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*To Victor Yudovich on the occasion of his 70th birthday*

# ASYMPTOTIC DIRECTIONS, MONGE-AMPÈRE EQUATIONS AND THE GEOMETRY OF DIFFEOMORPHISM GROUPS

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Abstract. In this note we obtain the characterization for asymptotic directions on various subgroups of the diffeomorphism group. We give a simple proof of non-existence of such directions for area-preserving diffeomorphisms of closed surfaces of non-zero curvature. Finally, we exhibit the common origin of the Monge-Ampère equations in 2D fluid dynamics and mass transport.

## 1. Asymptotic directions on diffeomorphism groups

The group of volume-preserving diffeomorphisms of a Riemannian manifold plays a fundamental role in the geometrical study of the Euler equation of hydrodynamics on the manifold [Ar]. In this paper we consider another equation, the Monge-Ampère equation, and discuss its universality in the context of diffeomorphism groups. This equation occurs in two main contexts: as the equation of asymptotic directions in 2D hydrodynamics and in the optimal transport problem in any dimension.

1.1. **Characterization of asymptotic directions on subgroups.** Let $M$ be a compact $n$-dimensional manifold without boundary and equipped with a Riemannian metric $g$. Let $\mathcal{D}(M)$ be (the connected component of the identity in) the group of all diffeomorphisms of $M$. Its tangent space at the identity diffeomorphism consists of smooth vector fields on $M$. The tangent space at a point $\eta$ consists of vector fields "reparameterized by $\eta$," i.e. of the maps $X_\eta : M \to TM$ with $X_\eta(x) \in T_{\eta(x)}M$. Define the "flat" $L^2$-metric on $\mathcal{D}(M)$ by assigning to each tangent space the inner product

$$g_\eta(X_\eta, Y_\eta) := \int_M g_{\eta(x)}\big(X_\eta(x), Y_\eta(x)\big)\, dV(x), \tag{1.1}$$

where $dV$ denotes the Riemannian volume form on $M$.

Let $\mathcal{SD}(M)$ be the subgroup of volume-preserving diffeomorphisms. The restriction of the $L^2$-metric to this subgroup is right-invariant and of particular importance in hydrodynamics. In [Ar] Arnold showed that geodesics in $\mathcal{SD}(M)$ correspond to motions of an ideal fluid in $M$ described by the Euler equations

$$\partial_t X + \nabla_X X = -\nabla p, \quad \operatorname{div} X = 0$$

for the fluid velocity field $X$.

As shown by Ebin and Marsden in [EM] the group $\mathcal{D}^s(M)$ of all diffeomorphisms of Sobolev class $H^s$ (as well as its various subgroups) can be viewed for $s > n/2+1$ as an infinite dimensional Hilbert manifold. All of the arguments in this paper can be rigorously developed in the Sobolev framework. However, to present the geometric ideas we will keep things formal and drop the index $s$ in what follows.

While the Euler equation depends only on the intrinsic Riemannian geometry of $\mathcal{SD}(M)$, it is also of interest to study its exterior geometry as a Riemannian submanifold in $\mathcal{D}(M)$. In particular, one can consider asymptotic directions in $\mathcal{SD}(M)$. A vector tangent to a Riemannian submanifold is *asymptotic* if the geodesics issued in the direction of this vector, one in the submanifold and the other in the ambient manifold, have a second order of tangency. (Note that in general two geodesics with a common tangent will have only a simple, i.e. first order, tangency.) More formally, asymptotic vectors are singled out by the condition that the second fundamental form evaluated on these vectors is zero. A curve whose tangent is asymptotic at each point is called an asymptotic line. An asymptotic line is a geodesic in the submanifold if and only if it is also a geodesic in the ambient manifold.

A description of asymptotic vectors in the group of volume-preserving diffeomorphisms was given by Bao and Ratiu in [BR], while asymptotic geodesics in $\mathcal{SD}(M)$ were studied in [M1] (as pressure-constant flows). We begin with the following convenient characterization of these vectors.

**Theorem 1.** [BR] *A vector field $X$ on a manifold $M$ is an asymptotic direction for $\mathcal{SD}(M)$ at the identity diffeomorphism if and only if*

$$\operatorname{div} \nabla_X X = \operatorname{div} X = 0. \tag{1.2}$$

Similarly, a vector $X_\eta$ is asymptotic to $\mathcal{SD}(M)$ at the diffeomorphism $\eta$ if and only if its right translation $X := X_\eta \circ \eta^{-1}$ is asymptotic to $\mathcal{SD}(M)$ at the identity. If $M$ is two-dimensional then more can be said:

**Corollary 2.** [BR] *The stream function $\psi$ of an asymptotic vector field $X$ in 2D satisfies a Monge-Ampère equation*

$$\det[D^2\psi] = \frac{g \cdot K}{2} |\nabla \psi|^2, \tag{1.3}$$

*where $g = \det(g_{ij})$, $K$ is the Gaussian curvature function of $M$ and $\det[D^2\psi]$ is the Hessian of $\psi$. Here the field $X = J\nabla\psi$ has the stream (or, Hamiltonian) function $\psi$ with respect to the symplectic area form $\omega = dV$ on $M$.*

Below we give a characterization of asymptotic vectors in a general setting. Let $\mathcal{B}$ be a subgroup of $\mathcal{D}(M)$ and let $\mathfrak{b}$ denote its Lie algebra. We assume that $\mathfrak{b}$ is a closed subspace, and therefore it has an orthogonal complement with respect to the $L^2$ inner product (1.1).

**Theorem 3.** *A vector field $X$ on $M$ is asymptotic for $\mathcal{B}$ if and only if*

$$X \in \mathfrak{b} \quad \text{and} \quad \nabla_X X \in \mathfrak{b}\,. \tag{1.4}$$

If $\mathcal{B} = \mathcal{SD}(M)$ then the Lie algebra $\mathfrak{b}$ consists of divergence-free vector fields and we recover Theorem 1. Moreover, if $M$ has a boundary then the diffeomorphisms from the subgroup $\mathcal{B}$ leave it invariant and hence the fields from the subalgebra $\mathfrak{b}$ are tangent to the boundary of $M$.

**Corollary 4.** [BR] *If $M$ has a boundary $\partial M$, then $X$ is asymptotic for $\mathcal{SD}(M)$ if in addition to equations (1.2) $X$ satisfies the conditions*

$$g(X, n) = g(\nabla_X X, n) = 0 \tag{1.5}$$

*where $n$ is the normal to $\partial M$.*

Suppose next that $M$ is a symplectic manifold of dimension $2n$ equipped with a symplectic 2-form $\omega$.

**Corollary 5. a)** *Let $\mathcal{B} = Symp(M)$ be the subgroup of symplectic diffeomorphisms with the corresponding Lie subalgebra $\mathfrak{b} = symp(M)$ of vector fields preserving $\omega$. Then a vector field $X$ is asymptotic if and only if*

$$L_X\omega = 0 \quad \text{and} \quad L_{\nabla_X X}\omega = 0.$$

**b)** *Let $\mathcal{B} = Ham(M)$ be the (generally speaking, smaller) subgroup of Hamiltonian diffeomorphisms. Then a vector field $X$ on $M$ is asymptotic if and only if*

$$\omega(X, \cdot) = d\psi \quad \text{and} \quad \omega(\nabla_X X, \cdot) = d\phi$$

*for some smooth functions $\psi$ and $\phi$ on $M$.*

If $\dim M = 2$ any divergence-free field is locally Hamiltonian and the latter equation is rewritten in (1.3) as the Monge-Ampère equation on its Hamiltonian, or stream function.

Curiously, this Monge-Ampère property does not survive when passing to Hamiltonian fields in higher dimensions, unlike what was conjectured at the end of [AK]. Indeed, already in the flat 4-dimensional case one obtains a system of three equations on pairwise products of second derivatives, while the corresponding Monge-Ampère with $\det[D^2\psi]$ would include the four-term products of second derivatives.

Another interesting example is provided by a contact manifold $M$ with a contact structure (i.e. maximally non-integrable distribution of hyperplanes) $\tau$.

**Corollary 6.** *Let $(M, \tau)$ be a contact manifold, and $\mathcal{B}$ a subgroup consisting of contact diffeomorphisms. Then the vector field $X$ is asymptotic to $\mathcal{B}$ if and only if*

*both $X$ and $\nabla_X X$ are contact.*

If $\alpha$ denotes a 1-form defining the contact structure $\tau$ then the Lie algebra $\mathfrak{b} := \{X : L_X\alpha = f\alpha\}$. For a contact vector field $X$ consider its contact Hamiltonian function $K_X := \alpha(X)$ with respect to the form $\alpha$. By rewriting the condition $\nabla_X X \in \mathfrak{b}$ for asymptotic vectors on $\mathcal{B}$ one can obtain an analog of the Monge-Ampère equation (1.3) on the Hamiltonian in the contact 3-dimensional case.

**Proof of Theorem 3.** The Lie algebra of $\mathcal{D}(M)$ splits orthogonally into $\mathfrak{b} \oplus \mathfrak{b}^\perp$ inducing the corresponding splitting at any point in $\mathcal{B}$ by right translations. Let $\bar\nabla$ denote the smooth Levi-Civita connection of the $L^2$-metric (1.1) on $\mathcal{D}(M)$ (see Ebin and Marsden [EM]). Let $X$ and $Y$ be two elements of $\mathfrak{b}$ and extend them to right-invariant vector fields $X_\eta = X \circ \eta$ and $Y_\eta = Y \circ \eta$ on $\mathcal{B}$. Decomposing into unique tangential and normal components (Gauss equation) we obtain

$$\bar\nabla_{X_\eta} Y_\eta = P_\eta(\bar\nabla_{X_\eta} Y_\eta) + (\bar\nabla_{X_\eta} Y_\eta)^\perp.$$

where $P_\eta$ is the projection onto the tangent space to $\mathcal{B}$ at the point $\eta$. The tangential component can be used to define a smooth right-invariant connection on $\mathcal{B}$ while the normal component $(\bar\nabla_{X_\eta} Y_\eta)^\perp$ is interpreted as the second fundamental form of $\mathcal{B}$ in $\mathcal{D}(M)$. Since

$$\bar\nabla_{X_\eta} Y_\eta = (\nabla_X Y) \circ \eta$$

we see that at the identity the $L^2$ covariant derivative $\bar\nabla$ is given by $\nabla_X Y$. Recall that a vector $X$ in $\mathfrak{b}$ is asymptotic if the second fundamental form evaluated at $X$ is zero. This implies that $(\nabla_X X)^\perp = 0$ and Theorem 3 follows. □

*Remark* 7. One can see how the above general consideration works in the case $\mathcal{B} = \mathcal{SD}(M)$. First, recall that an arbitrary vector field $X$ on $M$ can be decomposed into $L^2$-orthogonal divergence-free and gradient parts

$$X = P_{id}(X) + X^\nabla$$

where $P_{id}$ is now the projection onto the divergence-free fields, and $X^\nabla$ is the gradient part of $X$: explicitly $X^\nabla = \nabla\Delta^{-1}\mathrm{div}\, X$. By right invariance this induces a corresponding splitting at any point in $\mathcal{SD}(M)$. The tangential component of $\bar\nabla$ defines a smooth right-invariant connection on $\mathcal{SD}(M)$. Its normal component is the second fundamental form of the subgroup $\mathcal{SD}(M)$ in $\mathcal{D}(M)$. Thus, for any right-invariant vector fields $X_\eta = X \circ \eta$ and $Y_\eta = Y \circ \eta$ on $\mathcal{SD}(M)$ we again have the Gauss equation

$$\bar\nabla_{X_\eta} Y_\eta = P_\eta(\bar\nabla_{X_\eta} Y_\eta) + s(X_\eta, Y_\eta) \tag{1.6}$$

where

$$\bar\nabla_{X_\eta} Y_\eta = (\nabla_X Y) \circ \eta \quad \text{and} \quad s(X_\eta, Y_\eta) = \big(\nabla_X Y\big)^\nabla \circ \eta.$$

Since a vector $X$ tangent at the identity $\eta = id$ is asymptotic if $s(X, X) = 0$ we immediately obtain the characterization in Theorem 1.

*Remark* 8. Asymptotic geodesics (or pressure-constant flows) are of interest in the Lagrangian approach to hydrodynamic stability theory. Typical examples are plane-parallel flows on the flat torus $\mathbb{T}^2$. These flows can be considered unstable in the following sense. For any such flow it can be shown that sectional curvatures of $\mathcal{SD}(\mathbb{T}^2)$ along the corresponding geodesic are always non-positive. Therefore, by a suitable variant of the Rauch comparison argument, all linear perturbations in Lagrangian coordinates (Jacobi fields along the geodesic) must grow at least linearly in time. On the other hand, there are flows with non-positive curvatures for which

one can show that the growth must be at most polynomial (see Preston [Pr] for details).

1.2. **Non-existence of asymptotic directions.** Asymptotic directions are not always in good supply. In [BR] it is shown that if $M$ is a two-dimensional compact surface of revolution without boundary then any axially symmetric smooth solution of the Monge-Ampère equation is constant away from the cylindrical (i.e. fixed radius) bands of the surface. The strongest result in this direction so far states that for a compact closed surface $M$ of positive curvature there are no asymptotic directions on $\mathcal{SD}(M)$, see [Pa]. A similar result holds for surfaces with boundary. (Palmer also showed that there is no direct analog of this result in higher dimensions: e.g. every left-invariant vector field on a compact Lie group equipped with a bi-invariant metric solves the equations (1.2), i.e. is asymptotic. This had been observed previously for the three sphere $S^3$ in [M1].)

Here we prove the following generalization of Palmer's result.

**Theorem 9.** *If $M$ is a compact closed surface of nowhere zero curvature $K$, then the Monge-Ampère equation (1.3) admits no non-constant solutions. In particular, in this case $\mathcal{SD}(M)$ has no asymptotic directions.*

Note that the surface can be of any non-zero Euler characteristic (the case of the torus is ruled out by the Gauss–Bonnet theorem).

*Proof.* Recall that for any vector field $X$ on $M$ we have the identity

$$\operatorname{div}\nabla_X X = r(X,X) + \operatorname{tr}(DX)^2 + L_X(\operatorname{div} X) \tag{1.7}$$

where $r$ denotes the Ricci curvature of the metric $g$ and $L_X$ is the Lie derivative along $X$ (see for example [Ta]). If $X$ is divergence-free then the last term on the right side of (1.7) drops out.

Consider the length function $f := g(X,X)$. Since $M$ is compact $f$ must attain a maximum at some point $x_0$. Choosing normal coordinates at that point with $g_{ij}(x_0) = \delta_{ij}$ and $\partial g_{ij}/\partial x^k(x_0) = 0$, we obtain

$$0 = df(x_0) = 2\sum_{jk} X^j(x_0)\,\frac{\partial X^j}{\partial x^k}(x_0)\,dx^k\,.$$

Since at the point where $f$ has a maximum we must also have $X(x_0) \neq 0$, this implies that the Jacobi matrix $DX$ is degenerate at $x_0$. Therefore, rearranging the terms of the trace and using the fact that the divergence of $X$ is zero, we get

$$\operatorname{tr}(DX)^2(x_0) = \sum_{ij} \frac{\partial X^j}{\partial x^i}(x_0)\frac{\partial X^i}{\partial x^j}(x_0) = -2\det\left[DX(x_0)\right] = 0.$$

Substituting into the relation (1.7) we find that

$$\operatorname{div}\nabla_X X(x_0) = K(x_0)\,g(X,X)(x_0)\,,$$

because in two dimensions the Ricci and Gaussian curvatures coincide. However, if $X$ is asymptotic this implies that

$$0 = K(x_0)\, g(X,X)(x_0)$$

contradicting the assumption $K \neq 0$ on $M$, and in particular at the point $x_0$. □

*Remark* 10. Note that the proof above works for a $C^1$ field $X$, thus improving on the $C^2$ assumption used in [Pa].

The following is a similar result for the case of a manifold with boundary:

**Theorem 11.** *Let $M$ be a compact surface of nowhere zero curvature $K$ with smooth boundary. Assume that the geodesic curvature $k_g$ of $\partial M$ vanishes, at most, at finitely many points. Then the Monge-Ampère equation (1.3) along with the boundary condition (1.5) admits no non-constant solutions.*

*Proof.* The beginning of the proof follows [Pa]. Assuming that such an $X$ exists we show that $X$ has to vanish on the boundary. Indeed, for any point $x \in \partial M$ where $X(x) \neq 0$ we have that $X$ is tangent to $\partial M$ and

$$0 = (\nabla_X X, n)(x) = k_g(x) \cdot g(X,X)(x)\,.$$

The latter shows that the geodesic curvature $k_g$ must vanish at $x$. By the assumption, this implies that $X$ can be non-zero only at finitely many points of the boundary. Hence it is identically zero on $\partial M$ by continuity.

The proof of the boundary-free case is now applicable: the function $f := g(X,X)$ is also zero on the boundary $\partial M$ and must therefore attain a maximum in the interior of $M$. □

## 2. Mass transport and fibrations on diffeomorphism groups

2.1. **Monge-Ampère equation in optimal mass transport.** A somewhat different Monge-Ampère equation arises in the theory of mass transport. Namely, let $d\mu(x) = m(x)dV(x)$ and $d\nu(y) = n(y)dV(y)$ be two smooth measures on a manifold $M$ of any dimension. The Jacobian of a map $\eta$ which sends the measure $\mu$ to the other $\nu = \eta_*\mu$ satisfies the relation

$$n(\eta(x))\det[D\eta(x)] = m(x). \tag{2.1}$$

The property of optimality means that the 1-parameter family of maps $\eta_t(x)$ describes a geodesic curve on the space of densities with respect to the Kantorovich (or, Wasserstein) $L^2$-metric. The latter is the 'transport' $L^2$ metric on the space of densities: the distance between two densities on $M$ is the cost of transporting one of them to the other with the $L^2$-cost function.

If $M$ is a domain in $\mathbb{R}^k$ one can see that an optimal map $\eta : M \to M$ has to be the gradient $\eta = \nabla\phi$ of a convex function $\phi$, i.e. the equation on the potential $\phi$ assumes the Monge-Ampère form (see, e.g, [Br]):

$$\det[D^2\phi] = \frac{m(x)}{n(\nabla\phi(x))}. \tag{2.2}$$

In the case of an arbitrary manifold $M$ the optimality condition is expressed in terms of convexity of the function $\phi$ with respect to the metric on $M$, see [Mc]. The corresponding Monge-Ampère equation has a similar form.

2.2. **Universality of the (pre-) Monge-Ampère equation.** Although the two Monge-Ampère equations discussed above look rather different (equation (1.3) is for asymptotic *vector fields*, while equation (2.2) is for potentials of optimal *diffeomorphisms*), both have a common origin. It turns out that they can be viewed as projections of the *dispersionless Burgers equation*:

$$\partial_t X + \nabla_X X = 0\,, \tag{2.3}$$

which can be thought of as a *pre*-Monge-Ampère equation. This equation describes geodesics with respect to the "flat" $L^2$-metric (1.1) on the group of all diffeomorphisms $\mathcal{D}(M)$ for $M$ of any dimension. According to this equation each fluid particle moves along a geodesic in $M$.

Note that there is a natural fibration on $\mathcal{D}(M)$ given by the projection $\pi$ onto densities $\mathcal{P}(M)$. Two diffeomorphisms belong to the same fiber if they move a fixed density (say, the constant density $O$ for a compact $M$) to one and the same density. In particular, for a compact $M$ one considers densities with the same total mass, and $\mathcal{SD}(M) = \pi^{-1}(O)$ is one of the fibres. This projection is a Riemannian submersion onto $\mathcal{P}(M)$ equipped with the Wasserstein metric (see [Ot]).

Consider the "horizontal" geodesics in $\mathcal{D}(M)$, which project to geodesics in $\mathcal{P}(M)$, see Fig.1. This projection means that instead of solutions of the Burgers equation (2.3) we are interested only in the divergence of the corresponding fields, i.e. in how they act on densities. The latter is described by the infinitesimal version of the Monge-Ampère equation (2.2). The flow of this infinitesimal version delivers a solution to the Monge-Ampère equation for the potential of a gradient diffeomorphism, see [Br].

On the other hand, asymptotic directions on $\mathcal{SD}(M)$ correspond to "almost vertical" geodesics in the space $\mathcal{D}(M)$ with respect to the same flat $L^2$ metric. More precisely, an asymptotic *geodesic* (given by the Burgers equation) is "vertical" since it joins diffeomorphisms belonging to the same fiber $\mathcal{SD}(M)$, i.e. it joins different pre-images of the point $O$ on the quotient. Asymptotic *directions* are vertical initial vectors which correspond to geodesics having second order of tangency with the fiber. In other words, the projections of these geodesics to the base leave the initial point $O$ very slowly. Indeed, recalling that taking the projection is the same as taking the gradient part, we identify here both of equations (1.2): along with the Burgers equation they imply that $\mathrm{div} X = \mathrm{div} X_t = 0$. These relations mean that the projection of the corresponding geodesic to the base $\mathcal{P}(M)$ is a curve which starts at the point $O$ with zero velocity and zero acceleration, respectively. (For non-asymptotic directions only the velocity is zero, while the acceleration is not, since typical geodesics have the first order of tangency to $\mathcal{SD}(M)$.)

The above discussion can be summarized in the following statement:

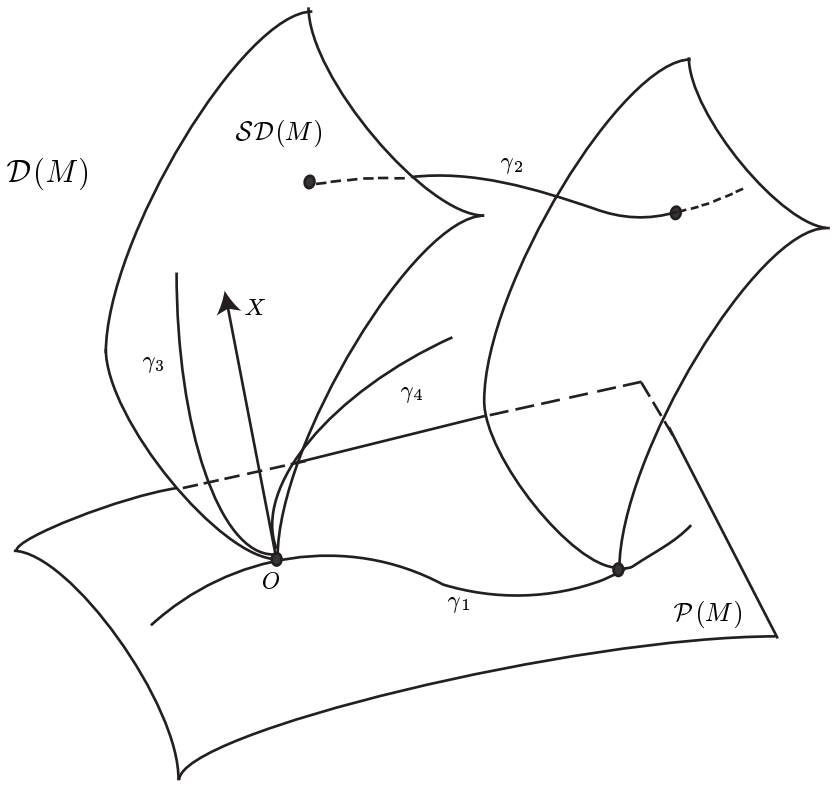


FIGURE 1. $\gamma_1$ is a geodesic in the space of densities $\mathcal{P}(M)$, $\gamma_2$ is a horizontal lift of $\gamma_1$ to $\mathcal{D}(M)$, $\gamma_3$ is an asymptotic (or 'vertical') geodesic in $\mathcal{S}D(M)$, $\gamma_4$ is an 'almost vertical' geodesic for $\mathcal{S}D(M) \subset \mathcal{D}(M)$ with the asymptotic direction $X$.

**Theorem 12.** *The Monge-Ampère equation (1.3) in 2D fluid dynamics and (the infinitesimal form of) the Monge-Ampère equation in mass transport (2.2) are projections of the same dispersionless Burgers equation.*

*Remark* 13. Yet another appearance of the Monge-Ampère equation is related to the discrete version of the 2D Euler equation for a special energy functional, due to Moser and Veselov [MV]. A solution of the discretized Euler equation for an ideal fluid is a recursive sequence of diffeomorphisms. The Monge-Ampère equation provides the constraint on the initial condition ensuring that all diffeomorphisms of the discrete geodesics are area-preserving. This restriction is similar to the Monge-Ampère condition singling out asymptotic directions among 2D divergence-free vector fields.

2.3. **Relations of diffeomorphism groups.** One of the most interesting problems in the context of diffeomorphism groups is the existence of a shortest path or a geodesic between any two diffeomorphisms. This question is interesting already in dimension $n = 1$, for the group $\mathcal{D}(S^1)$ equipped with either $L^2$ or $H^1$ metric. Locally such a geodesic always exists (see [M2]).

In finite-dimensional geometry such questions are answered using the Hopf-Rinow theorem, but in infinite dimensions things become complicated. Grossman and Atkin constructed examples which show that even complete infinite-dimensional Hilbert Riemannian manifolds have points that cannot be joined by any geodesic. For instance, for the groups $\mathcal{SD}(M^n)$ with $n \geq 3$ Shnirelman [Sh] constructed examples of pairs of diffeomorphisms for which a shortest path does not exist.

In the study of this problem the following relation of the groups of diffeomorphisms of one- and two-dimensional manifolds can be useful. Consider the fibration of all diffeomorphisms of a surface $M$ over the space of densities on $M$, which is exactly the context of optimal mass transport:

$$\mathcal{SD}(M) \hookrightarrow \mathcal{D}(M) \to C^\infty(M)\,.$$

Here the left arrow is an isometric embedding with respect to the corresponding $L^2$-metrics discussed above, and the right arrow is a Riemannian submersion into the space of densities $C^\infty(M)$ equipped with the Wasserstein $L^2$-metric.

Now, consider a surface $M^2$ whose boundary is the circle $S^1$. Similarly to the above, we have the following fibration

$$\mathcal{SD}_o(M^2) \hookrightarrow \mathcal{SD}(M^2) \to \mathcal{D}(S^1)$$

where $\mathcal{SD}_o(M^2)$ denotes the subgroup of smooth area-preserving diffeomorphisms of the surface that are pointwise fixed on the boundary $S^1$.

It would be interesting to study the Riemannian properties of this fibration. In this context the shortest path problem in the group $\mathcal{D}(S^1)$ looks like the problem for an optimal path in the group $\mathcal{SD}(M^2)$ or $\mathcal{D}(M^2)$, assuming that the projection to $\mathcal{D}(S^1)$ is a Riemannian submersion. The latter is a problem of optimal transport with a prescribed boundary map: we are connecting in an optimal way two disk diffeomorphisms which are liftings of the given diffeomorphisms of the circle. Here one could hope to employ the following reasoning. While the optimal map exists and is unique between any two convex domains, it automatically determines how the boundary is mapped. Hence one can almost never solve the same problem of finding the optimal map and simultaneously satisfy a particular boundary condition. This would allow one to conclude that the shortest path does not exist for almost all pairs of circle diffeomorphisms.

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## References

[Ar] V. Arnold, *Sur la geometrie differentielle des groupes de Lie de dimension infinie et ses applications a l'hydrodynamique des fluids parfaits*, Ann. Inst. Fourier, Grenoble **16** (1966), 316–361.

[AK] V. Arnold and B. Khesin, *Topological Methods in Hydrodynamics*, Annual Review of Fluid Mech. **24** (1992), 145–166.

[BR] D. Bao and T. Ratiu, *On the geometrical origin and the solutions of a degenerate Monge-Ampère equation*, Proc. Symp. Pure Math. AMS Providence **54** (1993), 55–68.

[Br] Y. Brenier, *Polar factorization and monotone rearrangement of vector-valued functions*, Comm. Pure Appl. Math. **44** (1991), no. 4, 375–417; *Some geometric PDEs related to hydrodynamics and electrodynamics*, Proc. ICM, Vol. III (Beijing, 2002), 761–772.

[EM] D. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. Math. **92** (1970), 102–163.

[Mc] R. McCann, *Polar factorization of maps on Riemannian manifolds*, Geom. Funct. Anal. **11** (2001), no. 3, 589–608.

[M1] G. Misiołek, *Stability of flows of ideal fluids and the geometry of the group of diffeomorphisms*, Indiana Univ. Math. J. **42** (1993), 215–235.

[M2] G. Misiołek, *Classical solutions of the periodic Camassa-Holm equation*, Geom. Funct. Anal. **12** (2002), 1080–1104.

[MV] J. Moser and A. Veselov *Two-dimensional 'discrete hydrodynamics' and Monge-Ampére equations*, Erg. Theory Dynam. Syst. **22** (2002), no. 5, 1575–1583.

[Ot] F. Otto, *The geometry of dissipative evolution equations: the porous medium equation*, Comm. PDE **26** (2001), no. 1-2, 101–174.

[Pa] B. Palmer, *The Bao-Ratiu equations on surfaces*, Proc. Roy. Soc. London Ser. A **449** (1995), no. 1937, 623–627.

[Pr] S. Preston, *Non-positive curvature on the area-preserving diffeomorphism group*, J. Geom. Phys. **53** (2005), no.2, 226–248.

[Sh] A. Shnirelman, *Generalized fluid flows, their approximation and applications*, Geom. Funct. Anal. **4** (1994), no.5, 586–620.

[Ta] M. Taylor, *Partial Differential Equations*, Appl. Math. Sci. v.117, Springer, New York 1996.

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