Polar de Rham Theorem

Boris Khesin,[*] Alexei Rosly[†] and Richard Thomas[‡]

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Abstract

We prove an analogue of the de Rham theorem for polar homology; that the polar homology $HP_q(X)$ of a smooth projective variety X is isomorphic to its $H^{n,n-q}$ Dolbeault cohomology group. This analogue can be regarded as a geometric complexification where arbitrary (sub)manifolds are replaced by complex (sub)manifolds and de Rham's operator d is replaced by Dolbeault's $\bar{\partial}$.

1 Introduction

The idea of polar homology can be explained as follows. In a complex manifold[1] X, consider a $(q+1)$-dimensional submanifold Y and such a meromorphic $(q+1)$-form β on Y that has only first order poles on a smooth q-dimensional submanifold $Z = \mathrm{div}_\infty \beta \subset Y \subset X$. Under these circumstances, the residue of β can be understood as a holomorphic q-form $\alpha = 2\pi i \,\mathrm{res}\, \beta$ on Z (we include a factor of $2\pi i$ for future convenience). In other words, to the pair (Y, β) we can associate another pair $(Z, \alpha) = (\mathrm{div}_\infty \beta, 2\pi i \,\mathrm{res}\, \beta)$ in one dimension less. We are going to extend this correspondence, $(Y, \beta) \mapsto (\mathrm{div}_\infty \beta, 2\pi i \,\mathrm{res}\, \beta)$, to the boundary map ∂ in a certain homological chain complex. Note that if we apply ∂ to the pair (Z, α) above, we get zero because α is holomorphic. This gives rise to the basic identity $\partial^2 = 0$. The formal definition of *the polar chain complex* given in the next section is somewhat lengthier, but its meaning should be already clear. In particular, the pairs (Z, α) correspond to q-cycles if α is a holomorphic q-form on a q-dimensional submanifold $Z \subset X$ and such a cycle is, in fact, a boundary if α is someone's residue.

In the above discussion we considered the situation when only smooth submanifolds occur. In general, the definition of the polar chain complex will have contributions from arbitrary subvarieties $Z \subset X$. Such a definition, which gives us a chain complex with homology groups to be denoted as $HP_q(X)$, was suggested in refs. [KR1, KR2]. In many

[*]Department of Mathematics, University of Toronto, Toronto, ON M5S 3G3, Canada; e-mail: khesin@math.toronto.edu

[†]Institute of Theoretical and Experimental Physics, B.Cheremushkinskaya 25, 117259 Moscow, Russia; e-mail: rosly@heron.itep.ru

[‡]Department of Mathematics, Imperial College, London SW7 2AZ, UK; e-mail: rpwt@ic.ac.uk

[1]All manifolds, varieties, their dimension, etc., are understood in this paper over $\mathbb{C}$.

aspects it is analogous to the definition of topological homology (say, singular homology). In the present paper, we are going to prove a theorem analogous to de Rham's theorem in the topological context. Namely, we shall prove that the groups $HP_q(X)$ for smooth projective X are dual to $H^q(X, \mathcal{O}_X)$, as it was conjectured in ref. [KR1]. In other words, we shall see that the Dolbeault $\bar{\partial}$-complex on $(0,q)$-forms interacts with the polar chain complex in the same way as the de Rham d-complex does with ordinary topological chains. The reader interested only in reading the main results should, after having a look at the definition 2.9, proceed directly to Theorem 3.1 and its proof in 3.13. The rest of the paper consists of technical preliminaries needed to deal with singularities.

One should note that there exists a more general polar complex, where the chains are complex subvarieties of dimension q with logarithmic p-forms on them. The corresponding polar homology groups, enumerated by two indices, are, in general, not isomorphic to any Dolbeault homology as simple examples show. From this point of view, the isomorphism for $p = q$ discussed in this paper is rather an exception than a rule.

2 Definitions

2.1 Poincaré residue. Let X be a smooth complex projective n-dimensional manifold and $V \subset X$ a smooth hypersurface in X. Consider a meromorphic n-form ω on X with first order poles on V. If $\{z = 0\}$ is a local equation for V, the form ω can be written as

$$\omega = \frac{dz}{z} \wedge \rho + \gamma \,,$$

where the locally defined holomorphic forms ρ and γ can be chosen in various ways. However, the restriction of ρ to V is defined uniquely and, therefore, becomes a global holomorphic $(n-1)$-form on V. It is denoted by $\operatorname{res}\omega = \rho|_V$ and is called the *Poincaré residue* of ω. This can be also described by the following exact sequence of sheaves:

$$0 \to K_X \to K_X(V) \to K_V \to 0 \,, \tag{1}$$

where K_X is the canonical sheaf on X, i.e., the sheaf of holomorphic n-forms, while $K_X(V)$ stands for n-forms with first order poles on V whose residues give us regular $(n-1)$-forms on V. The restriction map $K_X(V) \to K_V$ represents here the Poincaré residue for locally defined n-forms. The corresponding residue map for the globally defined forms, $res : H^0(X, K_X(V)) \to H^0(V, K_V)$, shows up in the cohomological long exact sequence implied by (1):

$$\begin{array}{ll} 0 \to & H^0(X, K_X) \to H^0(X, K_X(V)) \xrightarrow{res} H^0(V, K_V) \to \\ & H^1(X, K_X) \to H^1(X, K_X(V)) \longrightarrow \dots \end{array} \tag{2}$$

In this sequence we encounter elements of polar homology. Namely, the meromorphic n-forms $\omega \in H^0(X, K_X(V))$ will correspond (via the definitions in 2.9 below) to n-chains, the holomorphic $(n-1)$-forms $\rho \in H^0(V, K_V)$ will correspond to $(n-1)$-cycles, while the boundary map will be given by the map res in (2). We shall see that the contribution to the $(n-1)$-dimensional polar homology coming from a given (smooth) hypersurface V will correspond to the quotient $H^0(V, K_V)/res(H^0(X, K_X(V)))$. It remains to understand the contributions from arbitrary subvarieties in X.

2.2 Normal crossings. Since we are going to use the map $res : H^0(X, K_X(V)) \to H^0(V, K_V)$ in the definition of a boundary map on a vector space of chains we cannot restrict to the case of only smooth divisors of poles. As a matter of fact, it is sufficient to generalize to the case of normal crossings. We shall consider normal crossing divisors, as well as subvarieties with normal crossings in arbitrary codimension. We shall give a very restrictive definition of these which will suffice for our purposes. Let us explain our conventions in more detail. First of all, a (sub)variety will be always reduced, but not necessary irreducible. Thus, a subvariety in[2] X is just a Zariski closed subset of X. On the other hand, a smooth variety (= smooth manifold = manifold) will be always assumed irreducible (which is equivalent to connected for smooth varieties).

Let us consider a smooth n-dimensional manifold X. A hypersurface $V \subset X$ will be called a *normal crossing divisor* if V consists of smooth components that meet transversely, in the sense that $V = \cup_i V_i$, where each V_i is smooth and intersects transversely V_j, $V_j \cap V_k$, and so on, for all i, j, k,...[3]. In order to introduce the notion of a normal crossing subvariety of an arbitrary codimension, consider first a codimension two subvariety $W \subset V \subset X$ (where X and V are as above). Let us require that the part of W which resides in a smooth component of V is a normal crossing divisor there and that W intersects the normal crossing singularities of V transversely. More precisely, if $W \not\supseteq V_i \cap V_j, \forall i, j$, and $(W \cap V_i) \cup (V_i \cap (\cup_{k \neq i} V_k))$ is a normal crossing divisor in the smooth manifold V_i for all i, we shall say that W is a normal crossing divisor in V and a normal crossing subvariety in X. In such a way we obtain the notion of a normal crossing divisor in a variety, which is itself a normal crossing divisor in a bigger variety. Proceeding deeper in codimension we shall say that a subvariety Y of codimension m in X is a normal crossing subvariety if there exists a nested sequence

$$Y = V^m \subset V^{m-1} \subset \ldots \subset V^1 \subset V^0 = X\,, \tag{3}$$

such that V^{i+1} is a normal crossing divisor in V^i. We shall also say that two normal crossing divisors V and V' intersect transversely if $V + V'$ is a normal crossing divisor again. (This means in particular that V and V' have no common components and that $V \cap V'$ is a normal crossing divisor both in V and in V'.)

In fact, we shall need mainly the notion of an *ample* subvariety with normal crossings in a projective manifold X.

[2]In this paper the varieties are always projective or quasi-projective; the subvarieties are always closed.

[3]Near each point $x \in V$, one can choose local coordinates $z_1, \ldots, z_n$ in X in such a way that $z_1 \cdot \ldots \cdot z_p = 0$ is a local equation of V (where $p \leqslant n$ is the number of components of V passing through x). The latter local formulation could be used as a definition of a normal crossing divisor. We prefer, however, a stronger version, when the self-intersections of components are excluded.

Definition 2.3 *A normal crossing subvariety $Y \subset X$ in a projective manifold X is called ample if one can choose a flag (3) in such a way that V^{i+1} is an ample normal crossing divisor in V^i.*

2.4 Canonical line bundle. The canonical sheaf K_V is defined for a smooth variety V as the sheaf of holomorphic forms of the top degree on V and, if V is a hypersurface in some X, $i\colon V \hookrightarrow X$, the local properties are described by the sequence (1). In this case, one *has to show* that $i^*K_X(V) \simeq K_V$, while the Poincaré residue gives us a canonical choice of this isomorphism. In the case of a normal crossing divisor $i\colon V \hookrightarrow X$ we may take the sequence (1) as the *definition* of K_V. In other words, K_V is defined as $i^*K_X(V)$. By induction in codimension we obtain a definition that can be applied to any normal crossing subvariety Y; the result is a line bundle on Y which does not depend on the choice of the flag (3): invariantly, $K_Y = \mathcal{E}xt^m(\mathcal{O}_Y, K_X)$, where $m = \operatorname{codim} Y$. With such a definition, the global sections of K_V are regarded as "holomorphic" forms on V and the Poincaré residue, $res : H^0(X, K_X(V)) \to H^0(V, K_V)$, still maps meromorphic forms to holomorphic ones. This is precisely what we need to define a chain complex.

As a last preparation, it remains to check only the properties of the repeated residue map, as it has to support the identity $\partial^2 = 0$. Let V be a normal crossing divisor and suppose for simplicity that it consists of only two components, $V = V_1 \cup V_2$, so that V_1, V_2 are smooth and intersect transversely over a smooth variety $V_{12} = V_1 \cap V_2$. Then, a section $\alpha \in K_V$ can be described via its restrictions $\alpha_i = \alpha|_{V_i}$. Since $K_V|_{V_i} \simeq K_X(V_1 + V_2)|_{V_i} \simeq K_X(V_i)|_{V_i}(V_1 \cap V_2) \simeq K_{V_i}(V_{12})$, the α_i are in fact meromorphic forms, $\alpha_i \in H^0(K_{V_i}(V_{12}))$. Moreover, it follows from a local coordinate calculation with the definition that $\mathrm{res}_{V_{12}}\alpha_1 + \mathrm{res}_{V_{12}}\alpha_2 = 0$, which is summarized in the short exact sequence of sheaves

$$0 \to K_V \to K_{V_1}(V_{12}) \oplus K_{V_2}(V_{12}) \to K_{V_{12}} \to 0\,,$$

where the third arrow is taking the sum of residues. In other words, a holomorphic form $\alpha \in H^0(V, K_V)$ on a normal crossing variety V can be described as a collection of meromorphic forms α_i on V_i satisfying the pairwise cancellation of their residues at the intersections. (We shall say that the polar cycle (V, α) is the sum of two polar chains (V_1, α_1) and (V_2, α_2), whose boundaries cancel each other.)

2.5 Resolution of singularities. In the next section, our main tool will be the Hironaka theorem on resolution of singularities [H]. This theorem asserts that every algebraic variety Z admits a desingularization, that is there exists a smooth variety $\tilde{Z}$ and a regular projective birational morphism $\pi : \tilde{Z} \to Z$, which is biregular over $Z - Z_{\mathrm{sing}}$. Moreover, π can be obtained as a sequence of blowing up with smooth centers. If D is a subvariety in Z we can additionally require that $\pi^{-1}(D)$ is a normal crossing divisor in $\tilde{Z}$.

We shall also need the following important result, the (weak) factorization theorem for birational morphisms, proved recently by Abramovich, Karu, Matsuki and Włodarczyk [W, AKMW]. Below we cite only a part of their statement from ref. [AKMW] relevant to our needs (the complete proposition is much stronger).

Proposition 2.6 *Let $\phi : X \dashrightarrow X'$ be a birational map between smooth projective varieties X and X'. Then ϕ can be factored into a sequence of blowings up and blowings down with smooth irreducible centers, namely, there exists a sequence of birational maps between smooth projective varieties*

$$X = \tilde{X}_0 \overset{\varphi_1}{\dashrightarrow} \tilde{X}_1 \overset{\varphi_2}{\dashrightarrow} \cdots \overset{\varphi_i}{\dashrightarrow} \tilde{X}_i \overset{\varphi}{\dashrightarrow} \tilde{X}_{i+1} \overset{\varphi_{i+2}}{\dashrightarrow} \cdots \overset{\varphi_{l-1}}{\dashrightarrow} \tilde{X}_{l-1} \overset{\varphi_l}{\dashrightarrow} \tilde{X}_l = X'$$

where

1. $\phi = \varphi_l \circ \varphi_{l-1} \circ \cdots \circ \varphi_2 \circ \varphi_1$, *and*
2. *either* $\varphi_i : \tilde{X}_{i-1} \dashrightarrow \tilde{X}_i$, *or* $\varphi_i^{-1} : \tilde{X}_i \dashrightarrow \tilde{X}_{i-1}$ *is a morphism obtained by blowing up a smooth irreducible center.*

For the sake of brevity in what follows, under a 'blow-up' we shall understand 'a sequence of blowings up with smooth centers'. The following corollary of the Hironaka and Bertini theorems will also be useful in the sequel.

Proposition 2.7 *Let $Z \subset X$ be an arbitrary irreducible subvariety of codimension m in a smooth projective manifold X. Then, there exists such a blow-up $\pi : \tilde{X} \to X$ and such a flag of subvarieties*

$$\tilde{Z} \subset V^{m-1} \subset V^{m-2} \subset \ldots \subset V^1 \subset V^0 = \tilde{X} \tag{4}$$

that V^{i+1} is a smooth hypersurface in V^i and $\tilde{Z}$ is smooth and mapped birationally by π onto Z.

Proof. Firstly, by Hironaka, we can blow up X in such a way that the proper preimage of Z becomes smooth. If the codimension of Z is one, $m = 1$, the proposition is proved. We can thus proceed for $m > 1$ and assume that Z is already smooth. In this case, let us take a very ample divisor class H in X and consider hypersurfaces in this class containing Z. Such hypersurfaces are described as zero sets of global sections of the sheaf $\mathcal{I}_Z(H)$, where $\mathcal{I}_Z \subset \mathcal{O}_X$ is the ideal sheaf of the subvariety Z in X. By Bertini, the generic section $s \in H^0(X, \mathcal{I}_Z(H))$ defines a hypersurface $V = \{s = 0\} \subset X$ which is regular outside Z. As to the points of V which lie on Z, the singularities correspond to the zeros of the section $\bar{s} = ds \in H^0(Z, \mathcal{I}_Z/\mathcal{I}_Z^2(H))$ induced by s. Let us choose $H \gg 0$ in such a way that $H^0(Z, \mathcal{I}_Z/\mathcal{I}_Z^2(H)) \neq 0$, while $H^1(Z, \mathcal{I}_Z/\mathcal{I}_Z^2(H)) = 0$. Then we have a non-trivial section $\bar{s}$ in $H^0(Z, \mathcal{I}_Z/\mathcal{I}_Z^2(H))$ whose zeros form a proper closed subset $Z_0 \subsetneq Z$. Moreover, $H^1(Z, \mathcal{I}_Z/\mathcal{I}_Z^2(H)) = 0$ guarantees that the mapping $H^0(X, \mathcal{I}_Z(H)) \to H^0(Z, \mathcal{I}_Z/\mathcal{I}_Z^2(H))$, $s \mapsto \bar{s}$, is surjective. Hence, taking a generic s we can ensure that the resulting hypersurface $V = \{s = 0\}$ is regular outside $Z_0 = \{\bar{s} = 0\} \subsetneq Z$. Applying the Hironaka theorem, we can now resolve the singularities of V by blowing up X in centers belonging to $Z_0 \subset X$. Then, for the proper preimage $\tilde{Z}$ of Z, we have that $\tilde{Z} \subset V^1 \subset \tilde{X}$, where $\tilde{Z}$ and V^1 are smooth. We can then proceed in the same manner inside V^1 until the whole flag (4) obeying the required conditions is constructed. ■

2.8 Polar chains. The space of polar q-chains for a (not necessarily smooth) complex projective variety X, $\dim X = n$, will be defined as a $\mathbb{C}$-vector space with certain generators and relations.

Definition 2.9 *The space of polar q-chains $\mathcal{C}_q(X)$ is a vector space over $\mathbb{C}$ defined as the quotient $\mathcal{C}_q(X) = \hat{\mathcal{C}}_q(X)/\mathcal{R}_q$, where the vector space $\hat{\mathcal{C}}_q(X)$ is freely generated by the triples (A, f, α) described in (i),(ii),(iii) and $\mathcal{R}_q$ is defined as relations* (R1),(R2),(R3) *imposed on the triples.*

(i) A is a smooth complex projective variety, $\dim A = q$*;*

(ii) $f\colon A \to X$ is a holomorphic map of projective varieties;

(iii) α is a meromorphic q-form on A with first order poles on $V \subset A$, i.e., $\alpha \in H^0(A, K_A(V))$, where V is a normal crossing divisor in A.

The relations are generated by:

(R1) $\lambda(A, f, \alpha) = (A, f, \lambda\alpha)$,

(R2) $\sum_k (A_k, f_k, \alpha_k) = 0$ *provided that* $\sum_k f_{k*}\alpha_k \equiv 0$ *on* $\hat{A} - \hat{A}_{\mathrm{sing}}$,[4] *where* $f_k(A_k) = f_l(A_l) =: \hat{A},\ \forall\, k, l$ *and* $\dim \hat{A} = \dim f_k(A_k) = q,\ \forall k$*;*

(R3) $(A, f, \alpha) = 0$ *if* $\dim f(A) < q$.

Definition 2.10 *The boundary operator $\partial : \mathcal{C}_q(X) \to \mathcal{C}_{q-1}(X)$ is defined by*

$$\partial(A, f, \alpha) = 2\pi i \sum_k (V_k, f_k, \mathrm{res}_{V_k}\, \alpha)\,,$$

where V_k are the components of the polar divisor of α, $\mathrm{div}_\infty \alpha = \cup_k V_k$*, and the maps $f_k = f|_{V_k}$ are restrictions of the map f to each component of the divisor.*

Proposition 2.11 *The boundary operator ∂ is well defined, i.e. it is compatible with the relations* (R1),(R2),(R3).

For the proof see [KR1]. Now, by using the cancellation of repeated residues for forms α with normal crossing divisors of poles, one proves the following [KR1]:

Proposition 2.12 $\quad \partial^2 = 0$.

This allows one to define a homology theory.

[4] For a surjective holomorphic map $f : U \to V$ of two smooth complex manifolds of the same dimensions (that is to say, f is generically finite), we have a push-forward map f_* defined on differential forms by the summation over the preimages $P \in f^{-1}(Q)$ of a point $Q \in V$. This map is also called the trace map. See ref. [G, GH].

Definition 2.13 *For a complex projective variety* $X, \dim X = n$*, the chain complex*

$$0 \to \mathcal{C}_n(X) \xrightarrow{\partial} \mathcal{C}_{n-1}(X) \xrightarrow{\partial} \ldots \xrightarrow{\partial} \mathcal{C}_0(X) \to 0$$

is called the polar chain complex of X*. Its homology groups,* $HP_q(X), q = 0, \ldots, n,$ *are called the polar homology groups of* X*.*

2.14 Remark. It is useful to introduce the notion of the support of a q-chain $a \in \mathcal{C}_q(X)$. This is defined as the following minimal subvariety $\operatorname{supp} a = \bigcap \cup_k f_k(A_k) \subset X$ where the intersection $\bigcap$ is taken over all representatives $\sum_k (A_k, f_k, \alpha_k)$ in the equivalence class a. (In other words, $\operatorname{supp} a$ can be determined by taking $Z = \cup_k f_k(A_k)$ for an arbitrary representative $\sum_k (A_k, f_k, \alpha_k)$, removing those components of Z which are of dimension less than q or where the push-forwards $f_{k*}\alpha_k$ sum to zero as in (R2) in the Definition 2.9 above and taking closure.) This notion of the support of a polar chain coincides with the support of the current in X corresponding to that chain. (The relation with currents was discussed in ref. [KR1].)

If $a \in \mathcal{C}_q(X)$ then $Z = \operatorname{supp} a$ is either of pure dimension q, or empty. The smooth part of Z is provided with a meromorphic q-form α obtained by summation of $f_{k*}\alpha_k$. The meaning of the relation (R2) above is essentially that these data, $(\operatorname{supp} a, \alpha)$, define the equivalence class of (sums of) triples $a \in \mathcal{C}_q(X)$ in a unique way. By the Hironaka theorem, the subvariety Z can in fact be arbitrary, that is for an arbitrary q-dimensional $Z \subset X$, there exists a q-chain a such that $Z = \operatorname{supp} a$, but the meromorphic q-form α on $Z - Z_{\mathrm{sing}}$ cannot in general be arbitrary.

2.15 Relative polar homology. Let Z be a closed subvariety in a projective X. Analogously to the topological relative homology we can define the polar relative homology of the pair $Z \subset X$.

Definition 2.16 *The relative polar homology groups* $HP_q(X, Z)$ *are the homology groups of the following quotient complex of chains:*

$$\mathcal{C}_q(X, Z) = \mathcal{C}_q(X)/\mathcal{C}_q(Z).$$

Here we use the natural embedding of the chain groups $\mathcal{C}_q(Z) \hookrightarrow \mathcal{C}_q(X)$. This leads to the long exact sequence in polar homology:

$$\ldots \to HP_q(Z) \to HP_q(X) \to HP_q(X, Z) \xrightarrow{\partial} HP_{q-1}(Z) \to \ldots \tag{5}$$

2.17 The functorial properties of polar homology are straightforward. A regular morphism of projective varieties $h : X \to Y$ defines a homomorphism $h_* : HP_\bullet(X) \to HP_\bullet(Y)$. Analogously, for the relative polar homology we have $h_* : HP_\bullet(X, V) \to HP_\bullet(Y, W)$ if $V \subset X$, $W \subset Y$ are closed subsets and $h(V) \subset W$.

2.18 REMARK. In the case of a morphism of two pairs $h\colon (X,V) \to (X',V')$ as above, the induced homomorphisms h_* give us the homomorphism of the associated long exact sequences:

$$\begin{array}{ccccccccc} \ldots \to & HP_q(V) & \to & HP_q(X) & \to & HP_q(X,V) & \to & HP_{q-1}(V) & \to \ldots \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ \ldots \to & HP_q(V') & \to & HP_q(X') & \to & HP_q(X',V') & \to & HP_{q-1}(V') & \to \ldots \end{array} \tag{6}$$

We note that if any two of the three homomorphisms $HP_\bullet(V) \to HP_\bullet(V')$, $HP_\bullet(X) \to HP_\bullet(X')$, $HP_\bullet(X,V) \to HP_\bullet(X',V')$ are isomorphisms then the third one is an isomorphism as well.

3 Polar Homology and Dolbeault cohomology

We are going to show that the Dolbeault, or $\bar\partial$, cohomology on $(0,q)$-forms, $H^{(0,q)}_{\bar\partial}(X)$, plays the same role with respect to polar homology $HP_q(X)$ as does the de Rham cohomology in the topological context. First of all, there is an obvious pairing between $HP_q(X)$ and $H^{(0,q)}_{\bar\partial}(X)$. For $[(A,f,\alpha)] \in HP_q(X)$ and $[\omega] \in H^{(0,q)}_{\bar\partial}(X)$, we can write $\int_A \alpha \wedge f^*\omega$ and show that such a pairing descends to (co)homology classes. Recalling the isomorphism $H^{(0,q)}_{\bar\partial}(X) \simeq H^q(X,\mathcal{O}_X)$ and by the Serre duality, $H^q(X,\mathcal{O}_X)^* \simeq H^{n-q}(X,K_X)$, the above pairing is thus represented by the map

$$\rho : HP_q(X) \to H^{n-q}(X,K_X)\,, \tag{7}$$

where $n = \dim X$.

Theorem 3.1 (Polar de Rham theorem) *For a smooth projective n-dimensional X, the map ρ is an isomorphism for any q:*

$$HP_q(X) \simeq H^{n-q}(X,K_X)\,.$$

In the case of polar homology of X relative to a hypersurface $V \subset X$ we analogously have the pairing of $HP_q(X,V)$ and $H^q(X,\mathcal{O}_X(-V))$, or, by Serre's duality, the homomorphism

$$\rho : HP_q(X,V) \to H^{n-q}(X,K_X(V))\,, \tag{8}$$

and the corresponding relative version of the Theorem 3.1 is as follows.

Theorem 3.2 *Let V be a normal crossing divisor in a smooth projective X. Then*

$$HP_q(X,V) \simeq H^{n-q}(X,K_X(V))\,.$$

This more general assertion follows in fact from the Theorem 3.1 by comparing the long exact sequence in sheaf cohomology (2) with that in relative polar homology, cf., (5).

3.3 Remark. It follows from Theorem 3.1 that if two smooth projective manifolds X and X' are birationally equivalent, then $HP_q(X) = HP_q(X')$ since we have in this case that $H^{n-q}(X, K_X) = H^{n-q}(X', K_{X'})$. However, we in fact prove this and other similar results first without reference to sheaf cohomology, on the way to the proof of Theorem 3.1.

Lemma 3.4 *If two projective varieties X and X' are birationally equivalent and we have an isomorphism*

$$g : X - Z \xrightarrow{\sim} X' - Z' ,$$

where Z (resp. Z') is a Zariski closed subset in X (resp. in X'), then

$$HP_\bullet(X, Z) \simeq HP_\bullet(X', Z') .$$

Proof. We want to construct an isomorphism of complexes

$$g_\bullet : \mathcal{C}_\bullet(X, Z) \xrightarrow{\sim} \mathcal{C}_\bullet(X', Z'). \tag{9}$$

Let us take an arbitrary non-zero simple[5] chain $a \in \mathcal{C}_q(X, Z)$ and let the triple (A, f, α) be a representative of the equivalence class a. Since $a \neq 0$, the image $\hat{A} = f(A)$ of A in X has $\dim \hat{A} = q$ and $\hat{A} \not\subseteq Z$. Let us define $\hat{A}'$ as the closure of $g(\hat{A} - Z)$ in X'. By the Hironaka theorem (take the closure of the graph of $g|_{A-Z}$ in $A \times \hat{A}'$ and resolve), there exists a smooth q-dimensional variety A' with regular maps $f' : A' \to X'$ and $\pi : A' \to A$, where π is a birational map of A' onto A, such that they form together with f and g (on open dense subsets) a commutative square, namely:

$$\begin{tikzcd}
A - f^{-1}(Z) \arrow[r, "f"] & \hat{A} - Z \arrow[r, hook] \arrow[d, "\wr"] & X - Z \arrow[d, "g"', "\wr"] \\
A' - f'^{-1}(Z) \arrow[u, "\pi"] \arrow[r, "f'"] & \hat{A}' - Z' \arrow[r, hook] & X' - Z'
\end{tikzcd} \tag{10}$$

Moreover the triple (A', f', α'), which naturally arises here with $\alpha' := \pi^*\alpha$, is admissible: $\mathrm{div}_\infty \alpha'$ is a normal crossing divisor[6] in A' and the triple (A', f', α') defines a chain $a' \in \mathcal{C}_q(X', Z')$. We define the map g_q by setting $a' = g_q(a)$.

Note that the q-forms $f_*\alpha$ and $f'_*\alpha'$, which are defined on open dense subsets in $\hat{A}$ and $\hat{A}'$ respectively, coincide there (in the sense of the isomorphism $g : \hat{A} - Z \xrightarrow{\sim} \hat{A}' - Z'$) as follows from the commutative diagram (10). This observation shows us that $g_q : a \mapsto a'$ is well defined, because, in general, polar chains are uniquely defined in terms of the forms $f_*\alpha$ on the dense subsets in their supports (cf. Remark 2.14). It is obvious that the same construction applied to $g^{-1} : X' - Z' \xrightarrow{\sim} X - Z$ gives the inverse of $g_\bullet$. Compatibility with the boundary map ∂ is also obvious. Thus we have indeed constructed an isomorphism of complexes (9), which proves the lemma. ∎

[5]We call a chain simple if it is equivalent to a single triple rather than a sum of triples.

[6]We say '$\mathrm{div}_\infty \alpha$ is a normal crossing divisor' assuming that 'normal crossing divisor' implies 'first order poles' (i.e. $\mathrm{div}_\infty \alpha$ has no multiple components).

Lemma 3.5 *Let M be any projective variety, then*

$$HP_{\bullet}(M \times \mathbb{C}P^1) \simeq HP_{\bullet}(M),$$

where the isomorphism is induced by the projection $\pi : M \times \mathbb{C}P^1 \to M$.

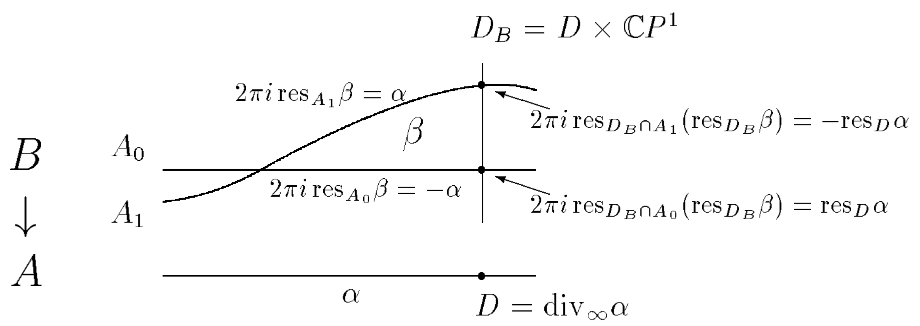
Proof. Choosing a point $0 \in \mathbb{C}P^1$, we will show that any cycle in $M \times \mathbb{C}P^1$ is homologous to one in the zero section $s = (\mathrm{id}, 0) : M \to M \times \mathbb{C}P^1$ by constructing a homotopy $h : \mathcal{C}_q(M \times \mathbb{C}P^1) \to \mathcal{C}_{q+1}(M \times \mathbb{C}P^1)$ from $s_* \circ \pi_*$ to the identity; that is

$$\partial \circ h + h \circ \partial = \mathrm{id} - s_* \circ \pi_* . \tag{11}$$

Let $a = (A, f, \alpha) \in \mathcal{C}_q(M \times \mathbb{C}P^1)$ be a simple chain; that is $\dim A = q$, α is a q-form on A whose poles form a normal crossing divisor in A, and $f = (f_M, g)$ with $f_M := \pi \circ f : A \to M$ a regular map and $g : A \to \mathbb{C}P^1$ a rational function on A. We would like to define the $(q+1)$-chain $h(a)$ by

$$h(a) = (A \times \mathbb{C}P^1, f_M \times \mathrm{id}_{\mathbb{C}P^1}, \beta), \qquad \text{where} \quad \beta = \frac{1}{2\pi i} \frac{g\, dz}{z(z-g)} \wedge \alpha .$$

Here z is an inhomogeneous coordinate on $\mathbb{C}P^1$ vanishing at $0 \in \mathbb{C}P^1$, and z, g and α are pulled back to the product $A \times \mathbb{C}P^1$. β has simple poles on the hypersurface $\mathrm{div}_\infty \beta = A_1 \cup A_0 \cup (\mathrm{div}_\infty \alpha \times \mathbb{C}P^1)$, where $A_1 = \{z = g\}$ and $A_0 = \{z = 0\}$ are two sections, so that, in particular, $A_1 \simeq A_0 \simeq A$.



The corresponding residues are as follows:

$$\begin{array}{rcl} 2\pi i\, \mathrm{res}_{A_1} \beta & = & \alpha\, , \\ 2\pi i\, \mathrm{res}_{A_0} \beta & = & -\alpha\, , \\ 2\pi i\, \mathrm{res}_{\mathrm{div}_\infty \alpha \times \mathbb{C}P^1} \beta & = & -\frac{g\, dz}{z(z-g)} \wedge \mathrm{res}\, \alpha\, . \end{array} \tag{12}$$

The only problem is that $\mathrm{div}_\infty \beta$ will not be a normal crossing divisor if A_0 does not meet A_1 or $A_1 \cap (\mathrm{div}_\infty \alpha \times \mathbb{C}P^1)$ transversely.

By changing z to z' (and so moving $0 \in \mathbb{C}P^1$) we can ensure that the new A_0' *does* meet A_1 and $A_1 \cap (\mathrm{div}_\infty \alpha \times \mathbb{C}P^1)$ transversely, and the resulting β' has normal crossing poles, but now the definition of $h'(a)$ appears to depend on the choice of A_0'. The solution is to take this new β' and add to it $\beta - \beta'$, which also has normal crossing poles (along $A_0 \cup A'_0 \cup (\mathrm{div}_\infty \alpha \times \mathbb{C}P^1)$). Thus $h(a) = \beta = \beta' + (\beta - \beta')$ is admissible in the sense of Definition 2.9, and h is well defined and linear.

From (12) we can now calculate:

$$\begin{aligned}
\partial h(a) &= (A_1\,,(f_M \times \mathrm{id}_{\mathbb{C}P^1})|_{A_1}\,,\alpha) \;-\; (A_0\,,(f_M \times \mathrm{id}_{\mathbb{C}P^1})|_{A_0}\,,\alpha) \\
&\qquad - (\mathrm{div}_\infty\,\alpha \times \mathbb{C}P^1, f_M|_{\mathrm{div}_\infty\,\alpha} \times \mathrm{id}_{\mathbb{C}P^1}\,, \tfrac{g\,dz}{z(z-g)} \wedge \mathrm{res}\,\alpha) \\
&= (A\,,f\,,\alpha) \;-\; (A\,, s\circ\pi\circ f\,,\alpha) \;-\; h(\mathrm{div}_\infty\,\alpha, f|_{\mathrm{div}_\infty\,\alpha}\,, 2\pi i\,\mathrm{res}\,\alpha) \\
&= a \;-\; s_*\pi_*(a) \;-\; h\partial(a)\,,
\end{aligned} \tag{13}$$

as in (11). ■

Lemma 3.6
(a) Let M be a smooth projective variety and E be the total space of a projective bundle over M, i.e. $\pi : E \to M$ is a locally trivial fibration with a projective space as a fiber. Then π induces an isomorphism in polar homology:

$$HP_\bullet(E) \simeq HP_\bullet(M)\,.$$

(b) The result (a) holds also for any projective M, that is without the assumption of smoothness.
(c) Let X and $\tilde{X}$ be two smooth projective manifolds and $\pi : \tilde{X} \to X$ be a sequence of blow-ups with smooth centers. Then

$$HP_\bullet(X) \simeq HP_\bullet(\tilde{X})\,.$$

(d) Let $X, \tilde{X}, \pi$ be the same as in (c) and let $Z \subset X$ be an arbitrary closed subset. Then

$$\begin{aligned}
HP_\bullet(Z) &\simeq HP_\bullet(\pi^{-1}(Z))\,,\\
HP_\bullet(X,Z) &\simeq HP_\bullet(\tilde{X},\pi^{-1}(Z))\,.
\end{aligned}$$

Proof. We shall prove the propositions (a)–(d) by a simultaneous induction in dimension. For $\dim E = 0$ and $\dim X = 1$ everything is obvious. Suppose that (a)–(d) are proved when $\dim X < n$ and $\dim E < n-1$. Let us prove these four propositions when $\dim E = n-1$ and $\dim X = \dim \tilde{X} = n$.

Consider a locally trivial fibration $\pi : E \to M$ where the fibers are all isomorphic to the projective space $\mathbb{C}P^k$ for some $k \leqslant n-1$. Since $\mathbb{C}P^k$ is birational to $(\mathbb{C}P^1)^{\times k}$ and by local triviality of π we conclude that E is birational to the direct product $E' := M \times (\mathbb{C}P^1)^{\times k}$. If M is smooth as in part (a) of our statement, both E and E' are smooth and the AKMW theorem (see Proposition 2.6) tells us that E and E' can be related by a sequence of blow-ups and blow-downs. But for $\dim E = \dim E' = n-1$, part (c) of the statement is applicable by our induction hypothesis and we conclude that $HP_\bullet(E) = HP_\bullet(E')$. Finally, $HP_\bullet(E') = HP_\bullet(M)$ according to Lemma 3.5. Thus, the induction step is proved in part (a).

Let us now consider the fibration $\pi : E \to M$, $\dim E = n-1$, for an arbitrary projective variety M as in part (b). If M is indeed singular (perhaps even with intersecting components) we denote its singular locus as $M_{\rm sing}$. By the Hironaka theorem there exists a desingularization $\sigma : \tilde{M} \to M$, where $\tilde{M}$ consists of smooth non-intersecting components and such that $M - M_{\rm sing} \simeq \tilde{M} - F$, where $F := \sigma^{-1}(M_{\rm sing})$. Let now $\tilde{\pi} : \tilde{E} \to \tilde{M}$ be the pull-back of π along σ. In this smooth situation, we have by proposition (a) that $HP_\bullet(\tilde{E}) = HP_\bullet(\tilde{M})$. Let us also consider the fibration $\tilde{\pi}^{-1}(F) \to F$ (the restriction of $\tilde{\pi}$). Although its base F may be singular, its dimension ($\dim \tilde{\pi}^{-1}(F) < \dim E = n-1$) allows us to use the induction hypothesis in part (b) to conclude that $HP_\bullet(\tilde{\pi}^{-1}(F)) = HP_\bullet(F)$. We want now to compare the polar homology of the pair $\tilde{M} \supset F$ to that of $\tilde{E} \supset \tilde{\pi}^{-1}(F)$. The isomorphisms $\pi_* : HP_\bullet(\tilde{E}) \simeq HP_\bullet(\tilde{M})$ and $\pi_* : HP_\bullet(\tilde{\pi}^{-1}(F)) \simeq HP_\bullet(F)$ imply (as in Remark 2.18) that

$$HP_\bullet(\tilde{E}, \tilde{\pi}^{-1}(F)) \simeq HP_\bullet(\tilde{M}, F)\,.$$

The varieties appearing in both sides of this equality have their birational counterparts:

$$\begin{array}{rcl} \tilde{M} - F & \simeq & M - M_{\rm sing}\,, \\ \tilde{E} - \tilde{\pi}^{-1}(F) & \simeq & E - \pi^{-1}(M_{\rm sing})\,. \end{array}$$

Hence, we can use Lemma 3.4 to conclude that

$$HP_\bullet(E, \pi^{-1}(M_{\rm sing})) \simeq HP_\bullet(M, M_{\rm sing}) \tag{14}$$

Since $\dim \pi^{-1}(M_{\rm sing}) < \dim E = n-1$, we can apply the induction hypothesis in part (b) to the fibration $\pi^{-1}(M_{\rm sing}) \to M_{\rm sing}$ and get the isomorphism

$$HP_\bullet(\pi^{-1}(M_{\rm sing})) \simeq HP_\bullet(M_{\rm sing}) \tag{15}$$

Finally, the isomorphisms (14) and (15) and the map of pairs $\pi : (E, \pi^{-1}(M_{\rm sing})) \to (M, M_{\rm sing})$ give the third isomorphism $HP_\bullet(E) = HP_\bullet(M)$ as in Remark 2.18, proving the induction step in part (b).

Now, we turn to part (c) with two smooth projective varieties X and $\tilde{X}$, where $\dim X = \dim \tilde{X} = n$. It is sufficient to consider the case when $\pi : \tilde{X} \to X$ is a single blow up with smooth center $M \subset X$. Let us denote by $E = \pi^{-1}(M) \subset \tilde{X}$ the exceptional divisor. Applying the proposition (a) to the fibration $\pi : E \to M$, we find that $HP_\bullet(E) = HP_\bullet(M)$, while, by Lemma 3.4, we find that $HP_\bullet(\tilde{X}, E) = HP_\bullet(X, M)$. These two isomorphisms imply the third one, $HP_\bullet(\tilde{X}) = HP_\bullet(X)$, and we obtain the proof for part (c).

In part (d), we again consider the case of a single blowing up. Let π, $X \supset M$, $\tilde{X} \supset E$ be the same as above and let $Z \subset X$ be any closed subset. The subvariety $\pi^{-1}(Z)$ in $\tilde{X}$ may have many components (even their dimensions may differ), so let us split these into two groups, $\pi^{-1}(Z) = Z' \cup F$, where

$$F = \pi^{-1}(Z \cap M)\,.$$

In other words, Z' is the union of the proper preimages of those components of Z not contained in M. So we have an isomorphism $Z - Z \cap M \simeq Z' - Z' \cap F$, which by Lemma 3.4 gives $HP_\bullet(Z, Z \cap M) = HP_\bullet(Z', Z' \cap F)$. Besides, for $\pi^{-1}(Z) = Z' \cup F$, we can write tautologically that $HP_\bullet(Z', Z' \cap F) = HP_\bullet(\pi^{-1}(Z), F)$ and, hence,

$$HP_\bullet(Z, Z \cap M) = HP_\bullet(\pi^{-1}(Z), F)\,.$$

Taking into account that $HP_\bullet(F) = HP_\bullet(Z \cap M)$, which follows from (b) for the fibration $F \to Z \cap M$, we conclude that

$$HP_\bullet(Z) = HP_\bullet(\pi^{-1}(Z))\,.$$

The remaining equality, $HP_\bullet(X, Z) = HP_\bullet(\tilde{X}, \pi^{-1}(Z))$ follows from (c), i.e. $HP_\bullet(X) = HP_\bullet(\tilde{X})$, and by consideration of the map of pairs $(\tilde{X}, \pi^{-1}(Z)) \to (X, Z)$. Thus we have proved (d) and the whole lemma. ■

3.7 If V is a closed hypersurface in X, the embedding $i : V \hookrightarrow X$ induces the corresponding homomorphisms in (co)homology. Namely, the polar homology maps forward,

$$i_* : HP_q(V) \to HP_q(X)\,. \tag{16}$$

We have also the restriction map in sheaf cohomology, $i^* : H^q(X, \mathcal{O}_X) \to H^q(V, \mathcal{O}_V)$. If V is smooth (or normal crossing), then by Serre duality, i^* produces the following covariant homomorphism:

$$i' : H^{n-1-q}(V, K_V) \to H^{n-q}(X, K_X)\,. \tag{17}$$

The proof of Theorem 3.1 will be achieved essentially by comparing the homomorphisms (16) and (17) and using (the simplest case of) Lefschetz's hyperplane theorem. To describe this we begin with a vanishing theorem.

Proposition 3.8 *Let V be an ample divisor and D be a normal crossing divisor in a smooth projective manifold X. Then*

$$H^p(X, K_X(V + D)) = 0\,,\ \ p > 0\,.$$

This mild generalization (i.e. to $D \neq \varnothing$) of the Kodaira vanishing theorem can be found in ref. [EV]. Now suppose also that V is a normal crossing divisor. Then the long exact sequence in cohomology of

$$0 \to K_X(D) \to K_X(V + D) \to K_V(D) \to 0\,, \tag{18}$$

gives the following.

Proposition 3.9 *If V and D are normal crossing divisors in a smooth projective X, with V ample, then*

$$\begin{aligned} i' :\ & H^p(V, K_V(D)) \xrightarrow{\sim} H^{p+1}(X, K_X(D)) \quad \text{for } p > 0\,,\\ i' :\ & H^0(V, K_V(D)) \twoheadrightarrow H^1(X, K_X(D))\,. \end{aligned}$$

Proposition 3.10 *If V is an ample normal crossing subvariety in a smooth projective X and $m = \operatorname{codim} V$, then*

$$\begin{array}{rlcll} i': & H^p(V, K_V) & \xrightarrow{\sim} & H^{p+m}(X, K_X) & \text{for } p > 0\,, \\ i': & H^0(V, K_V) & \twoheadrightarrow & H^m(X, K_X)\,. & \end{array}$$

This follows trivially from the Lefschetz theorem (Proposition 3.9) by considering a flag $V = V^m \subset V^{m-1} \subset \ldots \subset V^1 \subset V^0 = X$ with V^{i+1} being an ample normal crossing divisor in V^i (such a flag exists by definition).

Proposition 3.11 *Let $V = V^m \subset V^{m-1} \subset \ldots \subset V^1 \subset V^0 = X$ be as above and let $D \subset X$ be a normal crossing divisor which intersects each V^i transversely (so that $D \cap V^i$ is also a normal crossing divisor in V^i). Then*

$$\begin{array}{rlcll} i': & H^p(V, K_V(D)) & \xrightarrow{\sim} & H^{p+m}(X, K_X(D)) & \text{for } p > 0\,, \\ i': & H^0(V, K_V(D)) & \twoheadrightarrow & H^m(X, K_X(D))\,. & \end{array}$$

3.12 Remark. Suppose Theorem 3.2 is proven. Then Proposition 3.9 has also a similar implication in polar homology (with $D = \varnothing$), namely:

$$\begin{array}{rlcll} i_*: & HP_k(V) & \xrightarrow{\sim} & HP_k(X) & \text{for } k < n-1\,, \\ i_*: & HP_{n-1}(V) & \twoheadrightarrow & HP_{n-1}(X)\,. & \end{array}$$

It may be interesting to notice that this has the following topological analogue. For an n-dimensional CW-complex X and its $(n-1)$-skeleton $i\colon V \hookrightarrow X$, the map $i_*: H_q(V) \to H_q(X)$ is an isomorphism of cellular homology for $0 \leqslant q < n-1$ and is surjective for $q = n-1$.

Thus, by Lefschetz's theorem in the form of Proposition 3.9 one can view an *ample divisor* in the context of polar homology as an analogue of the $(n-1)$-*skeleton* in topology. Of course, the Morse theory proof of the Lefschetz theorem shows that the topological $(n-1)$-skeleton can indeed be taken to lie in the hyperplane.

3.13 Proof of Theorem 3.1. Let us show first that the map ρ in eq. (7) is surjective. Take an arbitrary ample smooth subvariety $i: V \hookrightarrow X$, $\dim V = q$. Then $i': H^0(V, K_V) \twoheadrightarrow H^{n-q}(X, K_X)$ is surjective by the Lefschetz theorem 3.10. But each element $\alpha \in H^0(V, K_V)$ corresponds, by definition, to a cycle $a = (V, i, \alpha)$ in $HP_q(X)$ and $\rho([a]) = i'(\alpha)$. Thus ρ is onto.

To prove injectivity we must show that for a q-cycle a the vanishing $\rho([a]) = 0 \in H^{n-q}(X, K_X)$ implies that $a = \partial b$ for some polar $(q+1)$-chain b. Let $a = \sum_k (A_k, f_k, \alpha_k) \in \mathcal{C}_q(X)$, $\partial a = 0$, be an arbitrary q-cycle. Its support, $\operatorname{supp} a = Z = \cup_k Z_k$, may be a singular reducible subvariety[7] in X. Let Z_{sing} be the subset of singular points of Z (including, of course, possible points of intersection of its components). By the Hironaka theorem we can find a blow-up $\pi: \tilde{X} \to X$ such that the following conditions are satisfied.

[7]We may suppose without loss of generality that Z has the same number of components as the number of terms in $a = \sum_k (A_k, f_k, \alpha_k)$.

a) There is a q-dimensional subvariety $\tilde Z \subset \tilde X$ which consists of smooth non-intersecting components and such that $\pi(\tilde Z) = Z$ and π gives us a birational map of $\tilde Z$ onto Z.

b) $\tilde Z$ is included into a nested sequence of subvarieties:

$$\tilde Z \subset \tilde Y = V^{n-q-1} \subset V^{n-q-2} \subset \ldots \subset V^1 \subset V^0 = \tilde X \tag{19}$$

where $\operatorname{codim} V^i = i$ (in particular, $\dim \tilde Y = q+1$) and each V^{i+1} is an ample normal crossing divisor in V^i, so that $\tilde Y$, in particular, is an ample normal crossing subvariety in $\tilde X$. (If $q = n$ our proposition is obvious: $HP_n(X) = H^0(X, K_X)$, while for $q = n-1$ we set simply $\tilde Y = \tilde X$.)

c) The preimage $D := \pi^{-1}(Z_{\text{sing}})$ of the singular locus of Z is a normal crossing divisor in $\tilde X$ which also intersects transversely $\tilde Z$, $\tilde Y$ as well as all other elements V^i of the flag (19).

We can ensure this by applying the Proposition 2.7 to each component of Z. The possibility to satisfy the condition (c) is also guaranteed by the Hironaka theorem. After that we can achieve the ampleness of $V^1, V^2, \ldots, \tilde Y$ by adding sufficiently ample components to them, which can be done preserving normal crossings.

We are now prepared to replace the original polar cycle $a \in \mathcal{C}_q(X)$, which has a singular support $Z \subset X$, with a cycle supported on $\tilde Z$ in $\tilde X$. Recall that $\tilde Z$ may have several components, $\tilde Z = \cup_k \tilde Z_k$, but these do not intersect. Each q-dimensional smooth subvariety $i_k : \tilde Z_k \hookrightarrow \tilde X$ acquires a meromorphic q-form $\tilde\alpha_k$ defined on $\tilde Z_k$, which has at worst simple poles on $\tilde Z_k \cap D$, that is only over the singularities of Z. This can be seen by noticing that there exists a smooth manifold $\tilde A_k$ birational to A_k with a commutative square

$$\begin{array}{ccc} \tilde A_k & \longrightarrow & \tilde Z_k \\ \big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ A_k & \xrightarrow{f_k} & Z_k \end{array}$$

which allows us to pull back α_k from A_k to $\tilde A_k$ and then to push it forward to $\tilde Z_k$. Thus, each triple $(\tilde Z_k, \tilde i_k, \tilde\alpha_k)$ is admissible and defines a q-chain in $\tilde X$. However, the sum of these triples, $\tilde a = \sum_k (\tilde Z_k, \tilde i_k, \tilde\alpha_k)$, does not necessarily form a cycle[8]. Nevertheless, $\tilde a$ has no boundary modulo D in $\tilde X$. In other words, we will consider $\tilde a$ as a q-cycle in $\mathcal{C}_q(\tilde X, D)$.

Now we suppose that $\rho([a]) = 0 \in H^{n-q}(X, K_X)$ and try to prove that $[a] = 0$ in $HP_q(X)$. Let us note first that by (5) it is enough to prove the vanishing of $[a]$ modulo $Z_{\text{sing}} \subset X$, that is in $HP_q(X, Z_{\text{sing}})$, because $\dim Z_{\text{sing}} < \dim Z = q$ and so $HP_q(Z_{\text{sing}}) = 0$. Secondly, since $\pi_* : HP_q(\tilde X, D) \xrightarrow{\sim} HP_q(X, Z_{\text{sing}})$ by Lemma 3.6(d)

[8]For example, a 1-cycle in X can be supported on a self-intersecting rational curve Z. Then the resolved smooth curve $\tilde Z \subset \tilde X$ will be equipped with a meromorphic 1-form which has simple poles at the resolution of the double point of Z and, hence, the resolved curve is no longer a cycle in $\tilde X$.

and since, obviously, $\pi_*[\tilde a] = [a]$ it is sufficient to prove that $[\tilde a] = 0 \in HP_q(\tilde X, D)$. To prove this latter vanishing we have to show first that $\tilde\rho([\tilde a]) = 0$, where

$$\tilde\rho : HP_q(\tilde X, D) \to H^{n-q}(\tilde X, K_{\tilde X}(D))$$

is the relative analogue of the map ρ, which is the subject of the proposition under consideration (cf. eqs. (7) and (8)). For this aim let us collect the relevant maps in polar homology recalling the isomorphisms in Lemma 3.6 as well as the isomorphism $H^{n-q}(\tilde X, K_{\tilde X}) \xrightarrow{\sim} H^{n-q}(X, K_X)$, which holds for smooth birationally equivalent X and $\tilde X$, in the following commutative diagram:

$$\begin{tikzcd}
{[a] \in HP_q(X)} \arrow[rr, hook] \arrow[ddd, "\rho"] & & HP_q(X, Z_{\mathrm{sing}}) & \\
& HP_q(\tilde X) \arrow[ul, "\pi_*"', "\sim"] \arrow[rr] \arrow[d] & & HP_q(\tilde X, D) \ni [\tilde a] \arrow[ul, "\pi_*"', "\sim"] \arrow[d, "\tilde\rho"] \\
& H^{n-q}(\tilde X, K_{\tilde X}) \arrow[rr] \arrow[dl, "\pi_*"', "\sim"] & & H^{n-q}(\tilde X, K_{\tilde X}(D)) \\
H^{n-q}(X, K_X) & & &
\end{tikzcd}$$

Then, from $\rho([a]) = 0$, it follows that $\tilde\rho([\tilde a]) = 0 \in H^{n-q}(\tilde X, K_{\tilde X}(D))$.

We are ready now to finish the proof. To simplify the notations let us write $\tilde a = (\tilde Z, \tilde\imath, \tilde\alpha)$ for the sum $\sum_k (\tilde Z_k, \tilde\imath_k, \tilde\alpha_k)$, where $\tilde\alpha \in H^0(\tilde Z, K_{\tilde Z}(D))$, while $\tilde\imath \colon \tilde Z \hookrightarrow \tilde X$ is the embedding of the union of smooth non-intersecting components $\tilde Z = \cup_k \tilde Z_k$ into $\tilde X$. The map $\tilde\rho$ applied to $\tilde a$ corresponds to the map $\tilde\imath' \colon H^0(\tilde Z, K_{\tilde Z}(D)) \to H^{n-q}(\tilde X, K_{\tilde X}(D))$, that is to say, $\tilde\rho([\tilde a]) = \tilde\imath'(\tilde\alpha)$. Thus, we have that $\tilde\imath'(\tilde\alpha) = 0$. Since the embedding $\tilde\imath \colon \tilde Z \hookrightarrow \tilde X$ can be described as a composition of two embeddings, $\tilde\imath_{\tilde Y} : \tilde Z \hookrightarrow \tilde Y$ and $\tilde\jmath \colon \tilde Y \hookrightarrow \tilde X$ the above map $\tilde\imath'$ factors in this case through $H^1(\tilde Y, K_{\tilde Y}(D))$:

$$H^0(\tilde Z, K_{\tilde Z}(D)) \xrightarrow{\tilde\imath'_{\tilde Y}} H^1(\tilde Y, K_{\tilde Y}(D)) \underset{\sim}{\xrightarrow{\tilde\jmath'}} H^{n-k}(\tilde X, K_{\tilde X}(D)), \tag{20}$$

where $\tilde\jmath' \circ \tilde\imath'_{\tilde Y} = \tilde\imath'$ and $\tilde\jmath'$ is an isomorphism by the ampleness of $\tilde Y$ (see Proposition 3.11). It follows that $\tilde\imath'_{\tilde Y}(\tilde\alpha) = 0$ and the problem reduces to a codimension one situation: $\tilde Z \subset \tilde Y$. We can consider now the following exact sequence:

$$0 \to K_{\tilde Y}(D) \to K_{\tilde Y}(D \cap \tilde Y + \tilde Z) \xrightarrow{res_{\tilde Z}} K_{\tilde Z}(D) \to 0 \tag{21}$$

and the corresponding long sequence in cohomology. The latter allows us to conclude that the vanishing $\tilde\imath'_{\tilde Y}(\tilde\alpha) = 0$, $\tilde\alpha \in H^0(\tilde Z, K_{\tilde Z}(D))$, implies that $\tilde\alpha = \mathrm{res}_{\tilde Z}\tilde\beta$ for some $\tilde\beta \in H^0(\tilde Y, K_{\tilde Y}(D \cap \tilde Y + \tilde Z))$. In terms of polar chains in $\tilde X$ (modulo D), this means that $\tilde a = (\tilde Z, \tilde\imath, \tilde\alpha) = \partial(\tilde Y, \tilde\jmath, \tilde\beta)$, or $[\tilde a] = 0 \in HP_q(\tilde X, D)$. As we explained above, this implies that $[a] = 0 \in HP_q(X)$, which proves the injectivity of ρ. ■

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References

[AKMW] D. Abramovich, K. Karu, K. Matsuki, and J. Włodarczyk, *Torification and factorization of birational maps*, J. Amer. Math. Soc. **15** (2002) 531–572 [math.AG/9904135]

[EV] H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, **20**, Birkhauser Verlag, Basel, (1992)

[G] P.A. Griffiths, *Variations on a Theorem of Abel*, Invent. Math., **35** (1976) 321–390

[GH] P.A. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, NY (1978)

[H] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, I*, Ann. of Math. (2) **79** (1964) 109–203;
H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero, II*, Ann. of Math. (2) **79** (1964) 205–326

[KR1] B. Khesin and A. Rosly, *Polar homology*, [math.AG/0009015]

[KR2] B. Khesin and A. Rosly, *Polar Homology and Holomorphic Bundles,* Phil. Trans. R. Soc. Lond., Ser. A, **359**, (2001) 1413–1428 [math.AG/0102152]

[W] J. Włodarczyk, *Combinatorial structures on toroidal varieties and a proof of the weak Factorization Theorem*, [math.AG/9904076]