapsto \mathcal{D}_+ \mathcal{D}_-^{-1}.
\]
A crucial part of the proof of Theorem 1.1 is solvability of (1) with respect to $\tilde{D}$, which in turn is related to a very special choice of exponents of the shift operator $T$ entering $D_-$ and $D_+$. We refer to the set of integers that are the exponents of $T$ entering a given difference operator $D$ as the support of $D$. It is easy to see that (1) is solvable if and only if the supports $J_\pm \subset \mathbb{Z}$ of the operators $D_\pm$ satisfy

$$|J_+ + J_-| < |J_-| + |J_+|,$$

where $J_+ + J_-$ is the Minkowski sum. Furthermore, for sets $J_\pm$ with $|J_\pm| > 1$ the latter inequality holds if and only if the $J_\pm$ are finite arithmetic progressions with the same common difference. Different choices of such pairs of progressions lead to different pentagram-type maps admitting a refactorization description. As we already saw, the choice $\{0, 2\}, \{1, 3\}$ corresponds to the usual pentagram map. More generally, the choice $\{0, 2, 4, \ldots\}, \{1, 3, 5, \ldots\}$ corresponds to short-diagonal maps of [10]. Similarly, $\{0, 1\}, \{2, 3\}$ leads to the inverse pentagram map, while $\{0, 1, \ldots, p\}, \{p + 1, p + 2, \ldots, q\}$ corresponds to the inverse dented map of [18]. Finally, the choice $\{0, d\}, \{1, d + 1\}$ leads to the pentagram map on corrugated polygons in $\mathbb{R}^d$ studied in [10].

One can also consider relation (1) for difference operators $D_\pm$ with non-disjoint supports. Such maps still admit a refactorization description, but they do not have a pentagram-like interpretation. Indeed, in this case the pair $D_\pm$ is not equivalent to a single operator $D_+ + D_-$, and because of that the phase space cannot be interpreted as the space of polygons. The simplest case $\{0, 1\}, \{1, 2\}$ corresponds to the leapfrog map defined in [10], while for other cases of non-disjoint supports the geometric interpretation is not known.

The structure of the paper is as follows. In Section 2 we define a general class of pentagram-type maps associated with pairs of disjoint arithmetic progressions with the same common difference. This class, in particular, includes all previously known integrable pentagram-type maps. In Section 3 we discuss difference and pseudo-difference operators, along with Poisson structures on such operators. Section 4 contains main results of the paper, namely we show that pentagram maps of Section 2 fit into an even bigger class of dynamical systems which are parametrized by pairs of not necessarily disjoint progressions and admit a refactorization description. As a corollary, all such maps admit an invariant Poisson structure and a Lax representation with Poisson-commuting spectral invariants. It is therefore very likely that all these maps are both Liouville and algebraically integrable. This integrability problem will be addressed in a separate publication. In addition to these results, in Section 4 we also discuss some applications, as well as relations to known constructions. In particular, in Section 4.2 we show how our approach yields the scaling invariance of pentagram-type maps, which was the central tool in the proof of integrability for the classical, as well as for short-diagonal and dented maps. Furthermore, in Section 4.3 we explicitly compute Poisson brackets for the short-diagonal pentagram map in $\mathbb{R}^4$, while in Section 4.4 we outline the connection between the approach of the present paper and the Y-meshes description of higher pentagram maps given in [12]. It turns out that Y-meshes are related to factorizations of difference operators. Finally, Section 5 is devoted to open questions.

The paper also contains an appendix (Section 6, joint with B. Khesin), where we introduce a new class of long-diagonal pentagram maps, which can be thought of as alternative geometric realizations of maps described in Section 2. This class not only encompasses all known integrable cases, but also includes some of the maps whose integrability was observed numerically in [17]. As a corollary, we describe the continuous limit of all pentagram-type maps appearing from refactorization: their limits turn out to be Boussinesq-type equations in the KdV hierarchy.

Acknowledgments. The author is grateful to Anton Alekseev, Michael Gekhtman, Boris Khesin, Nicolai Reshetikhin, Richard Schwartz, Alexander Shapiro, Yuri Suris, and Sergei Tabachnikov for fruitful discussions. A large part of this work was done during the author’s visit to Max Planck Institute for Mathematics (Bonn). The author would like to thank the Institute’s faculty and staff for their support and stimulating atmosphere.
2 Pentagram-type maps associated with pairs of arithmetic progressions

In this section we explain how to associate a pentagram-type map to any pair of finite disjoint arithmetic progressions $J_{\pm} \subset \mathbb{Z}$ with the same common difference. As particular cases of this construction, one obtains all known integrable pentagram-type maps. Later on, in Section 4.4, we will show that these maps fit into a more general class of dynamical systems which are parametrized by pairs of not necessarily disjoint progressions and admit a refactorization description.

All pentagram-type maps operate on polygons, i.e. ordered sequences of points in the projective space. We will only consider polygons satisfying the following natural condition:

**Definition 2.1.** A polygon in $\mathbb{R}P^d$ is a bi-infinite sequence of points $\{v_i \in \mathbb{R}P^d\}$ such that any $d + 1$ consecutive points $v_i, \ldots, v_{i+d+1}$ are in general position (i.e. do not belong to a subspace of dimension $d - 1$).

In contrast to the classical pentagram map, which is well-defined for all generic polygons, some of the maps that we will study operate on a more restricted class of polygons whose vertices satisfy certain additional coplanarity conditions, described in the following definition:

**Definition 2.2.** Let $J \subset \mathbb{Z}$, $|J| \geq 2$ be a finite set of integers containing at least two elements, and let $d := \max(J) - \min(J) - 1$. Then a polygon $\{v_i\}$ in $\mathbb{R}P^d$ is called $J$-corrugated if for any $i \in \mathbb{Z}$ the points $\{v_{i+j} \mid j \in J\}$ belong to a $|J| - 2$ dimensional plane (instead of a $|J| - 1$ dimensional plane, which is the generic case).

**Example 2.3.** Assume that $J$ consists of consecutive integers, $J = \{j, j+1, \ldots, j+d+1\}$. Then a $J$-corrugated polygon is any polygon in $\mathbb{R}P^d$.

**Example 2.4.** Assume that $J = \{0, 1, d, d+1\}$. Then $J$-corrugated polygons are corrugated polygons in $\mathbb{R}P^d$ in the sense of [10, Section 5.1.1].

**Example 2.5.** Assume that $J = \{0, 1, \ldots, l\} \cup \{m, m+1, \ldots, d+1\}$, where $l < m$, is a union of two disjoint sets of consecutive integers. Then $J$-corrugated polygons are partially corrugated polygons in $\mathbb{R}P^d$ in the sense of [13, Definition 6.3].

We now define an analogue of the pentagram map on the space of $J$-corrugated polygons. Such a map can be defined if $J \subset \mathbb{Z}$ can be partitioned as $J = J_+ \cup J_-$, where $J_{\pm} \subset \mathbb{Z}$ are finite arithmetic progressions with the same common difference.

**Definition 2.6.** Let $J_{\pm} \subset \mathbb{Z}$ be non-empty disjoint finite integral arithmetic progressions with the same common difference. Let also $J := J_+ \cup J_-$. Then the pentagram map associated with the pair $J_{\pm}$ is the map from the space of twisted $J$-corrugated $n$-gons to itself defined by

$$\tilde{v}_i := \langle v_{i+j} \mid j \in J_+ \rangle \cap \langle v_{i+j} \mid j \in J_- \rangle.$$ 

Here $v_i$'s are the vertices of the initial polygon, $\tilde{v}_i$'s are vertices of its image under the map, and the notation $\langle v_i \rangle$ stands for the projective subspace spanned by the points $\{v_i\}$.

**Remark 2.7.** This definition makes sense for arbitrary disjoint finite sets $J_{\pm} \subset \mathbb{Z}$, but for general $J_{\pm}$ the image of a $J$-corrugated polygon (where $J := J_+ \cup J_-$) under the so defined map is not $J$-corrugated. This property, however, does hold if $J_{\pm}$ are arithmetic progressions with the same common difference, as shown by the following proposition.

**Proposition 2.8.** For any non-empty disjoint finite arithmetic progressions $J_{\pm} \subset \mathbb{Z}$ with the same common difference, the corresponding pentagram map is a generically well-defined mapping from the space of $J$-corrugated $n$-gons to itself.

**Proof.** For a generic $J$-corrugated polygon $\{v_i \in \mathbb{R}P^d\}$, where $d = \max(J) - \min(J) - 1$, the subspaces $\langle v_{i+j} \mid j \in J_{\pm} \rangle$ have complementary dimensions $|J_{\pm}| - 1$ inside the $(|J| - 2)$-dimensional space $\langle v_{i+j} \mid j \in J \rangle$. Therefore, their intersection indeed defines a point $\tilde{v}_i \in \mathbb{R}P^d$. Furthermore, it is not hard to see that for generic $v_i$ any $d + 1$ consecutive points $\tilde{v}_i$ will be in general position, so $\{\tilde{v}_i\}$ is a polygon.
In the sense of Definition 2.7, so it remains to show that the new polygon \( \{ \tilde{v}_i \} \) is \( J \)-corrugated. To that end, for any \( i \in \mathbb{Z} \), consider the subspace

\[
L_i := \langle v_{i+j} \mid j \in J_+ + J_- \rangle,
\]

where \( J_+ + J_- \) is the Minkowski sum of \( J_+ \) and \( J_- \). Notice that for any \( i \in \mathbb{Z} \) and any \( j \in J \), we have \( \tilde{v}_{i+j} \in L_i \). Indeed, by construction of the polygon \( \{ \tilde{v}_i \} \), we have

\[
\tilde{v}_{i+j} = v_{i+j} + v_{i+j} \in J_+ + J_- \subset L_i.
\]

Assume that \( j \in J_+ \). Then \( \langle v_{i+j} \mid j \in J_- \rangle + \langle v_{i+j} \mid j \in J_+ \rangle \) is a subspace of \( L_i \), because \( j + j' \in J_+ + J_- \). Therefore, \( \tilde{v}_{i+j} \in L_i \). Analogously, if \( j \in J_- \), then \( \langle v_{i+j} \mid j \in J_- \rangle + \langle v_{i+j} \mid j \in J_+ \rangle \) is a subspace of \( L_i \), and we still have \( \tilde{v}_{i+j} \in L_i \). So, all points \( \tilde{v}_{i+j} \mid j \in J \) belong to \( L_i \). But the dimension of \( L_i \) does not exceed

\[
|J_+ + J_-| - 1 = |J_+| + |J_-| - 2 = |J| - 2,
\]

where we use that for finite arithmetic progressions \( J \subset \mathbb{Z} \) with the same common difference one has \( |J_+ + J_-| = |J_+| + |J_-| - 1 \). So, for every \( i \in \mathbb{Z} \), the points \( \{ \tilde{v}_{i+j} \mid j \in J \} \) belong to at most \( |J| - 2 \) dimensional subspace \( L_i \), which means the polygon \( \{ \tilde{v}_i \} \) is indeed \( J \)-corrugated, as desired.

**Example 2.9.** Examples of pentagram maps associated with pairs of arithmetic progressions are given in Table 1. Note that these examples cover all known integrable cases: all such cases fit into the above construction.

**Remark 2.10.** The classical pentagram is usually defined by \( \tilde{v}_i = \langle v_{i-1}, v_{i+1} \rangle \cap \langle v_i, v_{i+2} \rangle \) (right labelling scheme), or by \( \tilde{v}_i = \langle v_{i-2}, v_i \rangle \cap \langle v_{i-1}, v_{i+1} \rangle \) (left labelling scheme). This corresponds to progressions \( \{ -1,1 \}, \{ 0,2 \} \) for the right scheme, and \( \{ -2,0 \}, \{ -1,1 \} \) for the left scheme. Our choice \( \{ 0,2 \}, \{ 1,3 \} \) corresponds to the same map, but with a different labeling of vertices of the resulting polygon. More generally, shifting both \( J_+ \) and \( J_- \) by the same number results in the same map up to a shift of indices.

**Remark 2.11.** Note that except for the short-diagonal and inverse dented cases, our construction gives no maps which are defined on all generic polygons (with no additional coplanarity conditions). Indeed, such maps would correspond to sets \( J \) consisting of consecutive integers (cf. Example 2.3), and without loss of generality we can assume that \( J = \{ 0,1, \ldots \} \) (because we can always shift \( J \), as in Remark 2.10). But the only ways to represent this set \( J \) as a disjoint union of two arithmetic progressions with the same common difference are \( \{ 0,1, \ldots , k \} \cup \{ k+1, k+2, \ldots \} \) and \( \{ 0,2,4, \ldots \} \cup \{ 1,3,5, \ldots \} \), which corresponds to the inverse dented and short-diagonal maps respectively. There is, however, a way to modify our construction to produce other maps defined on generic polygons. This gives what we call long-diagonal pentagram maps, described in detail in the appendix (Section 6).
The space of $J$-corrugated polygons is infinite-dimensional for any $J$ with $|J| > 2$. One can still study pentagram-type maps on such spaces, but to obtain integrable dynamics one should impose some kind of boundary conditions on the polygon $\{v_i\}$. From the geometric perspective, the most natural condition is closedness, $v_{i+n} = v_i$. However, it turns out that the pentagram map, as well as similar maps studied in the present paper, have much better properties on a bigger space of polygons that are closed only up to a projective transformation. Such polygons as known as twisted:

**Definition 2.12.** A twisted $n$-gon is a polygon $\{v_i \in \mathbb{RP}^d\}$ such that $v_{i+n} = \phi(v_i)$ for every $i$ and a fixed (not depending on $i$) projective transformation $\phi: \mathbb{RP}^d \to \mathbb{RP}^d$, called the monodromy.

It is clear that all pentagram maps defined above (as well as any other map on polygons which is defined using only projectively natural operations) take twisted polygons to twisted polygons and, moreover, preserve the monodromy. Throughout the paper, all pentagram-type maps are assumed to operate on twisted polygons.

### 3 Difference and pseudo-difference operators

#### 3.1 Generalities on difference operators

In this section we recall some basic notions related to difference operators. Our terminology mainly follows that of [35]. Let $\mathbb{R}^\infty$ be the vector space of bi-infinite sequences of real numbers, and let $J \subset \mathbb{Z}$ be a finite collection of integers. A linear operator $\mathcal{D}: \mathbb{R}^\infty \to \mathbb{R}^\infty$ is called a difference operator supported in $J$ if it can be written as

$$\langle \mathcal{D} \xi \rangle_i = \sum_{j \in J} a_{j,i} \xi_{i+j},$$

or, equivalently, if

$$\mathcal{D} = \sum_{j \in J} a_j T^j,$$

where $T: \mathbb{R}^\infty \to \mathbb{R}^\infty$ is the left shift operator $(T\xi)_i = \xi_{i+1}$, and each coefficient $a_j$ is a sequence $\{a_{j,i} \mid i \in \mathbb{Z}\}$ acting on $\mathbb{R}^\infty$ by term-wise multiplication. Such sequences can per se be regarded as difference operators with $J = \{0\}$ and are called scalar difference operators.

The order of difference operator (3) is the number $\text{ord } \mathcal{D} := \max J - \min J$. Difference operator (3) is called properly bounded if none of the elements of sequences $a_{\min J}$, $a_{\max J}$ vanish. Clearly, for a properly bounded difference operator $\mathcal{D}$ one has $\dim \text{Ker } \mathcal{D} = \text{ord } \mathcal{D}$. A difference operator $\mathcal{D}$ is $n$-periodic if all its coefficients $a_j$ are $n$-periodic sequences, which is equivalent to saying that $\mathcal{D}$ commutes with the $n$'th power of the shift operator: $\mathcal{D} T^n = T^n \mathcal{D}$. Clearly, if $\mathcal{D}$ is an $n$-periodic operator, then its kernel is invariant under the action of $T^n$. The finite-dimensional operator $T^n|_{\text{Ker } \mathcal{D}}$ is called the monodromy of $\mathcal{D}$. Eigenvectors of the monodromy operator $T^n|_{\text{Ker } \mathcal{D}}$ are exactly quasi-periodic solutions of the equation $\mathcal{D} \xi = 0$, i.e. solutions which belong to the space of quasi-periodic sequences

$$QPS_n(z) := \{\xi \in \mathbb{R}^\infty \mid \xi_{i+n} = z\xi_i\}$$

for certain $z \in \mathbb{R}^\times$.

We denote the space of $n$-periodic difference operators supported in $J$ by $\text{DO}_n(J)$, while $\text{PBDO}_n(J) \subset \text{DO}_n(J)$ stands for the (dense) subset of properly bounded operators. Let also $\text{DO}_n$ be the associative algebra of all $n$-periodic difference operators (with arbitrary finite support).

**Remark 3.1.** The algebra $\text{DO}_n$ of $n$-periodic difference operators is isomorphic to the algebra $\text{Mat}_n \otimes \mathbb{R}[z, z^{-1}]$ of $\text{Mat}_n$-valued Laurent polynomials in one variable $z$ (here $\text{Mat}_n$ stands for the associative algebra of $n \times n$ matrices over the base field $\mathbb{R}$). Indeed, consider the natural action of $n$-periodic difference operators on the space (4) of all $n$-quasi-periodic bi-infinite sequences of real numbers with monodromy $z$. This gives a 1-parametric family $\rho_z$ of $n$-dimensional representations of the algebra $\text{DO}_n$. 


DO_\n. In each of the spaces \(\tilde{\mathcal{D}}\), take a basis \(\xi_1, \ldots, \xi_\n\) determined by the condition \(\xi_{ij} = \delta_{ij}\) for \(i, j = 1, \ldots, \n\) (where \(\delta_{ij}\) is the Kronecker delta). Written in this basis, the representation \(\rho_\n\) takes an \(n\)-periodic sequence \(a = \{a_1\}\), viewed as a scalar difference operator, to a diagonal matrix with entries \(a_1, \ldots, a_\n\), while the shift operator \(T\) becomes the matrix \(\sum_{i=1}^{\n-1} E_{i,i+1} + z E_{\n,1}\), where \(E_{i,j}\) is the matrix with a 1 at position \((i, j)\) and zeros elsewhere. Therefore, since the algebra of difference operators is generated by scalar operators, \(T\), and \(T^{-1}\), it follows that \(\rho_\n\) can be viewed as a homomorphism of difference operators into \(\text{Mat}_n \otimes \mathbb{R}[z, z^{-1}]\). Furthermore, it is easy to verify that this homomorphism is a bijection, and hence an isomorphism.

**Proposition 3.2.** Let \(\mathcal{D}\) be a properly bounded difference operator supported in \(J\), and let \(\mathcal{D}(z)\) be the associated element of the loop algebra. Then \(\det \mathcal{D}(z)\) is a constant multiple of the polynomial \(z^{\min J} \text{CPM}_D(z)\), where \(\text{CPM}_D\) is the characteristic polynomial of the monodromy of \(\mathcal{D}\).

**Proof.** If we multiply \(\mathcal{D}\) by \(T^m\), where \(m \in \mathbb{Z}\) then the characteristic polynomial of its monodromy does not change, while the polynomial \(\det T^m(z) = (\det T(z))^m = z^m\). So, it suffices to consider the case \(\min J = 0\). Furthermore, it is sufficient to prove the statement for generic properly bounded operators supported in \([0, \ldots, d]\), because within that set the coefficients of both polynomials \(\det \mathcal{D}(z)\) and \(\text{CPM}_D(z)\) are polynomial functions in terms of the coefficients of \(\mathcal{D}\). So, if one can show that these polynomials are proportional for generic operators, then it must be true for all operators. To establish the statement for generic \(\mathcal{D}\), observe that by definition of \(\mathcal{D}(z)\) the polynomial \(\det \mathcal{D}(z)\) vanishes for some \(z \neq 0\) if and only if \(\mathcal{D}\) has a kernel on the space \(\mathcal{QPS}_n(z)\), which is equivalent to saying that \(z\) is an eigenvalue of the monodromy of \(\mathcal{D}\). So, the roots of the polynomials \(\det \mathcal{D}(z)\) and \(\text{CPM}_D(z)\) are the same (as sets). Furthermore, for generic \(\mathcal{D}\) all roots of \(\text{CPM}_D(z)\) are distinct. So, to prove that the polynomials \(\det \mathcal{D}(z)\) and \(\text{CPM}_D(z)\) are proportional, it suffices to show that they have the same degree. In other words, we need to show that the degree of \(\det \mathcal{D}(z)\) is equal to the degree \(d\) of \(\mathcal{D}\). This can be checked by explicitly writing down the matrix \(\mathcal{D}(z)\), or by using the following argument. First of all, one easily checks that the statement holds for operators of degree 1. But a generic operator \(\mathcal{D}\) of degree \(d\) can be written as a product of operators of degree 1, so by multiplicativity for such operator we have \(\deg \det \mathcal{D}(z) = d\), as desired. \(\square\)

### 3.2 Difference operators and \(J\)-corrugated polygons

There is a close relation between difference operators supported in \(J\) and \(J\)-corrugated polygons. Denote by \(\mathcal{P}_n(J)\) the space of twisted \(J\)-corrugated \(n\)-gons, and let \(\mathcal{P}_n(J) / \text{PGL}\) be the quotient of that space by projective transformations. We will describe that space as a certain quotient of the space \(\text{PBDO}_n(J)\) of properly bounded \(n\)-periodic difference operators supported in \(J\). Namely, let \(H\) be the group of invertible \(n\)-quasi-periodic scalar difference operators, i.e.

\[
H := \{ \alpha \mid \exists z \in \mathbb{R}^* \text{ s.t. } \alpha \in \mathcal{QPS}_n(z) \text{ and } \alpha_i \neq 0 \forall i \in \mathbb{Z} \}.
\]

Further, let \(H \bowtie H\) be the subgroup of \(H \times H\) that consists of pairs of invertible \(n\)-quasi-periodic scalar difference operators with the same monodromy, i.e.

\[
H \bowtie H := \{ (\alpha, \beta) \mid \exists z \in \mathbb{R}^* \text{ s.t. } \alpha, \beta \in \mathcal{QPS}_n(z) \text{ and } \alpha_i, \beta_i \neq 0 \forall i \in \mathbb{Z} \}.
\]

This group acts on the space \(\text{DO}_n(J)\) of \(n\)-periodic difference operators with given support by means of the left-right action

\[
\mathcal{D} \mapsto \alpha \mathcal{D} \beta^{-1}.
\]

**Proposition 3.3.** For any finite subset \(J \subset \mathbb{Z}\) with \(|J| \geq 2\), there is a one-to-one correspondence (a homeomorphism) between the following spaces:

1. The space \(\mathcal{P}_n(J) / \text{PGL}\) of twisted \(J\)-corrugated \(n\)-gons modulo projective transformations.
2. The space \(\text{PBDO}_n(J) / H \bowtie H\) of properly bounded \(n\)-periodic difference operators supported in \(J\) modulo the left-right action \(5\) of the group \(H \bowtie H\) of pairs of invertible \(n\)-quasi-periodic scalar difference operators with the same monodromy.

8
Proof. The proof is analogous to that of [14, Proposition 2.2]. Let us just briefly outline the construction. Given a projective equivalence class of generic twisted \( J \)-corrugated \( n \)-gons, consider an arbitrary representative \( \{ v_i \in \mathbb{RP}^d \} \in \mathcal{P}_n(J) \) of that class (here \( d := \max(J) - \min(J) - 1 \)). Lift the quasi-periodic sequence of points \( v_i \in \mathbb{RP}^d \) to a quasi-periodic sequence of vectors \( V_i \in \mathbb{R}^{d+1} \). Then, from the \( J \)-corrugated condition it follows that for any \( i \in \mathbb{Z} \) the vectors \( \{ V_{i+j} \mid j \in J \} \) belong to a subspace of dimension \( |J| - 1 \) and, therefore, are linearly dependent:

\[
\sum_{j \in J} a_{j,i} V_{i+j} = 0. \tag{6}
\]

This is equivalent to \( DV = 0 \), where \( V \) is the bi-infinite sequence of \( V_i \)'s, and the operator \( D \) is given by [3]. Furthermore, since the sequence \( \{ V_i \} \) is quasi-periodic, the so-obtained operator \( D \) is periodic, while from the genericity condition for \( \{ v_i \in \mathbb{RP}^d \} \) it follows that \( D \) is properly bounded. Hence we obtain a properly bounded \( n \)-periodic difference operator supported in \( J \). To complete the proof, it suffices to notice that from the possibility to rescale each of the \( V_i \)'s and also multiply each of the equations [6] by a scalar it follows that \( D \) is defined up to the left-right action [5]. Details of the proof (in the case when \( J \) consists of four consecutive integers) can be found in [14].

As can be seen from this construction, one has the following relation between the monodromy of a \( J \)-corrugated polygon and the monodromy of the corresponding difference operator:

**Corollary 3.4.** Let \( P \in \mathcal{P}_n(J) \) be a twisted \( J \)-corrugated \( n \)-gon, and let \( D \in \text{PBDO}_n(J) \) be one of the corresponding difference operators. Then the monodromy of \( P \) is conjugate to the projectivization of the monodromy of \( D \).

**Proof.** Assume that the monodromy of the polygon \( \{ v_i \} \) in the proof of Proposition 3.3 is given by the projective transformation \( \phi \). Then the sequence of \( V_i \)'s satisfies \( V_{i+n} = MV_i \), where \( M \) is a matrix of \( \phi \) (i.e. \( \phi \) is a projectivization of \( M \)). At the same time, the components of the vectors \( V_i \) form a basis in the space \( \text{Ker} \, D \), and the monodromy matrix \( T^n \vert_{\text{Ker} \, D} \) written in that basis is the transpose of \( M \). So, the monodromy of the polygon \( \{ v_i \} \) is the projectivization of \( (T^n \vert_{\text{Ker} \, D})^t \), and hence is conjugate to the projectivization of \( T^n \vert_{\text{Ker} \, D} \).

In particular, one has the following relation between the eigenvalues of the monodromy and the determinant of the corresponding loop algebra element:

**Corollary 3.5.** Let \( P \in \mathcal{P}_n(J) \) be a twisted \( J \)-corrugated \( n \)-gon, and let \( D \in \text{PBDO}_n(J) \) be one of the corresponding difference operators. Then the eigenvalues of the monodromy of \( P \) coincide (with multiplicities) with non-zero roots of the polynomial \( \det D(z) \), where \( D(z) \) is an element of the loop algebra corresponding to the difference operator \( D \), as described in Remark 3.1.

**Proof.** This follows from Proposition 3.2.

**Remark 3.6.** Note that the monodromy of a twisted polygon is a projective transformation, so its eigenvalues are defined up to simultaneous multiplication by the same constant. However, the same is true for the roots of \( \det D(z) \), because taking \( a \) and \( b \) in [5] with non-trivial monodromy \( w \) leads to simultaneous rescaling of all the roots by a factor of \( w \).

### 3.3 The Poisson-Lie group of pseudo-difference operators

We define an \( n \)-periodic pseudo-difference operator as a formal Laurent series in terms of the left shift operator \( T \), whose coefficients are \( n \)-periodic sequences. In other words, every such operator is of the form

\[
\sum_{j=k}^{+\infty} a_j T^j, \tag{7}
\]

where \( k \in \mathbb{Z} \) is an integer, \( T \) is the left shift operator on \( \mathbb{R}^\infty \), while each \( a_j \) is an \( n \)-periodic bi-infinite sequence of real numbers. Such an expression can be regarded either as a formal sum, or as an actual operator acting on the space \( \{ \xi \in \mathbb{R}^\infty \mid \exists \, j \in \mathbb{Z} : \xi_i = 0 \forall \, i > j \} \) of eventually vanishing sequences.
We will denote the set of $n$-periodic pseudo-difference operators by $\Psi DO_n$. This set is an associative algebra. Moreover, almost every pseudo-difference operator in invertible. In particular, \(f(\alpha)\) is invertible if the coefficient $a_k$ of lowest power in $T$ is a sequence none of whose elements vanish. We will denote the set of invertible $n$-periodic pseudo-difference operators by $I\Psi DO_n$. This is a group with respect to multiplication. It can be regarded as an infinite-dimensional Lie group with Lie algebra $\Psi DO_n$.

**Remark 3.7.** One can also consider (and apply for the purposes of the present paper) pseudo-difference operators which have infinitely many terms of negative degree in $T$, but only finitely many terms of positive degree. This leads to an isomorphic algebra.

**Remark 3.8.** The isomorphism $DO_n \simeq \text{Mat}_n \otimes \mathbb{R}[z, z^{-1}]$ described in Remark 3.1 naturally extends to an isomorphism between the algebra of $n$-periodic pseudo-difference operators, and the algebra $\text{Mat}_n \otimes \mathbb{R}((z))$ of matrices over the field $\mathbb{R}((z))$ of formal Laurent series with real coefficients and finitely many terms of negative degree. Under this isomorphism, the group $I\Psi DO_n$ of invertible pseudo-difference operators is identified with the group of matrices over Laurent series with non-vanishing determinant (this group is one of the versions of the loop group of $GL_n$).

**Proposition 3.9.** There exists a natural Poisson structure $\pi$ on the group $I\Psi DO_n$ of $n$-periodic invertible pseudo-difference operators. This structure has the following properties:

1. It is multiplicative, in the sense that the group multiplication is a Poisson map. In other words, the group $I\Psi DO_n$, together with the structure $\pi$, is a Poisson-Lie group.

2. Assume that $J \subset \mathbb{Z}$ is a finite subset that consists of consecutive integers. Then the subset $IDO_n(J) := I\Psi DO_n \cap DO_n(J)$ of invertible difference operators supported in $J$ is a Poisson submanifold of $I\Psi DO_n$.

3. If $J$ is a one-point set, then the restriction of $\pi$ to $IDO_n(J)$ is zero. In particular, the Poisson structure $\pi$ vanishes on scalar operators.

4. The Poisson structure $\pi$ is invariant under the left-right action $\tilde{\pi}^J$ of pairs of invertible $n$-quasi-periodic scalar difference operators with the same monodromy.

5. Central functions on $I\Psi DO_n$ Poisson commute.

**Remark 3.10.** As explained in Remark 3.8, the group $I\Psi DO_n$ is isomorphic to a version of the loop group of $GL_n$. In the loop group language, the Poisson structure $\pi$ is well-known: it is the one associated with the trigonometric $r$-matrix. Here we will provide a construction of this Poisson structure which does not appeal to the loop group formalism. In fact, the language of (pseudo)difference operators seems to be more natural when dealing with the trigonometric $r$-matrix. We will, however, use the loop group language in some of the computations, see in particular Section 3.6.

Our Poisson structure on pseudo-difference operators can also be viewed as a natural discrete analogue of the Poisson-Lie structure on pseudodifferential operators $\Psi DO_n$.

**Remark 3.11.** For periodic sequences $\alpha$ and $\beta$, the fourth statement of Proposition 3.9 follows from the third one combined with the first. Indeed, by the third statement the Poisson structure $\pi$ vanishes on scalar operators, so from multiplicativity we get that both left and right multiplications by scalar operators are Poisson maps. However, if the sequences $\alpha$ and $\beta$ have non-trivial monodromy, then one cannot extract the fourth statement of the proposition from multiplicativity, because in that case $\alpha$ and $\beta$ are not elements of $I\Psi DO_n$.

**Remark 3.12.** As central functions on $I\Psi DO_n$, one can take expressions of the form $f_{ij}(D) := \text{Tr} T^m D^l$, where $i \in \mathbb{Z}, j \in \mathbb{Z}^+$, and the trace of a pseudo-difference operator is defined by formula (13) below. An alternative way to get the same functions is to consider the matrix-valued Laurent series $D(z) = \sum D_i z^i$ corresponding to the operator $D$ (cf. Remark 3.8), and then take the spectral invariants of the matrices $D_i$ (e.g. traces of powers or coefficients of the characteristic polynomial).

**Remark 3.13.** Take a subset $J \subset \mathbb{Z}$ which consists of $d+2$ consecutive integers. Then the $J$-corrugated condition is vacuous and, according to Proposition 3.3, the quotient of $PBDO_n(J)$ by the action (5) can be identified with the space of twisted polygons in $\mathbb{R}^d$, considered up to projective equivalence. So, restricting the Poisson structure $\pi$ to $PBDO_n(J)$ (which is an open subset of $IDO_n(J)$ and hence
a Poisson submanifold) and taking the quotient under the action \( \pi \) one gets a Poisson structure on the space of polygons. It seems, however, that this structure has nothing to do with pentagram maps. As will be explained below, Poisson structures invariant under pentagram maps arise from Poisson submanifolds of \( \text{I\hskip-3pt} \Psi \text{DO}_n \) given by rational pseudo-difference operators, i.e. operators that can be written as a quotient of two difference operators.

We will prove Proposition 3.9 in Section 3.5 after a brief general discussion of Poisson-Lie groups in Section 3.4.

### 3.4 Generalities on Poisson-Lie groups

This section is a brief introduction to the theory of Poisson-Lie groups. Our terminology follows that of [29]. Recall that a Lie group \( G \) endowed with a Poisson structure \( \pi \) is called a Poisson-Lie group if \( \pi \) is multiplicative, i.e. if the multiplication \( G \times G \to G \) is a Poisson map (it also follows from this that the inversion map \( i: G \to G \) is anti-Poisson, i.e. \( i \pi = -\pi \)). Assume that \( G \) is a Poisson-Lie group, and let \( \mathfrak{g} \) be its Lie algebra. Then, trivializing the tangent bundle of \( G \), one can identify the bivector field \( \pi \) with a map \( G \to \mathfrak{g} \wedge \mathfrak{g} \). Furthermore, one can show that multiplicativity of \( \pi \) is equivalent to that map being a cocycle on \( G \) with respect to the adjoint representation of \( G \) on \( \mathfrak{g} \wedge \mathfrak{g} \). If that cocycle is a coboundary, then \( G \) is called a coboundary Poisson-Lie group. A Poisson-Lie group \( G \) is coboundary if and only if there exists an element \( r \in \mathfrak{g} \wedge \mathfrak{g} \), called the classical \( r \)-matrix, such that the Poisson tensor \( \pi \) at every point \( g \in G \) is given by

\[
\pi_g = \frac{1}{2} \left( \langle \lambda_g \rangle_*, r - (\rho_g)_* r \right),
\]

where \( \lambda_g \) and \( \rho_g \) are, respectively, the left and right translations by \( g \). Note that although the bivector \( \pi \) is automatically multiplicative (since any coboundary is a cocycle), it does not need to satisfy the Jacobi identity. The necessary and sufficient condition for \( \pi \) to satisfy the Jacobi identity is a rather complicated equation in terms of \( r \) which is usually replaced by simpler sufficient conditions, such as the modified Yang-Baxter equation. We will state this condition under the assumption that the Lie algebra \( \mathfrak{g} \) is endowed with an invariant (under the adjoint action of \( G \)) inner product, in which case one can identify the bivector \( r \in \mathfrak{g} \wedge \mathfrak{g} \) with a skew-symmetric operator \( r: \mathfrak{g} \to \mathfrak{g} \). In terms of that operator, the modified Yang-Baxter equation reads

\[
[r x, r y] - r [x, y] - r [x, r y] = -[x, y] \quad \forall x, y \in \mathfrak{g}.
\]

It is well-known that this equation implies the Jacobi identity for \( \pi \). If the Lie algebra of a coboundary Poisson-Lie group \( G \) is endowed with an invariant inner product, and the corresponding \( r \)-matrix satisfies the modified Yang-Baxter equation, then \( G \) is called factorizable. In what follows, we will be interested in one particular type of \( r \)-matrices satisfying the modified Yang-Baxter equation:

**Proposition 3.14.** Let \( \mathfrak{g} \) be a Lie algebra endowed with an invariant inner product. Assume also that \( \mathfrak{g} \), as a vector space, can be written as a direct sum of three subalgebras \( \mathfrak{g}_+, \mathfrak{g}_0, \) and \( \mathfrak{g}_- \), such that \([\mathfrak{g}_0, \mathfrak{g}_ \pm] \subset \mathfrak{g}_ \mp \), the subalgebras \( \mathfrak{g}_ \pm \) are isotropic, and \( \mathfrak{g}_0 \) is orthogonal to both \( \mathfrak{g}_+ \) and \( \mathfrak{g}_- \). Then \( r := p_+ - p_- \), where \( p_\pm \) are projectors \( \mathfrak{g} \to \mathfrak{g}_ \pm \), satisfies the modified Yang-Baxter equation, thus turning the group \( G \) of the Lie algebra \( \mathfrak{g} \) into a factorizable Poisson-Lie group.

**Proof.** Direct verification of (9).

**Remark 3.15.** Formula (8) for the coboundary Poisson-Lie bracket can be written in a more explicit form when the Lie group \( G \) can be embedded, as an open subset, into an associative algebra \( A \) (in a typical situation \( G \) coincides with the group of invertible elements in \( A \), e.g. the group of invertible matrices inside the algebra of all \( n \times n \) matrices). In this case, the Lie algebra of \( G \) and, more generally, the tangent space to \( G \) at any point can be naturally identified with \( A \). Assume also that \( A \) is endowed with an invariant inner product, which in the context of associative algebras means that \( \langle x, y \rangle = \langle x y z \rangle \) for any \( x, y, z \in A \) (in particular, this inner product is invariant with respect to the adjoint action of \( G \subset A \) on \( A \)). In that case the \( r \)-matrix can be thought of as a skew-symmetric operator \( r: A \to A \),
and identifying the cotangent space \( T^* G \) with the tangent space \( T_g G = A \) by means of the invariant inner product, one can rewrite formula (8) for the corresponding Poisson tensor on \( G \) as

\[
\pi_g(x, y) = \frac{1}{2} \left( \langle r(xg), yg \rangle - \langle r(gx), gy \rangle \right) \quad \forall g \in G, x, y \in A.
\]  

(10)

The corresponding Poisson bracket is given by

\[
\{ f_1, f_2 \}(g) = \frac{1}{2} \left( \langle r(\text{grad} f_1(g) \cdot g), \text{grad} f_2(g) \cdot g \rangle - \langle r(g \cdot \text{grad} f_1(g)), g \cdot \text{grad} f_2(g) \rangle \right),
\]  

where the gradients are defined using the invariant inner product. Notice that the right-hand side of this formula is actually defined for every \( g \in A \), i.e., invertibility of \( g \) is not necessary. Therefore, this formula may be used to define a Poisson bracket on the whole of \( A \). This bracket is known as the second Gelfand-Dickey bracket on the associative algebra \( A \).

In what follows we will need the following standard facts about coboundary Poisson-Lie groups:

**Proposition 3.16.** Let \( G \) be a Lie group endowed with a coboundary Poisson structure \( \pi \), and let \( g \in G \). Then the Poisson structure \( \pi \) vanishes at \( g \) if and only if \( (\text{Ad}_g)_r = r \).

**Proof.** We have \( (\text{Ad}_g)_r = (\rho^{-1}_g)_* (\lambda_g)_* r \), so \( (\text{Ad}_g)_r = r \) if and only if \( (\lambda_g)_* r = (\rho_g)_* r \), i.e. \( \pi_g = 0 \). \(\square\)

**Proposition 3.17.** Let \( \sigma : G \to G \) be an automorphism of a coboundary Poisson-Lie group. Assume that the differential of \( \sigma \) at the identity preserves the \( r \)-matrix. Then \( \sigma \) is a Poisson map.

**Proof.** Since \( \sigma \) is an automorphism, we have \( \lambda_{\sigma(g)} = \sigma \lambda_g \sigma^{-1} \) and \( \rho_{\sigma(g)} = \sigma \rho_g \sigma^{-1} \), so

\[
\pi_{\sigma(g)} = \frac{1}{2} \left( (\lambda_{\sigma(g)})_* r - (\rho_{\sigma(g)})_* r \right) = \frac{1}{2} \left( \sigma_* (\lambda_g)_* r - \sigma_* (\rho_g)_* r \right).
\]

Since \( \sigma \) preserves the \( r \)-matrix, the latter expression can be rewritten as

\[
\frac{1}{2} \left( \sigma_* (\lambda_g)_* r - \sigma_* (\rho_g)_* r \right) = \sigma_* \pi_g.
\]

So, \( \pi_{\sigma(g)} = \sigma_* \pi_g \), which means that \( \sigma \) is a Poisson map. \(\square\)

**Proposition 3.18.** Central functions on a coboundary Poisson-Lie group Poisson commute.

**Proof.** Formula (8) is equivalent to

\[
\{ f_1, f_2 \}(g) = \frac{1}{2} \left( r(\lambda_g^* df_1(g), \lambda_g^* df_2(g)) - r(\rho_g^* df_1(g), \rho_g^* df_2(g)) \right) \quad \forall f_1, f_2 \in C^\infty(G), g \in G.
\]

But for central functions \( f_1, f_2 \) we have \( f_i \circ \lambda_g = f_i \circ \rho_g \Rightarrow \lambda_g^* df_i(g) = \rho_g^* df_i(g) \Rightarrow \{ f_1, f_2 \} = 0 \). \(\square\)

### 3.5 Existence and properties of the Poisson structure

In this section we prove Proposition 3.9 describing the Poisson structure on the group \( \PsiDO_n \) of \( n \)-periodic invertible pseudo-difference operators. To define that structure, we will use the construction described in Proposition 3.14. The Lie algebra of the group \( \PsiDO_n \) is the space \( \PsiDO_n \) of all \( n \)-periodic pseudo-difference operators. That is actually an associative algebra in which \( \PsiDO_n \) is embedded as the set of invertible elements. So, after we describe an invariant inner product and an \( r \)-matrix, we will be able to use formula (8). The invariant inner product is given by

\[
\langle D_1, D_2 \rangle = \text{Tr} D_1 D_2 \quad \forall D_1, D_2 \in \PsiDO_n,
\]

(12)
Analogously, rewriting (14) in terms of $r$ this decomposition clearly satisfies all the requirements of Proposition 3.14, so we get an $r := \sum_{i=1}^{n} a_{i,i}$. The product $\langle D_1, D_2 D_3 \rangle = \langle D_1 D_2, D_3 \rangle$. Furthermore, one can explicitly verify that $\text{Tr} D_1 D_2 = \text{Tr} D_2 D_1$, so the inner product $\langle D_1, D_2 \rangle$ is symmetric. Alternatively, this can be shown by using the isomorphism of $\Psi DO_n$ and the algebra $\text{Mat}_n \otimes \mathbb{R}((z))$ of matrices over formal Laurent series (see Remark 3.11). In the matrix language, the trace of an operator can be written as
\[
\text{Tr} D = \text{Res}_{z=0} \left( z^{-1} \text{Tr} D(z) \right),
\]
where $D(z)$ is a matrix with coefficients in $\mathbb{R}((z))$ associated to the operator $D$.

Now, we represent $\Psi DO_n$ as a sum of three subalgebras $\mathfrak{g}_-, \mathfrak{g}_0, \mathfrak{g}_+$. Let $\Psi DO_n(J)$ be the set pseudo-difference operators supported in $J \subset \mathbb{Z}$. By definition, (17) is supported in $J$ if $a_j \equiv 0$ for all $j \notin J$. Define
\[
\mathfrak{g}_- := \Psi DO_n(\mathbb{Z}_-), \quad \mathfrak{g}_0 := \Psi DO_n(\{0\}) = \Psi DO_n(\{0\}), \quad \mathfrak{g}_+ := \Psi DO_n(\mathbb{Z}_+).
\]
This decomposition clearly satisfies all the requirements of Proposition 3.14 so we get an $r$-matrix $r := p_+ - p_-$ and hence a factorizable Poisson-Lie structure on $I\Psi DO_n$. This proves the first statement of Proposition 3.14. To prove the second statement, we use formula (10). From that formula it follows that, when viewed as map $\Psi DO_n \rightarrow \Psi DO_n$, the Poisson tensor $\pi_D$ (where $D \in I\Psi DO_n$) reads
\[
\pi_D(Q) = D r(Q D) - r(D Q) D.
\]
To show that the set $IDO_n(J) \subset I\Psi DO_n$ of invertible difference operators supported in $J$ is a Poisson submanifold, one needs to prove that for $D \in IDO_n(J)$ the image of the Poisson tensor (14) belongs to the tangent space to $IDO_n(J)$ at $D$. The latter is the space $IDO_n(J)$ of all $n$-periodic difference operators supported in $J$, so we need to show that the operator (14) is supported in $J$ whenever $D$ is supported in $J$. To that end, notice that the right-hand side of (14) stays the same if $r$ is replaced by $r_{\pm} := r \pm \text{Id}$. But the image of $r_{\pm} = 2p_+ + p_0$ (where $p_0$ is the projector to $\mathfrak{g}_0$) is the space $\mathfrak{g}_0 + \mathfrak{g}_+$ of operators which only contain terms of non-negative power in $T$, so, rewriting (14) in terms of $r_{\pm}$, we get that
\[
\min \supp \pi_D(Q) \geq \min \supp D = \min J.
\]
Analogously, rewriting (14) in terms of $r_{-}$, we get $\max \supp \pi_D(Q) \leq \max J$. All in all, we have $\supp \pi_D(Q) \subset [\min J, \max J] = J$, as desired.

To prove the third statement, notice that if $D$ is supported in a one-point set, then conjugation by $D$ preserves the subalgebras $\mathfrak{g}_\pm$ and $\mathfrak{g}_0$, as well as the inner product on $\Psi DO_n$. Therefore, it preserves the $r$-matrix, and $\pi(D) = 0$ by Proposition 3.16.

To prove the fourth statement, we represent the left-right action (15) as a superposition of two actions: one is of the same form, but with periodic $\alpha$ and $\beta$, while the other one is conjugation action $D \mapsto \gamma D \gamma^{-1}$, with quasi-periodic $\gamma$. Then the former action is Poisson because the Poisson structure vanishes on scalar operators, while the latter is Poisson because conjugation by scalar operators preserves $\mathfrak{g}_\pm$ and $\mathfrak{g}_0$, as well as the inner product, and hence is Poisson by Proposition 3.17. So, the left-right action (15) is also Poisson.

Finally, the last statement of Proposition 3.16 directly follows from Proposition 3.18. So, Proposition 3.9 is proved.
3.6 Relation with the GL\(_n\) bracket

One can compute Poisson brackets of coordinate functions on \(I\Psi DO_n\) using formula (11). The resulting expressions are quite complicated and involve infinite series. However, only finitely many terms of those series are non-zero for every concrete pseudo-difference operator. Moreover, for a difference operator whose support is small compared to the period these series simplify to just one term. Below we explain how to compute the brackets in this case by using the standard Poisson-Lie structure on GL\(_n\).

Recall that the standard Poisson structure on GL\(_n\) is defined using the construction of Proposition 3.14 with \(\mathfrak{g}_-, \mathfrak{g}_+, \mathfrak{g}_0\) being the lower nilpotent, upper nilpotent, and the Cartan subalgebra respectively. Explicitly, the brackets of the matrix elements are given by

\[
\{x_{ij}, x_{kl}\} = \frac{1}{2}(\text{sgn}(k - i) + \text{sgn}(l - j))x_{il}x_{kj},
\]

where \(\text{sgn}(t)\) is +1 if \(t > 0\), −1 if \(t < 0\), and 0 if \(t = 0\). In other words, for any matrix elements \(a, b, c, d\) located as

\[
\begin{array}{cccc}
\vdots & \vdots & \cdots & \cdots \\
\ldots & a & \ldots & b \\
\vdots & \vdots & \cdots & \cdots \\
\ldots & c & \ldots & d \\
\vdots & \vdots & \cdots & \cdots \\
\end{array}
\]

we have

\[
\{a, b\} = \frac{1}{2}ab, \quad \{a, c\} = \frac{1}{2}ac, \quad \{a, d\} = bc, \quad \{b, c\} = 0.
\]

These rules also imply

\[
\{b, d\} = \frac{1}{2}bd, \quad \{c, d\} = \frac{1}{2}cd,
\]

because the relative position of \(b\) and \(d\) is the same as of \(a\) and \(c\), while the relative position of \(c\) and \(d\) is the same as of \(a\) and \(b\).

We will now explain the relation between the bracket on difference operators and the GL\(_n\) bracket. Consider the algebra \(\Psi DO_n(\mathbb{Z}_{\geq 0})\), where \(\mathbb{Z}_{\geq 0} := \mathbb{Z}_+ \cup \{0\}\), of \(n\)-periodic upper-triangular pseudo-difference operators. Any such operator

\[
\mathcal{D} = \sum_{j=0}^{+\infty} a_j T^j
\]

can be represented by an infinite upper-triangular matrix

\[
\begin{array}{cccccc}
& a_{0,i-1} & a_{1,i-1} & \cdots \\
& & a_{0,i} & a_{1,i} & \cdots \\
& & & a_{0,i+1} & a_{1,i+1} & \cdots \\
\end{array}
\]

Let \(\Phi_i(\mathcal{D})\) be the \(n \times n\) submatrix of this matrix which has the element \(a_{0,i}\) in its upper left corner.

**Proposition 3.19.** Each of the mappings \(\Phi_i\): \(\Psi DO_n(\mathbb{Z}_{\geq 0}) \to \text{Mat}_n\) takes the Poisson structure \(\pi\) on \(\Psi DO_n(\mathbb{Z}_{\geq 0})\) to the standard Poisson structure on \(\text{Mat}_n\).

**Remark 3.20.** Technically, we have defined Poisson structures only on invertible pseudo-difference operators and invertible matrices. However, since both pseudo-difference operators and matrices form associative algebras, the Poisson structures in fact extend to non-invertible elements (see Remark 3.15).

**Remark 3.21.** This proposition is saying that one can compute Poisson brackets of difference operator coefficients by sliding an \(n \times n\) window through the operator matrix. If the support of the operator is not too big compared to the period, then the size of the window is big enough to fit any pair of the coefficients, so all Poisson brackets can be computed in this way.
Example 3.22. Consider the space of operators of the form \( a + bT \). This corresponds to matrices

\[
\begin{pmatrix}
\ddots & \ddots & \ddots \\
& a_i & b_i \\
& a_{i+1} & b_{i+1} & \ddots \\
& \ddots & \ddots & \ddots
\end{pmatrix}
\]

If \( n = 1 \), then we cannot use the \( n \times n \) window to compute all the brackets. For \( n \geq 2 \), Proposition 3.19 gives

\[
\{a_i, b_i\} = \frac{1}{2} a_i b_i, \quad \{b_i, a_{i+1}\} = \frac{1}{2} b_i a_{i+1},
\]

while all other brackets are either obtained from these by shift of indices or vanish.

Example 3.23. Consider the space of operators of the form \( a + bT + cT^2 \). The matrix of such an operator is

\[
\begin{pmatrix}
\ddots & \ddots & \ddots \\
& a_i & b_i & c_i \\
& a_{i+1} & b_{i+1} & c_{i+1} & \ddots \\
& \ddots & \ddots & \ddots & \ddots \\
& a_{i+2} & b_{i+2} & c_{i+2} & \ddots & \ddots & \ddots
\end{pmatrix}
\]

For \( n \geq 3 \), Proposition 3.19 gives

\[
\{a_i, b_i\} = \frac{1}{2} a_i b_i, \quad \{a_i, c_i\} = \frac{1}{2} a_i c_i, \quad \{b_i, c_i\} = \frac{1}{2} b_i c_i, \quad \{b_i, a_{i+1}\} = \frac{1}{2} b_i a_{i+1},
\]

\[
\{b_{i+1}, b_{i+1}\} = a_{i+1} c_i, \quad \{c_i, b_{i+1}\} = \frac{1}{2} c_i b_{i+1}, \quad \{c_i, a_{i+2}\} = \frac{1}{2} c_i a_{i+2}.
\]

The proof of Proposition 3.19 is based on the following lemma.

Lemma 3.24. Consider the space \( \text{Mat}_n \otimes \mathbb{R}[[z]] \) of formal matrix power series endowed with the trigonometric \( r \)-bracket, and the space \( \text{Mat}_n \) endowed with the standard bracket. Then the mapping

\[
\Phi: \text{Mat}_n \otimes \mathbb{R}[[z]] \to \text{Mat}_n, \quad \Phi \left( \sum_{i=0}^{\infty} A_i z^i \right) := A_0,
\]

taking a matrix power series to its constant term is a Poisson map.

Proof of Lemma 3.24. The trigonometric \( r \)-bracket on the space \( \text{Mat}_n \otimes \mathbb{R}((z)) \) of formal matrix Laurent series is defined using the construction of Proposition 3.14, where \( \mathfrak{G}_+ \) consists of matrix power series with nilpotent upper-triangular constant term, \( \mathfrak{G}_- \) consists of matrix polynomials in \( z^{-1} \) with nilpotent lower-triangular constant term, while \( \mathfrak{G}_0 \) is the space of constant diagonal matrices. The invariant inner product on \( \text{Mat}_n \otimes \mathbb{R}((z)) \) is defined by

\[
\langle A(z), B(z) \rangle := \text{Res}_{z=0} \left( \frac{1}{z} \text{Tr} A(z) B(z) \right).
\]

The mapping \( \Phi \) is well-defined on the whole space \( \text{Mat}_n \otimes \mathbb{R}((z)) \) and maps both the \( r \)-matrix and the inner product on the latter space to the corresponding objects on \( \text{Mat}_n \). Also notice that for any function \( f \in C^\infty(\text{Mat}_n) \), we have \( \text{grad} \Phi^* f \in \text{Mat}_n \otimes \mathbb{R}[[z]] \). Indeed, the function \( \Phi^* f \) is constant on the subspace

\[
\text{Ker} \Phi = \left\{ \sum_{i=1}^{\infty} A_i z^i \right\},
\]
so \( \text{grad} \Phi^* f \in (\text{Ker} \Phi)^\perp = \text{Mat}_n \otimes \mathbb{R}[[z]]. \) Furthermore, since \( \Phi \) preserves the inner product, we have \( \Phi(\text{grad} (\Phi^* f)(A)) = \text{grad} f(\Phi(A)). \) Now, take two functions \( f_1, f_2 \in C^\infty(\text{Mat}_n) . \) Then the Poisson bracket of their \( \Phi \)-pullbacks at a point \( A = A(z) \in \text{Mat}_n \otimes \mathbb{R}[[z]] \) is given by formula (11):

\[
\{ \Phi^* f_1, \Phi^* f_2 \}(A) = \frac{1}{2} \left( r(\text{grad} \Phi^* f_1(A) \cdot A), \text{grad} \Phi^* f_2(A) \cdot A \right) - \left( r(A \cdot \text{grad} \Phi^* f_1(A)), A \cdot \text{grad} \Phi^* f_2(A) \right).
\]

Using that both \( A \) and the gradients of \( f_1, f_2 \) belong to \( \text{Mat}_n \otimes R[[z]], \) while the restriction of \( \Phi \) to \( \text{Mat}_n \otimes R[[z]] \) is a homomorphism of associative algebras preserving the inner product and the \( r \)-matrix, this can be rewritten as

\[
\{ \Phi^* f_1, \Phi^* f_2 \}(A) = \frac{1}{2} \left( r(\text{grad} f_1(\Phi(A)) \cdot \Phi(A)), \text{grad} f_2(\Phi(A)) \cdot \Phi(A) \right) - \left( r(\Phi(A) \cdot \text{grad} f_1(\Phi(A))), \phi(A) \cdot \text{grad} f_2(\Phi(A)) \right),
\]

which is exactly the \( \text{Mat}_n \) bracket of the functions \( f_1, f_2 \) at the point \( \Phi(A). \) Thus, the mapping \( \Phi \) is indeed Poisson, as claimed.

**Proof of Proposition 3.19.** In the loop algebra language, the space \( \Psi DO_n(\mathbb{Z}_{\geq 0}) \) of upper-triangular \( n \)-periodic pseudo-difference operators is the space of formal matrix power series of the form

\[
A(z) = \sum_{i=0}^{\infty} A_i z^i,
\]

where \( A_0 \) is upper-triangular. The infinite matrix corresponding to such power series is

\[
\begin{array}{ccccccc}
A_0 & A_1 & \cdots &    \\
A_0 & A_1 & \cdots &    \\
A_0 & A_1 & \cdots &
\end{array}
\]

with upper-left corners of \( A_0 \) blocks located at positions \((1, 1)\), \((n + 1, n + 1)\), etc. Thus, the mapping \( \Phi_1 \) takes \( A(z) \) to \( A_0 \) and is, therefore, a restriction of the mapping \( \Phi \) from Lemma 3.24. So, \( \Phi_1 \) is a Poisson map. Furthermore, we have \( \Phi_{i+1} = \Phi_1 \circ \text{Ad}_T \), where \( \text{Ad}_T(D) := TDT^{-1} \). Therefore, since \( \Phi_1 \) and \( \text{Ad}_T \) are Poisson maps, it follows that all \( \Phi_i \)'s are also Poisson, as desired. \( \square \)

### 3.7 The subgroup of sparse operators

We say that a pseudo-difference operator is \( k \)-sparse if its support is an arithmetic progression with step \( k \). For example, the operator \( T^{-1} + T + T^3 + T^5 + \ldots \) is 2-sparse. Denote the set of invertible \( k \)-sparse pseudo-difference operators by \( \Psi DO_n(\mathbb{Z} + *) \). This is a Lie subgroup of \( \Psi DO_n \), whose Lie algebra is the space \( \Psi DO_n(\mathbb{Z}) \) of pseudo-difference operators supported in \( \mathbb{Z} \). It is not, however, a Poisson submanifold and hence not a Poisson-Lie subgroup \( \Psi DO_n \). One can, however, define a different Poisson structure on \( \Psi DO_n(\mathbb{Z} + *) \), which has all the same properties as the Poisson structure on \( \Psi DO_n \) described above. More precisely, we have the following:

**Proposition 3.25.** There exists a natural Poisson structure \( \pi^{(k)} \) on the group \( \Psi DO_n(\mathbb{Z} + *) \) of invertible \( k \)-sparse pseudo-difference operators. It has all the same properties as the Poisson structure \( \pi \) described in Proposition 3.9 except for the second property which is replaced by the following: if \( J \subset \mathbb{Z} \) is an arithmetic progression with common difference \( k \), then \( DO_n(J) \) is a Poisson submanifold of \( \Psi DO_n(\mathbb{Z} + *) \).

**Proof.** This Poisson structure is given by the following decomposition of the Lie algebra \( \Psi DO_n(\mathbb{Z}) \):

\[
\mathfrak{g}_- := \Psi DO_n(\mathbb{Z}_-), \quad \mathfrak{g}_0 := \Psi DO_n(\{0\}), \quad \mathfrak{g}_+ := \Psi DO_n(\mathbb{Z}_+).
\]

All necessary properties are established in the same way as in the proof of Proposition 3.9. \( \square \)
Remark 3.26. A more constructive way to describe the Poisson structure $\pi^{(k)}$ is as follows. When $n$ and $k$ are coprime, there is a group isomorphism $\Psi DO_n(k\mathbb{Z}) \simeq \Psi DO_n$ given by the action of $\Psi DO_n(k\mathbb{Z})$ on eventually vanishing sequences whose non-zero entries are contained in an arithmetic progression with common difference $k$. Explicitly, this isomorphism is given by

$$\sum a_{kj} T^{kj} \mapsto \sum \tilde{a}_{j} T^{j},$$

where $\tilde{a}_{j,i} = a_{kj,ki}$. The Poisson structure $\pi^{(k)}$ can be defined as the pull-back of the structure $\pi$ by this isomorphism. Furthermore, $\pi^{(k)}$ uniquely extends to the whole group $\Psi DO_n(k\mathbb{Z} + *)$ if we require that the resulting structure is invariant under multiplication by $T$. This gives a structure which coincides with the one described in the proof of Proposition 3.25. Furthermore, this construction can also be applied when $n$ and $k$ are not coprime, in which case $\Psi DO_n(k\mathbb{Z})$ is isomorphic to a product of $m := \gcd(n, k)$ copies of $\Psi DO_{n/m}$. The corresponding $m$ maps $\Psi DO_n(k\mathbb{Z}) \to \Psi DO_{n/m}$ are given by (15) with $\tilde{a}_{j,i} = a_{kj,ki+t}$, where $l = 0, \ldots, m-1$.

Example 3.27. Consider sparse operators of the form $a + bT^2$. The Poisson bracket $\pi^{(2)}$ on such operators may be obtained from the bracket $\pi$ on operators of the form $a + bT$ using the following mnemonic rule (justified by Remark 3.26): take the formulas (15) for brackets on $a + bT$ and replace all indices of the form $i + j$ with $i + 2j$. This gives

$$\{a_i, b_i\} = \frac{1}{2} a_i b_i, \quad \{b_i, a_{i+2}\} = \frac{1}{2} b_i a_{i+2}.$$  

4 General refactorization maps associated with pairs of arithmetic progressions

4.1 The main theorem

In this section we describe a class of maps parametrized by a pair of finite arithmetic progressions $J_{\pm} \subset \mathbb{Z}$ with the same common difference. For disjoint $J_{\pm}$ these maps coincide with pentagram maps on $J$-corrugated polygons described in Section 2. All these maps, regardless of whether $J_{\pm}$ are disjoint or not, admit a refactorization description in terms of the group $\Psi DO_n$ of periodic pseudo-difference operators. As a corollary, all such maps admit an invariant Poisson structure and a Lax representation with Poisson-commuting spectral invariants. This makes us believe that all these maps are both Liouville and algebraically integrable. In order to actually prove that, one needs to accurately verify certain technical conditions, which is beyond the scope of the present paper.

Recall that pentagram maps defined in Section 2 act on the space $\mathcal{P}_n(J)/\mathrm{PGL}$ of twisted $J$-corrugated $n$-gons modulo projective transformations. By Proposition 3.3, that space can be identified with the space $\Psi DO_n(J)/H \times H$ of properly bounded $n$-periodic difference operators supported in $J$ modulo the left-right action of the group $H \times H$ of pairs of invertible $n$-quasi-periodic scalar difference operators with the same monodromy. Furthermore, decomposing a difference operator $D \in \Psi DO_n(J)$ into a sum $D_++D_-$, where $D_\pm \in DO_n(J_\pm)$ are difference operators supported in $J_\pm$, one can identify a dense subset in the quotient $\Psi DO_n(J)/H \times H$ with $\Psi DO_n(J_+ \times \Psi DO_n(J_-))/H \times H$, where $H \times H$ acts on both factors by the simultaneous left-right action. Thus, our pentagram maps (considered on sufficiently generic polygons) can be thought of as transformations defined on the left-right quotient $\Psi DO_n(J_+ \times \Psi DO_n(J_-))/H \times H$ with $J_\pm$ being disjoint. Note, however, that the latter quotient is well-defined regardless of whether $J_\pm$ are disjoint or not. If they are not, one can think of it as some kind of a "virtual polygon space" (in some cases, this space can be interpreted as the space of pairs of polygons, see Remark 1.6 below). Below we describe certain dynamics on that space which in the disjoint case coincides with the pentagram dynamics.

We define a Poisson structure on $\Psi DO_n(J_-)$ as the restriction of the structure $\pi^{(k)}$ on $k$-sparse operators, where $k$ is the common difference of $J_-$ (see Proposition 3.25). When $k = 1$, this is just the usual Poisson structure on pseudo-difference operators described in Proposition 4.2. Further, on $\Psi DO_n(J_-)$...
we take the same Poisson structure but with an opposite sign. This endows \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) \) with a product Poisson structure. Furthermore, the quotient \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) / H \times H \) (i.e. the “virtual polygon space”) inherits the Poisson structure because the left-right action is Poisson.

The following theorem is the main result of the paper.

**Theorem 4.1.** Let \( J_{\pm} \subset \mathbb{Z} \) be a pair of non-empty finite arithmetic progressions with the same common difference. Let also \( \mathcal{D}_{\pm} \in \text{PBDO}_n(J_{\pm}) \) be \( n \)-periodic difference operators supported in \( J_{\pm} \). Consider the multivalued map of the space \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) \) to itself that assigns to \( \mathcal{D}_{\pm} \) new difference operators \( \tilde{\mathcal{D}}_{\pm} \in \text{PBDO}_n(J_{\pm}) \) defined by the equation

\[
\tilde{\mathcal{D}}_{+}\mathcal{D}_{-} = \mathcal{D}_{-}\mathcal{D}_{+}.
\]

Then the following statements hold.

1. This map \( \mathcal{D}_{\pm} \mapsto \tilde{\mathcal{D}}_{\pm} \) descends to a generically defined single-valued transformation \( \Psi_{J_{\pm}} \) of the quotient \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) / H \times H \), where \( H \times H \) is the group of pairs of invertible \( n \)-quasi-periodic scalar difference operators with the same monodromy acting on \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) \) by simultaneous left-right action \( \{ \alpha \} \).

2. If \( J_{\pm} \) are disjoint, then the so-obtained map \( \Psi_{J_{\pm}} \) coincides with the pentagram map associated with the progressions \( J_{\pm} \).

3. Associate with an element of \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) / H \times H \) the pseudo-difference operator \( \mathcal{L} = \mathcal{D}^{-1}_+ \mathcal{D}_+ \), defined up to the conjugation action of \( H \). Then, in terms of \( \mathcal{L} \), the mapping \( \Psi_{J_{\pm}} \) is a refactorization map

\[
\mathcal{L} = \mathcal{D}^{-1}_+ \mathcal{D}_+ \mapsto \tilde{\mathcal{L}} = \mathcal{D}_+ \mathcal{D}^{-1}_-.
\]

In other words, the mapping \( \Psi_{J_{\pm}} \) has a Lax representation

\[
\mathcal{L} \mapsto \mathcal{D}_+ \mathcal{L} \mathcal{D}^{-1}_+.
\]

4. The mapping \( \Psi_{J_{\pm}} \) is Poisson.

5. Suitably normalized central functions on the space of Lax operators \( \mathcal{L} \) are Poisson commuting first integrals of \( \Psi_{J_{\pm}} \).

**Remark 4.2.** Let us explain what we mean by normalization of central functions. Recall that as central function we can take functions of the form \( f_{ij}(\mathcal{L}) := \text{Tr} T^{in} \mathcal{L}^j \), where \( i \in \mathbb{Z}, j \in \mathbb{Z}_+ \) (see Remark 3.12). Upon conjugation of \( \mathcal{L} \) by a quasi-periodic scalar operator \( \alpha \in H \) with monodromy \( z \), the function \( f_{ij} \) transforms as

\[
f_{ij}(\alpha \mathcal{L} \alpha^{-1}) = \text{Tr} T^{in} \alpha \mathcal{L}^j \alpha^{-1} = z^i \text{Tr} \alpha T^{in} \mathcal{L}^j \alpha^{-1} = z^i f_{ij}(\mathcal{L}).
\]

Thus, the functions \( f_{ij} \) do not descend to the quotient of Lax operators by the conjugation action of \( H \). One can, however, fix this issue dividing \( f_{ij} \), say, by \( f_{11} \). This is what we mean by normalization.

**Remark 4.3.** As explained in Remark 3.8 periodic pseudo-difference operators can be identified with matrices over the field \( \mathbb{R}(z) \) of formal Laurent series. Thus, Lax representation (19) can be viewed as a Lax representation with spectral parameter \( z \). Note, however, that since \( \mathcal{L} \) is only defined up to conjugation by quasi-periodic scalar operators \( \alpha \in H \), the corresponding \( z \)-dependent matrix \( \mathcal{L}(z) \) is also not uniquely defined. Namely, conjugation by periodic scalar operators translates to conjugation by \( z \)-independent diagonal matrices, while conjugation by a scalar operator \( \alpha \) defined by \( \alpha_{t,i} := t^{(i-1)/n} \) becomes the action \( \mathcal{L}(z) \mapsto \mathcal{L}(tz) \). Since \( H \) can be written as a direct product of periodic operators and the subgroup \( \{ \alpha_t \mid t \in \mathbb{R} \} \), it follows that the Lax matrix \( \mathcal{L}(z) \) is defined up to transformations of the form \( \mathcal{L}(z) \mapsto A \mathcal{L}(tz) A^{-1} \), where \( A \) is a constant diagonal invertible matrix. Also note that since \( \mathcal{L} \) is defined as a quotient of two difference operators, the corresponding matrix \( \mathcal{L}(z) \) is not just a formal Laurent series but a rational function of \( z \).

One can also characterize first integrals of the maps \( \Psi_{J_{\pm}} \) provided by Theorem 4.1 as follows:

**Corollary 4.4.** The characteristic polynomial \( CPM_{\mathcal{D}+ + w \mathcal{D}_-}(z) \) of the monodromy of \( \mathcal{D}_+ + w \mathcal{D}_- \), defined up to transformations of the form \( z \mapsto tz \) and a constant factor, is invariant under the map \( \Psi_{J_{\pm}} \).
which we were able to identify with a familiar integrable system $J$.

$L$ can lift such two polygons to two bi-infinite sequences $V$ in lattice $[33]$. The phase space of the leapfrog map is, by definition, the space of pairs of twisted operators supported in the Minkowski sum $J_+ + J_-$. But for arithmetic progressions with the same common difference we have $|J_+ + J_-| = |J_+| + |J_-| - 1$, so

$$\dim \text{DO}_n(J_+ + J_-) = n(|J_+| + |J_-| - 1) < n(|J_+| + |J_-|) = \dim \text{DO}_n(J_+ \oplus \text{DO}_n(J_-)).$$

This identifies the space of pairs of twisted $n$-gons in $\mathbb{RP}^1$ with the same monodromy, considered up to simultaneous projective transformations, with the left-right quotient $\text{PD}O_n(J_+) \times \text{PD}O_n(J_-) / H \times H$, while the leapfrog map gets identified with the map $\Psi_{J_+}$. The only proof of this we were able to find consists of expressing both maps in coordinates. However, we do believe that it should be possible to directly identify the equation (17) with the geometric “leapfrogging” definition, similarly to how we identify it with pentagram-type dynamics in the case disjoint $J_{\pm}$.

**Proof of Theorem 4.1**. For fixed $D_{\pm} \in \text{PD}O_n(J_{\pm})$ consider the linear map

$$\tilde{D}_-, \tilde{D}_+ \mapsto \tilde{D}_+ D_- - \tilde{D}_- D_+. \quad (20)$$

Observe that the image of this map is contained in the space of operators supported in the Minkowski sum $J_+ + J_-$. But for arithmetic progressions with the same common difference we have $|J_+ + J_-| = |J_+| + |J_-| - 1$, so

$$\dim \text{DO}_n(J_+ + J_-) = n(|J_+| + |J_-| - 1) < n(|J_+| + |J_-|) = \dim \text{DO}_n(J_+ \oplus \text{DO}_n(J_-)).$$

Therefore, the map $[20]$ should have a non-trivial kernel, which is equivalent to saying that for any $D_{\pm} \in \text{PD}O_n(J_{\pm})$ there exist operators $\tilde{D}_{\pm} \in \text{DO}_n(J_{\pm})$ which do not vanish simultaneously and satisfy (17). Furthermore, it is easy to see that the set $X$ of $D_{\pm} \in \text{PD}O_n(J_{\pm})$ for which there exist properly bounded operators $\tilde{D}_{\pm} \in \text{PD}O_n(J_{\pm})$ satisfying (17) is Zariski open. Moreover, this set $X$ is non-empty, because for $D_{\pm}$ with constant coefficients we have a solution given by $\tilde{D}_{\pm} = D_{\pm}$. So, the set $X$ is dense, which means that for almost all $D_{\pm} \in \text{PD}O_n(J_{\pm})$ there exist properly bounded operators $\tilde{D}_{\pm} \in \text{PD}O_n(J_{\pm})$ satisfying (17).
We will now show that for generic $D_{\pm} \in \text{PBDO}_n(J_{\pm})$ the properly bounded operators $\hat{D}_{\pm} \in \text{PBDO}_n(J_{\pm})$ satisfying (17) are unique up to the simultaneous left action of the group $H_0 := \text{IDO}_n(\{0\}) \subset H$ of invertible $n$-periodic scalar operators. To that end, assume that $\text{Ker} \ D_+ \cap \text{Ker} \ D_- = 0$ (this condition is satisfied for generic $D_{\pm} \in \text{PBDO}_n(J_{\pm})$). Applying both sides of (17) to any $\xi \in \text{Ker} \ D_+$ we get that $\hat{D}_+ D_+ \xi = 0$. So, we have $D_-(\text{Ker} \ D_+) \subset \text{Ker} \ D_+$. But since $\text{Ker} \ D_+ \cap \text{Ker} \ D_- = 0$, it follows that $\text{dim} \ D_- (\text{Ker} \ D_+) = \text{dim} \ D_+$. Furthermore, since both $D_+$ and $\hat{D}_+$ are properly bounded and have the same support, it follows that $\text{dim} \ Ker \ D_+ = \text{dim} \ D_+$, and thus $\text{Ker} \ D_+ = \text{Ker} \ D_-. (\text{Ker} \ D_+)$. But an $n$-periodic properly bounded periodic difference operator is uniquely determined by its kernel up to left multiplication by elements $\alpha \in H_0$. So, a properly bounded operator $\hat{D}_{\pm}$ solving (17) is uniquely determined up to the left action of $H_0$. Furthermore, since $D_- = D_+ D_- D_+^{-1}$, it follows that properly bounded $\hat{D}_{\pm}$ satisfying (17) are unique up to the simultaneous left action of $H_0$, as claimed.

We now see that (17) can be viewed as generically defined and generically single-valued map $\text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}) \to H_0 \setminus \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm})$, where $H_0 \setminus \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm})$ is the left quotient of $\text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm})$ by $H_0$. Furthermore, the solution set of (17) is invariant under the right action of $H_0$ on $D_{\pm}$, so (17) can be as well viewed as a map

$$\text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}) / H_0 \to H_0 \setminus \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}).$$

Finally, observe that (17), viewed as such a map, commutes with the conjugation action of $H$. Thus, it indeed descends to a (generically defined and generically single-valued) map

$$\Psi_{J_{\pm}} : \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}) / H \to \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}) / H \to \text{PBDO}_n / \text{Ad}H,$$

proving the first statement of the theorem.

The proof of the second statement repeats, word for word, the proof of the corresponding part of Theorem [11] so we proceed to the third statement. The fact that the operator $\mathcal{L}$ transforms as (18) follows directly from its definition and relation (17). Indeed, we can rewrite (17) as

$$\hat{D}_-^{-1} \hat{D}_+ = D_+ D_-^{-1},$$

(21)

which precisely means that the operator $\hat{L} = \hat{D}_-^{-1} \hat{D}_+$ associated with $\hat{D}_{\pm}$ is obtained from the operator $\mathcal{L} := D_-^{-1} D_+$ associated with $D_{\pm}$ by means of refactorization (18). What needs to be proved, though, is that the dynamics in terms of $\mathcal{L}$ is equivalent to the dynamics in terms of $D_{\pm}$. In other words, we need to show that the map

$$\mathcal{L} : \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}) / H \to \text{PBDO}_n / \text{Ad}H,$$

(22)

(where $\text{PBDO}_n / \text{Ad}H$ is the quotient by the conjugation action of $H$) given by $\mathcal{L}(D_{\pm}) = D_-^{-1} D_+$ is generically injective. To prove that, it suffices to establish generic injectivity of the map $H_0 \setminus \text{PBDO}_n(J_{\pm}) \times \text{PBDO}_n(J_{\pm}) \to \text{PBDO}_n$ given by the same formula $D_{\pm} \mapsto D_-^{-1} D_+$. We will use the following lemma:

**Lemma 4.7.** Let $D \in \text{PBDO}_n$ be a properly bounded $n$-periodic difference operator. Then

$$D_-^{-1} = (\text{CPM}_D(T^n))^{-1} \hat{D},$$

where $\text{CPM}_D$ is the characteristic polynomial of the monodromy of $D$, and $\hat{D}$ is a difference operator.

**Remark 4.8.** As a matter of fact, one can replace the characteristic polynomial by the minimal polynomial, cf. [20] Lemma 1.3.

**Proof of Lemma 4.7.** Without loss of generality assume that $\min \text{supp}(D) = 0$. Consider the element $D(z)$ of the loop algebra corresponding to $D$. Then $D(z)^{-1} = (\det D(z))^{-1} \hat{D}(z)$, where $\hat{D}(z)$ is the comatrix of $D(z)$ and hence a Laurent polynomial in $z$. Furthermore, by Proposition 3.2 we have $\det D(z) = CPM_D(z)$, so

$$D(z)^{-1} = (\text{CPM}_D(z))^{-1} \hat{D}(z).$$

Translating this back into the language of pseudo-difference operators, we get the desired statement. □
We now return to the proof of Theorem 4.1. We need to show that a generic rational operator \( D^{-1}_+D_+ \), where \( D_{\pm} \in \text{PBDO}_n(J_{\pm}) \), has a unique presentation in that form up to simultaneous left multiplication of \( D_{\pm} \) by invertible \( n \)-periodic scalar operators \( \alpha \in H_n \) (cf. Proposition 5). Assume that \( D^{-1}_+D_+ = D'^{-1}_+D'_+ \). Rewriting this as \( D'_-D'_-D'_+ = D'_+ \). Applying Lemma 4.7, this can be further rewritten as

\[
D'_-D_-D_+ = D'_+ \text{CPM}_D_-(T^n),
\]

where \( \text{CPM}_D_-(T^n) \) is the characteristic polynomial of the monodromy of \( D_- \). Assume that the monodromies of \( D_{\pm} \) have disjoint spectra (this is clearly so for generic operators). Then we must have \( \text{Ker} D_+ \cap \text{Ker} \text{CPM}_D_-(T^n) = 0 \). Indeed, let \( W := \text{Ker} D_+ \cap \text{Ker} \text{CPM}_D_-(T^n) \). Then \( W \) is invariant under the action of \( T^n \), so \( T^n \) must have an eigenvector in \( W \). In other words, there exists \( \xi \in W \) such that \( T^n \xi = z \xi \). But this means that \( z \) is an eigenvalue of the monodromy of \( D_+ \) (since \( \xi \in \text{Ker} D_+ \)) and also an eigenvalue of the monodromy of \( D_- \) (since \( \text{CPM}_D_-(T^n) \xi = \text{CPM}_D_-(z) \xi = 0 \)). But that is impossible, so we must have \( W = 0 \). Therefore, \( \text{CPM}_D_-(T^n) \) is an isomorphism of \( \text{Ker} D_+ \) to itself, and applying (23) to an arbitrary element of \( \text{Ker} D_+ \) we see that \( \text{Ker} D'_+ = \text{Ker} D_+ \), which implies the desired uniqueness result. So, the third statement of the theorem is proved.

To prove the fourth statement, consider the diagram

\[
\begin{array}{ccc}
\text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) / H \times H & \xrightarrow{\Psi_{J_{\pm}}} & \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) / H \times H \\
D_+D_-^{-1} & \mapsto & D_-^{-1}D_+
\end{array}
\]

This diagram is commutative by (21). Furthermore, the diagonal arrows are Poisson, since multiplication in \( \text{I\PsiDO}_n \) is Poisson, inversion is anti-Poisson, and the Poisson structure on the space \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) \) of pairs of operators is defined by reversing the structure on the factor corresponding to \( D_- \). Also notice that as proved above the right diagonal arrow is generically one-to-one. So, \( \Psi_{J_{\pm}} \) is a Poisson map, proving the fourth statement of the theorem.

Finally, we prove the fifth statement. Central functions on \( \text{I\PsiDO}_n \) applied to \( \mathcal{L} \) are preserved by the map \( \Psi_{J_{\pm}} \) due to representation (19) so it suffices to prove that they commute. More precisely, we need to establish Poisson commutativity for the pull-backs of central functions on \( \text{I\PsiDO}_n \) by the map (22). But that follows from commutativity of central functions on \( \text{I\PsiDO}_n \) along with the fact that (22) is a Poisson map. So, Theorem 4.1 is proved.

### 4.2 Scaling invariance

Most of the known constructions of first integrals and Lax representations for pentagram-type maps are based on scaling symmetries. A scaling symmetry is a 1-parametric group of transformations of the polygon space which commutes with the pentagram map. In most cases such symmetries were guessed by studying explicit formulas for the corresponding map, and their geometric meaning is not known. The aim of this section is to show that the scaling symmetry is an immediate corollary of our construction.

**Proposition 4.9.** The map \( \Psi_{J_{\pm}} \), described in Theorem 4.1, commutes with a 1-parametric group \( R_w \) of transformations which is defined, in terms of difference operators, as

\[
D_+, D_- \mapsto D_+, wD_-.
\]

In terms of the Lax operator, this transformation is simply rescaling:

\[
\mathcal{L}(z) \mapsto w^{-1} \mathcal{L}(z).
\]

**Remark 4.10.** Transformation (24) commutes with the left-right \( H \times H \) action (while (25) commutes with the conjugation action) and hence can be viewed as a map from the space \( \text{PBDO}_n(J_+) \times \text{PBDO}_n(J_-) / H \times H \) (which is where the map \( \Psi_{J_{\pm}} \) is defined) to itself.
Proof of Proposition 4.9. Indeed, the defining equation (17) of the map $\Psi_{J_+}$ is invariant under the transformation $\mathcal{D} \mapsto w \mathcal{D}$, while the Lax form (19) is invariant under rescaling.

Proposition 4.11. In the case of the classical pentagram map, as well as in short-diagonal and dented cases, transformations $R_w$ defined in Proposition 4.9 coincide with scaling transformations introduced for these maps in [21, 16, 18].

Proof. The proof is achieved by introducing coordinates on the polygon space and rewriting the scaling symmetry in those coordinates. As an example, let us consider short-diagonal maps in $\mathbb{RP}^{2k}$. This corresponds to $J_+ = \{0, 2, 4, \ldots, 2k\}$, $J_- = \{1, 3, 5, \ldots, 2k+1\}$ (see Table 1). The phase space of the associated short-diagonal map is the space $\mathcal{P}_n(J)/\text{PGL}$ with $J = J_+ \sqcup J_- = \{0, \ldots, 2k+1\}$ of arbitrary polygons in $\mathbb{RP}^{2k}$, considered modulo projective equivalence. In terms of difference operators, it is the space of operators supported in $J$ and considered modulo the left-right action (5) of $H \times H$. As can be seen from [16, Section 3.2], as well as from [24, Section 8.2], if $\gcd(2k+1, n) = 1$, then every orbit of the $H \times H$ action has a unique representative of the form

$$\mathcal{D} = 1 + \sum_{j=1}^{2k} a_j T^j - T^{2k+1}. \quad (26)$$

Thus, one can take entries of the sequences $a_j, j = 1, \ldots, 2k$, as coordinates on the polygon space. To write our scaling transformation $R_w$, in these coordinates, we need to apply it to operator (26), which gives

$$\mathcal{D}' = 1 + \sum_{j=1}^{2k-1} w a_j T^j + \sum_{j=2}^{2k} a_j T^j - w T^{2k+1}, \quad (27)$$

and then normalize, i.e. find an operator $\tilde{\mathcal{D}}$ of the form (26) which belongs to the same orbit of the $H \times H$ action as (22). Note that since the constant term of $\mathcal{D}'$ is already of necessary form, it remains to normalize the coefficient of $T^{2k+1}$, which can be done using only the conjugation action of $H$. The condition for $\alpha \mathcal{D}' \alpha^{-1}$, where $\alpha \in H$, to have coefficient of $T^{2k+1}$ equal to $-1$ is $\alpha + 2k+1 \alpha^{-1} = w$. This has a quasi-periodic solution $\alpha_i = \lambda^i$, where $\lambda$ is such that $\lambda^{2k+1} = w$. Computing $\tilde{\mathcal{D}} = \alpha \mathcal{D}' \alpha^{-1}$ with such $\alpha$, we find that its coefficients $\tilde{a}_k$ are given by

$$\tilde{a}_j = w \lambda^{-j} a_j = \lambda^{2k+1-j} a_j$$

when $j$ is odd, and

$$\tilde{a}_j = \lambda^{-j} a_j$$

when $j$ is even. Upon a parameter change $s = \lambda^{-2}$, this coincides with formulas for the scaling given in [16, Section 9].

Remark 4.12. In [16], the invariance of short-diagonal maps under scaling was only established in dimensions $\leq 6$, while the general case was proved in [22]. With our definition, the invariance of pentagram maps under scaling is immediate.

Corollary 4.13. For the classical, as well as short-diagonal and dented maps, first integrals obtained from our construction coincide with the ones obtained in [27, 16, 18].

Proof. Indeed, according to Corollary 4.4, our integrals can be interpreted as spectral invariants of the monodromy for polygons obtained from the initial one by means of scaling $R_w$. But this is exactly the definition of first integrals in [27, 16, 18].

Remark 4.14. For corrugated maps of [10] our first integrals also coincide with the known ones. In fact, one can show more: for these maps, our refactorization description [18] is equivalent to the one given in [10, Proposition 4.10]. The refactorization description of [10] looks more complicated because it is given in terms of actual loop group elements (equivalently, pseudo-difference operators) $A_1(z)$, $A_2(z)$, as opposed to elements of the quotient by the $H \times H$ action. Rewriting refactorization on the quotient as operator refactorization involves choosing a section of the action, which complicates the resulting formulas.
4.3 Poisson brackets for the short-diagonal map in 3D

In this section we derive explicit formulas for Poisson brackets preserved by the short-diagonal pentagram map in 3D. The corresponding sets $J_\pm$ are $J_+ = \{-2, 0, 2\}$, $J_- = \{-1, 1\}$ (the choice $J_+ = \{0, 2, 4\}$, $J_- = \{1, 3\}$ indicated in Table 1 leads to the same map up to the shift of indices and hence gives rise to the same Poisson bracket). The phase space of the associated map is the space of all twisted $n$-gons in $\mathbb{RP}^3$ modulo projective equivalence. We coordinatize that space as in [16, Section 5.2], namely we assign to a twisted $n$-gon $\{v_i \in \mathbb{RP}^3\}$ three periodic $n$-sequences $x_i, y_i, z_i$ defined as the following negative cross-ratios:

$$
x_i := -[v_i, v_{i-1}, v_{i+1}, v_{i+2}] \cap \langle v_i, v_{i+1}, v_{i+2} \rangle \cap \langle v_i, v_{i+1}, v_{i+2} \rangle \cap \langle v_i, v_{i+1}, v_{i+2} \rangle,
$$

$$
y_i := -[v_i, v_{i-1}, v_{i+2}, v_{i+3}] \cap \langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle \cap \langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle \cap \langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle,
$$

$$
z_i := -[v_i, v_{i+1}, v_{i+2}, v_{i+3}] \cap \langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle \cap \langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle \cap \langle v_i, v_{i+1}, v_{i+2}, v_{i+3} \rangle.
$$

**Proposition 4.15.** In these coordinates, the Poisson structure for the short-diagonal pentagram map in $\mathbb{RP}^3$ takes the following form:

$$
\{x_i, x_{i+1}\} = x_i x_{i+1}, \quad \{x_i, x_{i+2}\} = x_i x_{i+2} u_i+1, \quad \{y_i, y_{i+2}\} = y_i y_{i+2} u_i+1, \quad \{z_i, z_{i+2}\} = z_i z_{i+2} u_i
$$

$$
\{x_i, y_{i-2}\} = x_i y_{i-2} u_i-1, \quad \{x_i, y_{i+2}\} = -x_i y_{i+2} u_i+1, \quad \{x_i, z_{i-1}\} = x_i z_{i-1} (u_i-1 - 1), \quad \{x_i, z_{i+1}\} = x_i z_{i+1} (i), \quad \{x_i, z_{i+3}\} = -x_i z_{i+3} u_i+1, \quad \{y_i, z_{i-1}\} = y_i z_{i-1} (1 - u_i-1), \quad \{y_i, z_{i+1}\} = -y_i z_{i+1} (i), \quad \{y_i, z_{i+3}\} = y_i z_{i+3} u_i+1,
$$

where $w_i := y_i+1 z_i$.

**Proof.** A direct computation shows that for any difference operator $D = aT^{-2} + bT^{-1} + c + dT + eT^2$ representing the polygon $\{v_i\}$, the coordinates $x_i, y_i, z_i$ can be expressed in terms of coefficients of $D$ as follows:

$$
x_{i-2} = -\frac{c_{i+1} c_i}{d_i a_{i+1}}, \quad y_{i-2} = -\frac{a_{i+1} d_i}{b_i c_{i+1}}, \quad z_{i-2} = -\frac{b_{i+1} c_i}{e_i d_{i+1}}, \quad (28)
$$

The Poisson bracket between coefficients of $D$ is, by construction, the product bracket corresponding to the decomposition $D = D_+ + D_-$, where $D_+ = aT^{-2} + c + eT^2$, $D_- = bT^{-1} + cT$. The bracket on operators $D_+$ is defined as the restriction of the bracket $\pi(2)$ on 2-sparse operators, while the $D_-$ part is endowed with the negative of that bracket. Similarly to Example 3.27 we get

$$
\{a_i, c_i\} = \frac{1}{2} a_i c_i, \quad \{a_i, e_i\} = \frac{1}{2} a_i e_i, \quad \{c_i, e_i\} = \frac{1}{2} c_i e_i, \quad \{c_i, a_{i+2}\} = \frac{1}{2} c_i a_{i+2}, \quad \{c_i, c_{i+2}\} = a_i c_{i+2}, \quad \{e_i, c_{i+2}\} = \frac{1}{2} c_i e_{i+2}, \quad \{e_i, a_{i+4}\} = \frac{1}{2} c_i a_{i+4}
$$

and

$$
\{b_i, d_i\} = -\frac{1}{2} b_i d_i, \quad \{d_i, b_{i+2}\} = -\frac{1}{2} b_i a_{i+2}.
$$

It now remains to compute the brackets of functions (28) using these formulas. This is done by a straightforward calculation.

**Remark 4.16.** As shown in [16, Theorem 5.6], the short-diagonal map in $xyz$-coordinates reads

$$
\tilde{x}_i = x_i, \quad \tilde{y}_i = y_i, \quad \tilde{z}_i = z_i.
$$

where

$$
\alpha_i := 1 + y_i-1 + z_i+2 + y_i-1 z_i+2 - y_i+1 z_i, \quad \beta_i := 1 + y_i-1 + z_i.
$$

It follows from our construction that this map preserves the above bracket. This can of course be verified with a computer algebra system.
4.4 Refactorization and Y-meshes

In this section we outline the connection between the refactorization description of higher pentagram maps and the description in terms of Y-meshes given in [12]. We consider the example of a short-diagonal pentagram map in \( \mathbb{RP}^3 \), but all the same arguments work for more general maps in any dimension.

Let us briefly recall the Y-mesh description of the short-diagonal map from [12]. A Y-pin \( S \) is four distinct points \( S = \{a, b, c, d \in \mathbb{Z}^3\} \), satisfying certain technical conditions. A Y-mesh of type \( S \) and dimension \( d \) is a map \( v: \mathbb{Z}^{2*} \rightarrow \mathbb{P}^d \) such that the points \( v(r + a), v(r + b), v(r + c), v(r + d) \) are collinear for any \( r \in \mathbb{Z}^2 \). One can view any Y-mesh as a polygon depending on a discrete time variable \( t \in \mathbb{Z} \). By definition, the \( i^{th} \) vertex of the polygon at time \( t \) is given by \( v(i, t) \). In what follows, we will only consider Y-meshes such that \( v(i + n, t) = \phi(v(i, t)) \) for a fixed projective transformation \( \phi \). In other words, we assume that all the polygons defined by the Y-mesh are twisted \( n \)-gons with the same monodromy.

The collinearity assumption on \( v(r + a), v(r + b), v(r + c), v(r + d) \) defines a relation between the polygon \( v(*, t) \) and the polygons corresponding to several previous time instances. Thus, Y-meshes can be regarded as dynamical systems. Since the polygon \( v(*, t) \) may be expressed in terms of polygons corresponding to several previous values of time, such a dynamical system is, generally speaking, defined on the space of \( k \)-tuples of polygons (as opposed to pentagram maps which are defined on polygons). Furthermore, those polygons need to satisfy certain additional restrictions. As an example, consider the Y-pin \( S := \{(-1, 0), (1, 0), (0, 1), (0, 2)\} \) depicted in Figure 2. In this case, the horizontal level \( v(*, t + 2) \) may be expressed in terms of the previous two levels. Indeed, by definition of a Y-mesh, the vertex \( v(i, t + 2) \) may be reconstructed as the intersection of lines \( \langle v(i - 1, t), v(i + 1, t) \rangle \cap \langle v(i - 1, t + 1), v(i + 1, t + 1) \rangle \). Thus, in this case the Y-mesh may be viewed as a dynamical system on pairs of polygons. These polygons satisfy two additional conditions:

- The vertex \( v(i, t + 1) \) of the second polygon lies on the diagonal \( \langle v(i - 1, t), v(i + 1, t) \rangle \) of the first polygon.
- The respective diagonals \( \langle v(i - 1, t), v(i + 1, t) \rangle \) and \( \langle v(i - 1, t + 1), v(i + 1, t + 1) \rangle \) of the two polygons are coplanar.

Further, the authors of [12] observe that in dimension \( d = 3 \) the square of the map

\[
(v(*, t), v(*, t + 1)) \mapsto (v(*, t + 1), v(*, t + 2))
\]

defined by the Y-pin depicted in Figure 2 is precisely the short-diagonal pentagram map. Indeed, we have \( v(i - 1, t + 1) \in \langle v(i - 2, t), v(i, t) \rangle \) and \( v(i + 1, t + 1) \in \langle v(i, t), v(i + 2, t) \rangle \), so the point \( v(i, t + 2) \in \langle v(i - 1, t + 1), v(i + 1, t + 1) \rangle \) belongs to the plane \( \langle v(i - 2, t), v(i, t), v(i + 2, t) \rangle \). Given also that \( v(i, t + 2) \in \langle v(i - 1, t), v(i + 1, t) \rangle \), we get

\[
v(i, t + 2) \in \langle v(i - 1, t), v(i + 1, t) \rangle \cap \langle v(i - 2, t), v(i, t), v(i + 2, t) \rangle,
\]

which is precisely the definition of the short-diagonal map. Thus, the map defined by the Y-pin depicted in Figure 2 can be viewed as the “square root” of the short-diagonal map. This square root, however,

\[
\text{Figure 2: A Y-pin corresponding to the short-diagonal map in 3D.}
\]
is not defined on the space of polygons itself, but on a certain extension of that space which consists of pairs of polygons satisfying two above-mentioned conditions. It can be shown, using purely geometric arguments, that this extended space is generically a finite cover of the space of polygons. In other words, given a level \( v(*,t) \) of a Y-mesh of type depicted in Figure 2, there are generically finitely many ways how to reconstruct the next level \( v(*,t+1) \) and thus all subsequent levels. Below we show that this reconstruction problem is equivalent to a factorization problem for the difference operator \( D_+ \) corresponding to the initial polygon \( v(*,t) \).

Recall that the short-diagonal map in 3D corresponds to progressions \( J_+ = \{-2, 0, 2\} \), \( J_- = \{-1, 1\} \). To every twisted \( n \)-gon in \( \mathbb{R}^3 \) we can assign two operators \( D_+ \in \text{DO}_n(J_+) \) supported in those sets, which identifies the short-diagonal map with refactorization dynamics \((17)\). Assume now that the polygon encoded by the operators \( D_+ \) is realized as a level \( v(*,t) \) of a Y-mesh of type depicted in Figure 2. Let \( V(i,t) \) be the lifts of points \( v(i,t) \) to \( \mathbb{R}^4 \). Since the levels \( v(*,t) \) and \( v(*,t+2) \) are related by the short-diagonal map, their lift \( V(*,t) \), \( V(*,t+2) \) may be chosen in such a way that

\[
D_+ V(*,t) = -D_- V(*,t) = V(*,t+2)
\]

(cf. the proof of Theorem 4.1). Furthermore, since \( v(i,t+2) \in \langle v(i-1,t+1), v(i+1,t+2) \rangle \), there exists a difference operator \( D_+^{(1)} \) supported in \( \{-1,1\} \) such that

\[
V(*,t+2) = D_+^{(1)} V(*,t+1)
\]

Analogously, there exists a difference operator \( D_+^{(2)} \) supported in \( \{-1\} \) such that

\[
V(*,t+1) = D_+^{(1)} V(*,t)
\]

(29)

Therefore, we have

\[
D_+ = D_+^{(1)} D_+^{(2)}.
\]

Conversely, given such a factorization of \( D_+ \), we can reconstruct the level \( v(*,t+1) \) of the Y-mesh by using \((29)\), and hence reconstruct all the subsequent levels.

**Proposition 4.17.** A generic operator difference operator \( D \) supported in \( \{-2, 0, 2\} \) has two factorizations of the form \( D = D_1 D_2 \), where \( D_i \)'s are supported in \( \{-1,1\} \), if \( n \) is odd, and four such factorizations if \( n \) is even. Two factorizations \( D_1 D_2 \) and \( D_1 D_2 \) are considered the same if \( D_1 = D_1 \alpha^{-1} \) and \( D_2 = \alpha D_2 \) for a certain \( n \)-periodic invertible scalar operator \( \alpha \).

**Remark 4.18.** The coefficients of the factors are, in general, complex numbers, even if the initial operator \( D \) is real.

**Proof of Proposition 4.17.** The problem is equivalent to representing an operator supported in \( \{0,2,4\} \) as a product of two operators supported in \( \{0,2\} \). If \( n \) is odd, this problem further reduces, using the isomorphism described in Remark 4.20, to representing an operator \( D \) supported in \( \{0,1,2\} \) as a product \( D_1 D_2 \) of two operators supported in \( \{0,1\} \). The latter problem has two different solutions for generic \( D \) since \( D_2 \) is a right divisor of \( D \) if and only it annihilates a certain element of \( \text{Ker} \ D \), and since \( D_2 \) must be periodic, this element has to be of the two eigenvectors of the monodromy operator. Similarly, if \( n \) is even, an operator supported in \( \{0,2,4\} \) can be identified with two \((n/2)\)-periodic operators supported in \( \{0,1,2\} \) (see Remark 4.20), each of which has two different factorizations. Hence, in this case we generically have \( 2 \times 2 = 4 \) distinct factorizations.

Therefore, the square root of the short-diagonal map defined by the Y-pin depicted in Figure 2 acts on the space which is generically a 2-to-1 or 4-to-1 covering of the space of polygons. This space can be described as the space of triples of operators \( D_+^{(1)}, D_+^{(2)}, D_- \), all of which are supported in \( \{-1,1\} \). These operators should be considered up to the action

\[
D_+^{(1)} \mapsto \alpha D_+^{(1)} \beta^{-1}, \quad D_+^{(2)} \mapsto \beta D_+^{(2)} \gamma^{-1}, \quad D_- \mapsto \alpha D_- \gamma^{-1},
\]
where $\alpha, \beta, \gamma$ are $n$-quasi-periodic operators with the same monodromy. This space projects to the space of polygons in $\mathbb{P}^3$ be means of the map

$$D^{(1)}_+, D^{(2)}_+, D_- \mapsto D^{(1)}_+ D^{(2)}_+, D_-.$$  

Furthermore, the Y-mesh dynamics (i.e. the square root of the short-diagonal map) can be expressed in terms of difference operators as follows:

$$\bar{D}_- D^{(2)}_+ = D^{(1)}_+ D_-,$$

which can also be described as the following refactorization:

$$D^{-1}_+ D^{(1)}_+ D^{(2)}_+ \mapsto D^{(2)}_+ D^{-1}_+ D^{(1)}_+.$$

Since $\bar{D}^{(2)}_+ = D^{(1)}_+$, applying this refactorization twice we obtain the operator $D^{(1)}_+ D^{(2)}_+ D^{-1}_+$, which is equivalent to the short-diagonal map. Thus, the Y-mesh interpretation of higher pentagram maps can be regarded as a step-by-step refactorization, where on each step one needs to solve a refactorization problem for binomial difference operators (i.e. operators whose support consists of two elements). As shown in [12], each of these individual steps can be identified with a sequence of mutations in an appropriately defined cluster algebra.

## 5 Open problems

**1. Relation to cluster algebras.** The classical pentagram map, as well as pentagram maps on corrugated polygons, can be described as sequences of cluster mutations [11, 10]. It would be interesting to find a similar description for more general pentagram maps on $J$-corrugated polygons or, even more generally, the maps $\Psi_{J\pm}$ associated with arbitrary pairs of progressions with the same common difference.

Short-diagonal and dented maps were recently treated from the cluster perspective in [12] (see also Section 4.4 above), where the authors introduced certain variables which transform, under the corresponding pentagram map, according to a cluster rule. However, the definition of those variables involves introduction of the $k$’th root of the corresponding map, which in general results in multivalued functions on the space of polygons (as we show in Section 4.4, computation of such a root is equivalent to a factorization problem for a certain difference operator; in general, this operation cannot be performed using only rational functions). Do there exist single-valued cluster variables for short-diagonal, dented, and more general maps studied in the present paper?

A related question is whether our maps fit into a construction of [13] of integrable systems associated with dimer models on bipartite graphs, or perhaps some generalized version of it.

**2. Refactorization and Y-meshes.** Generalize the approach of Section 4.4 to all types of Y-meshes. What is the precise relation between maps described in the present paper and maps that admit a Y-mesh description? In particular, is it possible to interpret the cluster dynamics of [12] as refactorization of ratios of binomial difference operators, as in Section 4.4 above?

**3. Maps associated with pairs of non-disjoint progressions.** In this paper we constructed refactorization maps associated with pairs of progressions $J\pm \subset \mathbb{Z}$ with the same common difference. When these progressions are disjoint, such maps can be interpreted as pentagram-type maps. What is a geometric interpretation in the non-disjoint case?

**4. The leapfrog map.** Give a geometric proof of the fact that for $J_+ = \{-1, 0\}$, $J_- = \{0, 1\}$ our construction leads to the leapfrog map of [10] (cf. Remark 4.6).

**5. Integrability.** For all maps $\Psi_{J\pm}$ associated with pairs of progressions we constructed a Lax representation with spectral parameter and a Poisson structure such that the first integrals
coming from the Lax representation Poisson-commute. This suggests that all these maps are both algebraically and Liouville integrable. Find a proof of this fact, i.e. show that the joint levels sets of first integrals are Lagrangian submanifolds of symplectic leaves, that each of those submanifolds can be identified with an open subset in the Jacobian of the corresponding spectral curve, and that a suitable power of the map $\Psi_{f_\pm}$ is a translation relative to the natural group structure on the Jacobian.

6. Difference operators with matrix coefficients and pentagram maps on Grassmannians. The construction of the present paper can be generalized to difference operators with matrix coefficients. Does this lead to pentagram maps on Grassmannians defined in [3]? How are the corresponding Poisson structures related to double brackets of [26]?

7. Partial difference operators and Laplace transform. One can generalize the construction of the present paper to partial difference operators supported in arithmetic progressions $J_\pm \subset \mathbb{Z}^2$. This leads to pentagram-type maps defined on polyhedra. The simplest example of such a map is the discrete Laplace transform of [6] corresponding to $J_+ = \{(0,0), (1,0)\}, J_- = \{(0,1), (1,1)\}$. Are maps of this type integrable?

8. Poisson structures on reductions of difference operators. Poisson structures studied in the present paper arise as reductions of structures on rational pseudo-difference operators. One can also study Poisson structures on polygons arising as reductions of difference operators, see Remark 3.13. For example, taking $d = 1$ and coordinatizing the moduli space of polygons in $\mathbb{RP}^1$ by means of cross-ratios of quadruples of consecutive vertices, one gets the following Poisson bracket:

$$\{x_i, x_{i+1}\} = x_i x_{i+1} (x_i + x_{i+1} - 1), \quad \{x_i, x_{i+2}\} = x_i x_{i+1} x_{i+2}.$$ 

This bracket is well-known in relation to the Volterra lattice and also arises in the study of cross-ratio dynamics on polygons [2, 34]. Furthermore, this structure is often considered as a lattice analogue of the Virasoro algebra [7]. Similarly, computing the bracket on polygons in $\mathbb{RP}^2$, one recovers the Belov-Chaltikian lattice $W_3$-algebra [3]. More generally, we believe that Poisson structures on polygons obtained by reduction from difference operators can be viewed as lattice versions of classical $W$-algebras. In particular, we conjecture that these structures coincide with the ones constructed by means of difference Drinfeld-Sokolov reduction [23]. One interesting property that such structures have is that, in contrast to Poisson brackets studied in the present paper, they restrict to the space of closed polygons.

6 Appendix: Refactorization as long-diagonal pentagram maps and its continuous limit (by Anton Izosimov and Boris Khesin\textsuperscript{1})

In this appendix we introduce a new class of long-diagonal pentagram maps in arbitrary dimension, and derive their complete integrability from the results on refactorization. Furthermore, it turns out that any factorization case corresponds to an appropriate long-diagonal pentagram map. As a corollary, one can describe the continuous limit of all integrable pentagram-type maps appearing from refactorization: this limit gives the equations of the Boussinesq type in the KdV hierarchy.

6.1 Long-diagonal pentagram maps

**Definition 6.1.** [13] Let $I = (i_1, \ldots, i_{d-1})$ be a $(d-1)$-tuple of positive integers (a jump tuple). Given a (twisted) $n$-gon $\{v_k \mid k \in \mathbb{Z}\}$ in $\mathbb{RP}^d$, its **diagonal hyperplanes associated with $I$** are defined by

$$P_k := \text{span}(v_k, v_{k+i_1}, \ldots, v_{k+i_1+\ldots+i_{d-1}}).$$

Let also $R = (r_1, \ldots, r_{d-1})$ be another $(d-1)$-tuple of positive integers (an intersection tuple). Then the pentagram map $T_{I,R}$ is defined as follows: the vertex $\hat{v}_k$ of the image of the polygon $\{v_k \mid k \in \mathbb{Z}\}$ under $T_{I,R}$ is given by

$$\hat{v}_k := P_k \cap P_{k+r_1} \cap \ldots \cap P_{k+r_1+\ldots+r_{d-1}}.$$  

–\textsuperscript{1}Department of Mathematics, University of Toronto, Toronto, Canada, khesin@math.toronto.edu
Example 6.2. For \( I = (2) \) and \( R = (1) \), the associated map \( T_{I,R} \) is the usual pentagram map in \( \mathbb{R}P^2 \).

Example 6.3. For \((d-1)\)-tuples \( I = (2, \ldots, 2) \) and \( R = (1, \ldots, 1) \), the map \( T_{I,R} \) is the short-diagonal pentagram map in \( \mathbb{R}P^d \) [16].

A natural generalization of the latter example is "long-diagonal pentagram maps", defined below. These maps correspond to jump tuples \( I \) of the form \((m, \ldots, m)\). It turns out that to preserve integrability the intersection tuple \( R \) also needs to be of special form.

Definition 6.4. Take two disjoint arithmetic \( m \)-progressions (i.e. arithmetic progressions with common difference \( m \)) \( R_+ \) and \( R_- \) with \( d \) elements in total. Let \( R \) be the sequence of differences between consecutive elements in the union \( R_+ \cup R_- \). Then the associated long-diagonal pentagram map \( T_{long} \) is \( T_{I,R} \) for such an \( R \) and \( I = (m, \ldots, m) \).

Remark 6.5. More explicitly, the set \( R \) should be of one of the following forms: \( R = (m, m, \ldots, m, m) \) for any \( p \in \mathbb{Z}_+ \), \( R = (m, m, m, q, m - q, q, m - q, m, \ldots, m) \) or \( R = (m, m, m, q, m - q, q, m, \ldots, m) \) for any positive integer \( q < m \).

Example 6.6. For \( I = (2, 2) \) and \( R = (1, 2) \) we recover the pentagram map in \( \mathbb{R}P^3 \) which was known to be numerically integrable, see [17] Section 6. Now it can be regarded as a long-diagonal pentagram map with \( R_- = \{0\} \) and \( R_+ = \{1, 3\} \), while \( R = (1, 2) \) is the sequence of differences between elements of \( R_- \cup R_+ = \{0, 1, 3\} \), the union of two arithmetic 2-progressions.

Example 6.7. Set \( I = (3, 3) \) and \( R = (1, 2) \) in \( \mathbb{R}P^3 \). Here \( R = (1, 2) \) is again the sequence of differences between elements of \( \{0, 1, 3\} \), but the latter set is now regarded as a union of 3-progressions \( R_+ = \{0, 3\} \) and \( R_- = \{1\} \). Then the corresponding long-diagonal map is

\[
\hat{v}_k = T_{long} v_k = (v_k, v_{k+3}, v_{k+6}) \cap (v_{k+1}, v_{k+4}, v_{k+7}) \cap (v_{k+3}, v_{k+6}, v_{k+9}) = (v_{k+1}, v_{k+4}, v_{k+7}) \cap (v_{k+3}, v_{k+6}),
\]

see Figure 3.

![Figure 3: Integrable long-diagonal map in \( \mathbb{R}P^3 \) for \( I = (3, 3) \) and \( R = (1, 2) \).](image)

Remark 6.8. The above definition generalizes all previously known examples of integrable pentagram maps [27 [10 [16 [22 [18]. Indeed, the short-diagonal pentagram maps with \( I = (2, \ldots, 2) \) and \( R = (1, \ldots, 1) \) corresponds to \( m = 2 \) and \( R = (m, q, m - q, q, m - q, m) \) with \( q = 1 \). The dual (deep) dented or corrugated maps correspond to \( m = 1, I = (1, \ldots, 1) \), and \( R = (1, \ldots, 1, p, 1, \ldots, 1) \) with any \( p \in \mathbb{Z}_+ \).

Theorem 6.9. The long-diagonal pentagram maps \( T_{long} \) are completely integrable discrete dynamical systems on generic twisted \( n \)-gons in \( \mathbb{R}P^d \). Namely, each of those maps admits a Lax representation with spectral parameter and an invariant Poisson structure such that the spectral invariants of the Lax matrix Poisson commute.
This theorem covers all known examples of integrable pentagram maps defined by intersections of diagonals: short-diagonal, dented, deep-dented (including corrugated) cases. The case of pentagram maps on Grassmanians of \( \mathbb{S} \) is apparently related to a “matrix version” of the above theorem.

**Proof of Theorem 6.9.** Consider separately two sequences \( R_+ \) and \( R_- \) defining the jump \((d-1)\)-tuple \( R \) for hyperplanes in \( \mathbb{R}^d \). Without loss of generality, assume that the minimal element in the union \( R_+ \cup R_- \) is equal to 0. (This can always be arranged by simultaneously shifting \( R_+ \) and \( R_- \).) Such a shift does not change the jump tuple \( R \) and hence the corresponding long-diagonal map. Then formula (30) can be rewritten as

\[
\hat{v}_k := \bigcap_{r \in R_+ \cup R_-} P_{k+r} = L_{k,+} \cap L_{k,-},
\]

where

\[
L_{k,\pm} := \bigcap_{r \in R_\pm} P_{k+r}
\]

are planes of complementary dimensions in \( \mathbb{R}^d \). Notice that since each of \( R_\pm \) is an arithmetic progression, with the same common difference \( m \), while \( I = (m, \ldots, m) \), it follows that each of the planes \( L_{k,\pm} \) is also spanned by vertices \( v_j \) with indices \( j \) forming arithmetic \( m \)-progressions \( k + J_+ \) and \( k + J_- \). Explicitly, one has

\[
J_\pm = \bigcap_{r \in R_\pm} \{r, r + m, \ldots, r + m(d-1)\} = \{\max(R_\pm), \max(R_\pm) + m, \ldots, \min(R_\pm) + m(d-1)\}. \tag{31}
\]

This, in particular, implies \(|J_\pm| = d+1 - |R_\pm|\), so \(|J_-| + |J_+| = d + 2\). It is also easy to see that \( J_\pm \) are disjoint. (The only exception is the case of \( R_+ = (0, m, \ldots, km) \), \( R_- = ((k+1)m, \ldots, (d-1)m) \), which implies \( J_+ = \{km, \ldots, (d-1)m\} \), \( J_- = \{(d-1)m, \ldots, (d+k)m\} \), so the intersection is non-empty. This corresponds to the identity pentagram map \( T_{1,R} \) up to a shift of indices.)

Now consider the space of \( J \)-corrugated polygons in \( \mathbb{R}^d \) for \( J := J_+ \cup J_- \) and \( \bar{d} := \max(J) - \min(J) - 1 \). The spaces of generic \( n \)-gons in \( \mathbb{R}^d \) and \( J \)-corrugated \( n \)-gons in \( \mathbb{R}^d \) are defined by the same linear relation on vertices and are locally diffeomorphic, while globally it is a map \( N \)-to-1.

For instance, the standard corrugated condition means that the vectors \( V_j, V_{j+1}, V_{j+d} \) and \( V_{j+d+1} \) in \( \mathbb{R}^{d+1} \), which are lifts of vertices \( v_j, v_{j+1}, v_{j+d} \) and \( v_{j+d+1} \) in \( \mathbb{R}^d \), are linearly dependent for all \( j \in \mathbb{Z} \). Thus the subset of corrugated polygons is singled out in the space of generic twisted polygons by the relations

\[
a_j V_{j+d+1} + b_j V_{j+d} + c_j V_{j+1} + d_j V_j = 0, \quad j \in \mathbb{Z}.
\]

Note that this relation also allows one to define a map \( \Phi_{J_\pm} \) from generic twisted \( n \)-gons in \( \mathbb{R}^2 \) to corrugated ones in \( \mathbb{R}^d \) for any dimension \( d \): consider a lift of vertices \( v_j \in \mathbb{R}^2 \) to vectors \( V_j \in \mathbb{R}^3 \) so that they satisfy these relations for all \( j \in \mathbb{Z} \), see [10]. Now by considering solutions \( V_j \in \mathbb{R}^{d+1} \) of these linear relations modulo the natural action of \( \text{SL}_{d+1}(\mathbb{R}) \) we obtain a polygon in the projective space \( \mathbb{P}^d \) satisfying the corrugated condition. The constructed map \( \Phi_{J_\pm} \) commutes with the pentagram maps (since all operations are projectively invariant) and is a local diffeomorphism. A similar consideration is applicable to any \( J \)-corrugated maps, where the relations are on vertices \( V_{j+k} \) with indices \( k \in J \) for all \( j \in \mathbb{Z} \).

Now we apply the main refactorization theorem (Theorem 4.1) to obtain integrability of the lift of generic polygons from \( \mathbb{R}^d \) to \( J \)-corrugated ones in \( \mathbb{R}^d \) with \( \bar{d} := \max(J) - \min(J) - 1 \). This lift commutes with the pentagram map, as follows from refactorization or consideration above. Since the lift is a local diffeomorphism on the space of twisted polygons, this implies integrability of the long-diagonal pentagram map \( T_{1,R} = T_{\text{long}} \).

A separate interesting issue is to understand the ramification of the lift of the space of polygons to corrugated ones in higher-dimensional projective spaces, as well as the global behavior of orbits for the pentagram map before and after the lift.
Example 6.10. For the above Example 6.7 in $\mathbb{R}P^3$, we get $J_+ = \{0, 3, 6\}$ and $J_- = \{2, 5\}$ with period 3. Indeed, the intersection of two planes $P_1 = (v_1, v_2, v_3)$ and $P_2 = (v_2, v_5, v_8)$ is the line $L_+ = (v_2, v_5)$, which is intersected with another plane $L_- = (v_0, v_3, v_6)$. The refactorization theorem establishes its integrability in $\mathbb{R}$. Similarly, the long-diagonal maps in $\mathbb{R}^3$ with $I = (m, m)$ and $R = (m-k, k)$ for $1 \leq k \leq m-1$ correspond to the $m$-progressions: $J_+ = \{0, m, 2m\}$ and $J_- = \{k, k+m\}$, while the maps with $I = (m, m)$ and $R = (k, m)$ for $k \neq 0, m$ correspond to the $m$-progressions: $J_+ = \{0, m, 2m\}$ and $J_- = \{k + m, k + 2m\}$, cf. Example 6.6.

Remark 6.11. In [17] numerical non-integrability was observed for a pentagram map $T_{I,R}$ in $\mathbb{R}^3$ with $I = (3, 3)$ and $R = (1, 1)$. It turns out, however, that for $I = (3, 3)$ and $R = (1, 2)$ the map becomes integrable!

It turns out that not only long-diagonal pentagram maps can be described in terms of the factorization, but also any refactorization corresponds to a certain pentagram map:

Theorem 6.12. Every pentagram-type map associated with a pair of nonintersecting arithmetic $m$-progressions $(J_\pm)$ can be uniquely realized by a long-diagonal pentagram map.

Proof. Indeed, for $d = |J_-| + |J_+| - 2$ set $(d-1)$-tuple $I = (m, \ldots, m)$. Next, one can recover the progressions $R_\pm$ from $J_\pm$ using formula (31). Namely, one has

$$R_\pm = \{\max(J_\pm) - m(d-1), \max(J_\pm) - m(d-2), \ldots, \min(J_\pm)\}.$$

Finally, the arithmetic progressions $R_\pm$ define $R$ and hence the map $T_{I,R}$. The fact that this long-diagonal pentagram map is equivalent to the $(J_\pm)$-pentagram map is proved in Theorem 6.9.

Corollary 6.13. The pentagram-type maps associated with pairs of nonintersecting arithmetic $m$-progressions $J_\pm$ of total length $d + 2$ are in one-to-one correspondence with long-diagonal pentagram maps in $\mathbb{R}^d$ for the jump tuple $I = (m, \ldots, m)$.

This follows by combining the arguments of Theorems 6.9 and 6.12.

Remark 6.14. In the case of closed $n$-gons with $m$ mutually prime with $n$, by using renumeration of vertices, the case of $I = (m, \ldots, m)$ and $R$ being the steps between two arithmetic $m$-progressions $R_\pm$, can be regarded as $I = (1, \ldots, 1)$ and $R = (1, \ldots, 1, p, 1, \ldots, 1)$, i.e. dual of deep dented maps. (It is not clear if a similar consideration is applicable to a twisted case.) Thus all those integrable cases described in the refactorization theorem are somewhat similar to the “deep dented” pentagram cases.

Furthermore, consider any pair of arithmetic $m$-progression $(J_\pm)$ and $(I_+, I_-)$, where $J_+ \subset I_+$, while $J_- \subset I_-$. Then the map associated to the first pair is a restriction of the map associated to the second pair, since all maps are given by the same formula. In particular, any map related to a pair $(J_\pm)$ with $m = 1$ is a restriction of the (dual) dented map. The same is true for any $m$, provided that the convex hulls of $J_+$ and $J_-$ do not intersect. If the period $m$ is not equal to 1, then the corresponding embedding is not Poisson. The case of an arbitrary $m$ can be reduced to the case $m = 1$ by using renumeration of vertices.

6.2 Continuous limit of refactorization pentagram maps

Theorem 6.15. The continuous limit of all refactorization pentagram maps is equivalent to the $(2, d + 1)$-KdV equation, generalizing the Boussinesq equation for $d = 2$.

To define the continuous limit of refactorization maps we use their realization by long-diagonal pentagram maps on $n$-gons in $\mathbb{R}P^d$. In the limit as $n \to \infty$ a generic twisted $n$-gon becomes a smooth non-degenerate quasi-periodic curve $\gamma(x)$. The limit of pentagram maps is an evolution on such curves constructed as follows. Consider the lift of $\gamma(x)$ in $\mathbb{R}^d$ to a curve $G(x)$ in $\mathbb{R}^{d+1}$ defined
by the conditions that the components of the vector function \( G(x) = (G_1, ..., G_{d+1})(x) \) provide the homogeneous coordinates for \( \gamma(x) = (G_1 : ... : G_{d+1})(x) \) in \( \mathbb{RP}^d \) and

\[
\det(G(x), G'(x), ..., G^{(d)}(x)) = 1
\]

for all \( x \in \mathbb{R} \). Furthermore, \( G(x + 2\pi) = MG(x) \) for a given \( M \in SL_{d+1}(\mathbb{R}) \). Then \( G(x) \) satisfies the linear differential equation of order \( d + 1 \):

\[
G^{(d+1)} + u_{d-1}(x)G^{(d-1)} + ... + u_1(x)G' + u_0(x)G = 0
\]

with periodic coefficients \( u_i(x) \), which is a continuous limit of difference equation defining a space \( n \)-gon. Here \( l^t \) stands for \( d/dx \).

Fix a small \( \varepsilon > 0 \) and let \( \bar{I} \) be any \((d-1)\)-tuple \( \bar{I} = (i_1, ..., i_{d-1}) \) of positive integers. For the \( \bar{I} \)-diagonal hyperplane

\[
P_k := (v_k, v_{k+i_1}, v_{k+i_1+i_2}, ..., v_{k+i_1+...+i_{d-1}})
\]

its continuous analogue is the hyperplane \( P^\varepsilon(x) \) passing through \( d \) points \( \gamma(x), \gamma(x + i_1 \varepsilon), ..., \gamma(x + (i_1 + ... + i_{d-1}) \varepsilon) \) of the curve \( \gamma \). In what follows we are going to make a parameter shift in \( x \) (equivalent to shift of indices) and define \( P^\varepsilon(x) := (\gamma(x + k_0 \varepsilon), \gamma(x + k_1 \varepsilon), ..., \gamma(x + k_{d-1} \varepsilon)) \), for any real \( k_0 < k_1 < ... < k_{d-1} \) such that \( \sum k_i = 0 \).

Let \( \zeta_\varepsilon(x) \) be the envelope curve for the family of hyperplanes \( P^\varepsilon(x) \) in \( \mathbb{RP}^d \) for a fixed \( \varepsilon \). (Geometrically the envelope can be thought of as the intersection of infinitely close “consecutive” hyperplanes of this family along the curve.) The envelope condition means that \( P^\varepsilon(x) \) are the osculating hyperplanes of the curve \( \zeta_\varepsilon(x) \), that is the point \( \zeta_\varepsilon(x) \) belongs to the hyperplane \( P^\varepsilon(x) \), while the vector-derivatives \( \zeta'_\varepsilon(x), ..., \zeta^{d-1}_\varepsilon(x) \) span this hyperplane for each \( x \). It means that the lift of \( \zeta_\varepsilon(x) \) in \( \mathbb{RP}^d \) to \( Z^\varepsilon(x) \) in \( \mathbb{RP}^{d+1} \) satisfies the system of \( d \) equations:

\[
\det(G(x + k_0 \varepsilon), ..., G(x + k_{d-1} \varepsilon), Z^{(j)}(x)) = 0, \quad j = 0, ..., d-1.
\]

Here the lift \( Z^\varepsilon(x) \) is again defined by the constraint \( \det(Z^\varepsilon(x), Z'_\varepsilon(x), ..., Z^{(d)}_\varepsilon(x)) = 1 \) for all \( x \in \mathbb{R} \).

One can show that the expansion of the lift \( Z^\varepsilon(x) \) has the form

\[
Z^\varepsilon(x) = G(x) + \varepsilon^2 B(x) + O(\varepsilon^3),
\]

where there is no term linear in \( \varepsilon \) due to the condition \( \sum k_i = 0 \).

**Definition 6.16.** A continuous limit of the pentagram map is the evolution of the curve \( \gamma \) in the direction of the envelope \( \zeta_\varepsilon \), as \( \varepsilon \) changes: \( dG/dt = B \). More explicitly, the lift \( Z^\varepsilon(x) \) satisfies the family of differential equations:

\[
Z^{(d+1)}_\varepsilon + u_{d-1,\varepsilon}(x)Z^{(d-1)}_\varepsilon + ... + u_{1,\varepsilon}(x)Z'_\varepsilon + u_{0,\varepsilon}(x)Z_\varepsilon = 0,
\]

where \( Z_0(x) = G(x) \), i.e. \( u_{j,0}(x) = u_j(x) \). Then the corresponding expansion of the coefficients \( u_{j,\varepsilon}(x) \) as \( u_{j,\varepsilon}(x) = u_j(x) + \varepsilon^2 w_j(x) + O(\varepsilon^3) \), defines the continuous limit of the pentagram map as the system of evolution differential equations \( du_j(x)/dt = w_j(x) \) for \( j = 0, ..., d-1 \).

This definition of limit via an envelope assumes that we are dealing with consecutive hyperplanes in the pentagram map, i.e. the intersection tuple is \( R = 1 := (1, ..., 1) \) or its multiple. We are going to apply this to the case \( \bar{R} = m \bar{1} := (m, ..., m) \), which gives the same limit upon rescaling \( \varepsilon \mapsto m\varepsilon \).

We start with reminding the following theorem, a variation of a result from [18], which is the main ingredient of the proof of Theorem 6.15.

**Remark 6.17.** Below we will use the following property of the pentagram map \( T_{I, R} \) for arbitrary \((d-1)\)-tuples \( \bar{I} = (i_1, ..., i_{d-1}) \) and \( \bar{R} = (r_1, ..., r_{d-1}) \): its inverse \( T_{I, R}^{-1} \) coincides with the map \( T_{R^*, I^*} \) (modulo shift of indices), where \( R^* \) and \( I^* \) are respectively \((d-1)\)-tuples \( R \) and \( I \) read backwards: \( R^* = (r_{d-1}, ..., r_1) \) and \( I^* = (i_{d-1}, ..., i_1) \), see [18].
Theorem 6.18. (cf. [13]) The continuous limit of any generalized pentagram map $T_{\bar{l}, \bar{R}}$ for any $\bar{l} = (l_1, \ldots, l_{d-1})$ and $\bar{R} = m1$ (and in particular, of the inverse of any long-diagonal pentagram map) in dimension $d$ defined by the system $d u_j(x)/dt = w_j(x)$, $j = 0, \ldots, d - 1$ for $x \in S^1$ is the $(2, d + 1)$-KdV flow of the Adler-Gelfand-Dickey hierarchy on the circle.

Remark 6.19. Recall that the $(k, d + 1)$-KdV flow is defined on linear differential operators $L = \partial^{d+1} + u_{d-1}(x)\partial^{d-1} + u_{d-2}(x)\partial^{d-2} + \cdots + u_1(x)\partial + u_0(x)$ of order $d + 1$ with periodic coefficients $u_j(x)$, where $\partial^k$ stands for $d^k/dx^k$. One can define the fractional power $L^{k/d+1}$ as a pseudo-differential operator for any positive integer $n$ and take its pure differential part $Q_k := (L^{k/d+1})_+$. In particular, for $k = 2$ one has

$$Q_2 = \partial^2 + \frac{2}{d+1} u_{d-1}(x).$$

Then the $(k, d + 1)$-KdV equation is the evolution equation on (the coefficients of) $L$ given by $dL/dt = [Q_k, L]$, see [1].

For $k = 2$ this gives the $(2, d + 1)$-KdV system

$$\frac{dL}{dt} = [Q_2, L] := \left[ \partial^2 + \frac{2}{d+1} u_{d-1}(x), L \right].$$

(32)

For $d = 2$ and $k = 2$ the $(2,3)$-KdV system gives evolution equations on the coefficients $u$ and $v$ of the operator $L = \partial^3 + u(x)\partial + v(x)$. Upon elimination of $v$ this reduces to the classical Boussinesq equation on the circle: $u_t + 2(u^2)_{xx} + u_{xxxx} = 0$, which appears as the continuous limit of the 2D pentagram map [27].

Proof of Theorem 6.18. The proof is based on the expansion of the envelope $Z_{\varepsilon}(x)$ in the parameter $\varepsilon$: one can show that

$$Z_{\varepsilon}(x) = G(x) + \varepsilon^2 C_{d,m,\bar{l}} \left( \partial^2 + \frac{2}{d+1} u_{d-1}(x) \right) G(x) + \mathcal{O}(\varepsilon^3)$$

as $\varepsilon \to 0$, for a certain non-zero constant $C_{d,m,\bar{l}}$. This gives the following evolution of the curve $G(x)$ given by the $\varepsilon^2$-term of this expansion:

$$\frac{dG}{dt} = \left( \partial^2 + \frac{2}{d+1} u_{d-1} \right) G,$$

or which is the same, $dG/dt = Q_2 G$.

To find the evolution of the differential operator $L$ tracing it recall that for any $t$, the curve $G$ and the operator $L$ are related by the differential equation $L G = 0$. In particular, $d(LG)/dt = (dL/dt)G + L(dG/dt) = 0$, which, in view of $dG/dt = Q_2 G$, implies

$$\left( \frac{dL}{dt} + LQ_2 \right) G = 0,$$

and hence

$$\left( \frac{dL}{dt} - [Q_2, L] \right) G = 0,$$

where $[Q_2, L] := Q_2 L - LQ_2$. But $dL/dt - [Q_2, L]$ is an operator of order $\leq d$, so it can only annihilate the vector-function $G(x) \in \mathbb{R}^{d+1}$ if $L$ satisfies the $(2, d + 1)$-KdV equation

$$\frac{dL}{dt} = [Q_2, L].$$

which proves Theorem 6.18.
Remark 6.20. A similar argument can be used to prove the refactorization theorem: one first shows that the left hand-side of (1) applied to the bi-infinite sequence $V$ is equal to the right hand-side of (1) applied to $V$, and then observes that the difference of two sides is an operator of order 2 and hence can only annihilate $V$ if it is zero. This argument replaces the count of number of equations in the proof of the refactorization theorem (and is essentially equivalent to it). Notice also that not just the proofs are similar, but also the statements: Theorem 4.1 says that the pentagram is refactorization (which is, essentially, discrete Lax representation), while Theorem 6.18 states that its continuous limit has a Lax representation. It would be interesting to expand this similarity.

Proof of Theorem 6.15. To complete the proof we will show that for any pair of arithmetic progressions $J_+$ and $J_-$ with step $m$ in the refactorization theorem one can find an appropriate problem with intersection of $m$-consecutive hyperplanes, thus reducing the corresponding continuous limit to the known case. Indeed, given arithmetic $m$-progressions $J_+$ and $J_-$ we first consider the associated long-diagonal pentagram map which gives the same dynamics on polygons. This pentagram map has the diagonal tuple $I = (m, ..., m)$ and a certain intersection tuple $R$. Now pass to the inverse long-diagonal map. For such a map the new diagonal tuple is $\bar{I} = R^*$, and the new intersection tuple is $\bar{R} = I^* = (m, ...m)$. According to Theorem 6.18 the continuous limit for pentagram maps $T_{\bar{I}, \bar{R}}$ in $\mathbb{RP}^d$ with such $\bar{I}$ and $\bar{R}$ (including the inverses of long-diagonal maps) is equivalent to the $(2, d+1)$-KdV equation.

Finally, note that the inverse of Equation (32) is the same differential equation with the reversed time variable. Thus the continuous limit of the long-diagonal (and hence refactorization) pentagram maps is given by the same $(2, d+1)$-KdV equations upon the changing time $t \rightarrow -t$, which we treat on equal footing with the original KdV flows.

Remark 6.21. For intersection of hyperplanes indexed by more than one parameter the continuous limit is not an envelope and it can be arranged rather arbitrarily. For instance, for some special choices, one can obtain higher equations of the KdV hierarchy, see [21]. Furthermore, even in the case of intersecting but not $m$-consecutive hyperplanes $P_k$, i.e. for the intersection tuple different from $\bar{R} = (m, ..., m)$, there remains some freedom in the definition of the continuous limit. In our case the continuous limit turned out to be a familiar system thanks to the regular structure of the diagonal plane $I = (m, ..., m)$.

Remark 6.22. It would be interesting to obtain the $(2, d+1)$-KdV equation as the continuous limit directly from the Lax form for the pentagram maps, by formally passing to the limit from the linear difference equations defining polygons to the linear differential equations defining curves in $\mathbb{R}^{d+1}$.

References


